A stochastic variational condition for the Maxwell connections

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Abstract: The infinitesimal reduced variation of the stochastic parallel displacement on the commutative principal bundle is martingale if and only if the connections satisfy the Maxwell equations.

Key words: Stochastic parallel displacement; Malliavin calculus; Maxwell connection; Yang-Mills connection.

1. Introduction and results. In this brief note we shall show some connections between the Maxwell equations and the stochastic parallel displacement along the Brownian path as an application of the stochastic variational theory by P. Malliavin ([2]).

R. O. Bauer ([1]) recently showed that some variational problems of stochastic parallel displacement give the non-commutative Maxwell equations or the Yang-Mills equations. He perturbed the connections of the principal bundle which determine the stochastic parallel displacement to get the Yang-Mills equations; but we shall show that we can substitute the variational problem on the path space in Malliavin's sense for the perturbation of the connections at least in commutative situation.

We can start with the simple setting:

Let $(X_t^1, X_t^2, \dots, X_t^n)$ be the standard Brownian motion on the n dimensional Euclidian space and X_t^{n+1} be the stochastic parallel displacement along the Brownian path $(X_t^1, \dots, X_t^n)$. Our process $X_t = (X_t^1, \dots, X_t^n, X_t^{n+1})$ is defined on the trivial principal bundle $\mathbf{R}^n \times U(1)$ by the following stochastic differential equations (S.D.E.):

$$\begin{aligned} dX_t^i &= dw_t^i, \quad (i = 1, 2, \dots, n) \\ dX_t^{n+1} &= \sum_{\mu=1}^{n} A_\mu(X_t^1, \dots, X_t^n) X_t^{n+1} \circ dw_t^\mu, \\ X_0^i &= x^i, \quad (i = 1, 2, \dots, n+1) \end{aligned}$$

where $(w_t^1, \dots, w_t^n)$ is the standard Brownian motion on $\mathbf{R}^n$ and $A_\mu(x)$ are the pure imaginary smooth functions $A_\mu : \mathbf{R}^n \to i\mathbf{R}$, which are called the connections on the principal bundle or the vector potentials. The initial datas $x^1, \dots, x^n$ are real valued and x^{n+1} is an element of $U(1)$. We denote by $\circ dw_t$ the Stratonovitch differential. Let us denote simply the system of the S.D.E. above in the one line:

$$dX_t^i = \sum_{\mu=1}^{n} \tilde{A}_\mu^i(X_t) \circ dw^\mu$$

for $i = 1, 2, \dots, n+1$, where $\tilde{A}_\mu^i$ is naturally defined by A_μ and the initial conditions are the same above. We consider the derivative on the path space in Malliavin's sense. We know that the derivative $D_w X_t$ for the process $X_t = X_t(x, w)$ is given by

$$\begin{aligned} D_w X_t[h] &= \langle f_s, h \rangle \\ &= Y_t \int_0^t (Y_s^{-1} \tilde{A}(X_s)) \dot{h}(s) ds, \end{aligned}$$

where $Y_t = Y_j^i(t) = \partial X^i / \partial x^j(x, w)$ and $h \in \mathbf{CM}$ (the Cameron-Martin space), that is, h is absolutely continuous and its Radon-Nikodym derivative $\dot{h}$ is square integrable.

Note that f_s means the infinitesimal reduced variation of the stochastic parallel displacement (see [3]).

In this simple setting, our result is the following.

Theorem 1. *f_s is a martingale if and only if A_μ satisfies the Maxwell equations.*

Remark. We can automatically extend this to the theorem for the trivial principal bundles of differentiable Riemannian manifolds and commutative groups because our proof below can also be applied to the situation in the same way. The essential point of our proof is that we can explicitly solve the S.D.E. of the stochastic parallel displacement, and it works in the general commutative case.

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2. Proof of the theorem. Our proof is by explicit calculations. Of course we can solve the S.D.E. on the preceding page:

$$X^i(t) = x^i + w_t^i \quad (i = 1, 2, \ldots, n)$$

and

$$X^{n+1}(t) = x^{n+1} \exp \left\{ \int_0^t \sum_{\mu=1}^n A_\mu(X^1(s), \ldots, X^n(s)) \circ dw_s^\mu \right\}.$$

By this explicit form of $X^i(t)$, we can get the expression of the matrix valued process $Y_j^i(t)$:

$$Y_j^i(t) = \begin{pmatrix} 1 & 0 & 0 \ldots & 0 \\ 0 & 1 & 0 \ldots & 0 \\ 0 & 0 & 1 \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 \; 1 & 0 \\ Y_1^{n+1} & Y_2^{n+1} & \ldots\ldots & Y_{n+1}^{n+1} \end{pmatrix},$$

where Y_j^{n+1} are

$$Y_j^{n+1}(t) = X_t^{n+1} \int_0^t \frac{\partial A_\mu}{\partial x^j} \circ dw^\mu$$

for $j = 1, 2, \ldots, n$, and

$$Y_{n+1}^{n+1}(t) = \exp \int_0^t \sum_{\mu=1}^n A_\mu \circ dw_s^\mu.$$

Remember that $\tilde{A}_\mu^i(x)$ is the $(n+1, n)$ -matrix, whose entries are

$$\tilde{A}_\mu^i(x) = \begin{cases} \delta_{i\mu}, & (1 \leq i, \mu \leq n) \\ A_\mu^{n+1}(x^1, \ldots, x^n) x^{n+1}, & (i = n+1) \end{cases}$$

where $\delta_{i\mu}$ is 1 if $i = \mu$ and is 0 the otherwise. Here we can get the form of $f_s = Y^{-1}\tilde{A}$ by easy calculations. Then the non-constant entries of f_s are

$$r_\nu = x^{n+1} \left(\int_0^t \sum_\mu \partial_\nu A_\mu \circ dw^\mu - A_\nu \right)$$

where ∂_ν denotes the derivative by ν-th coordinate. By Itô formula,

$$\begin{aligned} \frac{r_\nu}{x^{n+1}} &= \int_0^t \sum_\mu \partial_\nu A_\mu \circ dw^\mu \\ &\qquad - A_\nu(0) - \int_0^t \sum_\mu \partial_\mu A_\nu \circ dw^\mu \\ &= -A_\nu(0) \\ &\qquad + \int_0^t \sum_\mu (\partial_\nu A_\mu - \partial_\mu A_\nu) \circ dw^\mu \\ &= -A_\nu(0) \\ &\qquad + \int_0^t \sum_\mu (\partial_\nu A_\mu - \partial_\mu A_\nu) dw^\mu \\ &\qquad + \frac{1}{2} \int_0^t \sum_\mu \partial_\mu (\partial_\nu A_\mu - \partial_\mu A_\nu) dt. \end{aligned}$$

In the last line above, we rewrote the Stratonovitch type integral to the Itô type one. If f_s is a martingale, the third term of the right hand side vanishes for any coordinate ν and now we get the Maxwell equations on the n dimensional state space with imaginary time:

$$\sum_\mu \partial_\mu F_{\nu\mu} = 0,$$

where

$$F_{\nu\mu} = (\partial_\nu A_\mu - \partial_\mu A_\nu)$$

is the curvature of the principal bundle or the field strength. If the Maxwell equations are satisfied, f_s is a martingale. The proof of the theorem is complete.

The author have not found the suitable setting for non-commutative case at present, but we shall discuss this problem in the following article.

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