A note on $K3$ surfaces and sphere packings

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Abstract: We study the Mordell-Weil lattices ([9]) of the elliptic $K3$ surfaces which have been introduced by Inose [4] and Kuwata [6]. The point is that the lattices (of rank up to 18) arising this way can be of some interest in terms of sphere packings. In this note, we treat the case of rank 16, 17 or 18, and show that these Mordell-Weil lattices are *potentially comparable* with the record lattices in these dimensions ([2]). The detailed account is in preparation, which will include the corresponding results for other dimensions as well.

Key words: Mordell-Weil lattice; $K3$ surface; sphere packings.

1. Elliptic $K3$ surfaces of Inose and Kuwata. Recently Kuwata [6] has constructed elliptic $K3$ surfaces with high Mordell-Weil rank (up to rank 18), starting from the Kummer surfaces $\mathrm{Km}(A)$ of the product abelian surfaces $A = C_1 \times C_2$, where C_1 and C_2 are two elliptic curves defined over k, an algeraically closed field contained in the field of complex numbers $\mathbf{C}$. More precisely, he defines six elliptic $K3$ surfaces $F^{(n)}$ over $\mathbf{P}^1$ $(1 \le n \le 6)$ for a given A, with the property that (i) $F^{(2)}$ is isomorphic to $\mathrm{Km}(A)$ and (ii) for each divisor d of n $(n = de)$, $F^{(n)}$ is obtained as the base change of $F^{(e)}$ by the map $t \mapsto t^d$ of degree d. Then he determines the Mordell-Weil rank $r^{(n)}$ of $F^{(n)}$ by the wellknown relation of the Picard number and the Mordell-Weil rank, using Inose's theorem ([3]) that two $K3$ surfaces X, Y have the same Picard number $\rho(X) = \rho(Y)$ if there is a rational map of finite degree between them. The latter implies

$$(1) \quad \begin{aligned} &\rho(F^{(n)}) = \rho(\mathrm{Km}(C_1 \times C_2)) = 18 + h, \\ &h = \mathrm{rk}\,\mathrm{Hom}(C_1, C_2). \end{aligned}$$

Actually Inose [4] has earlier studied the elliptic fibration $F^{(2)}$ on the Kummer surface $\mathrm{Km}(C_1 \times C_2)$, and constructed the elliptic $K3$ surface $F^{(1)}$, together with its defining equation:

$$(2) \quad \begin{aligned} y^2 = x^3 &- (3\sqrt[3]{j_1 j_2})t^4 x \\ &+ t^5(t^2 + 1 - 2t\sqrt{(j_1 - 1)(j_2 - 1)}) \end{aligned}$$

where j_1, j_2 denote the j-invariant of C_1, C_2 respectively. The reducible fibers of $F^{(1)}$ are as follows. There are two reducible fibers of Kodaira type II^* at $t = 0$ and ∞, and no other ones if $j_1 \ne j_2$. If $j_1 = j_2 \ne 0, 1$, there is an additional one of type I_2; if $j_1 = j_2 = 1$, then two of type I_2; finally if $j_1 = j_2 = 0$, then there is one ot type IV.

This information determines the structure of trivial lattice of $F^{(n)}$ $(n \le 6)$ to be $U \oplus V^-$, where U is the hyperbolic lattice of rank 2 and $V = V^{(n)}$ is a sum of the root lattices. In case $j_1 \ne j_2$, we have, according as $n = 1, \ldots, 6$,

$$(3) \qquad V^{(n)} = E_8^{\oplus 2}, E_6^{\oplus 2}, D_4^{\oplus 2}, A_2^{\oplus 2}, \{0\}, \{0\}.$$

In case $j_1 = j_2 \ne 0, 1$ (resp. $= 1$ or $= 0$), $V^{(n)}$ has an additional factor $A_1^{\oplus n}$ (resp. $A_1^{\oplus 2n}$ or $A_2^{\oplus n}$).

In view of (1), the rank $r^{(n)} = \rho(F^{(n)}) - 2 - \mathrm{rk}\,V^{(n)}$ is given as follows (cf. [6, Th.4.1]): for any $n \le 6$, we have

$$(4) \qquad r^{(n)} = \mathrm{Min}\{4(n-1), 16\} + h - \begin{cases} 0 & \text{if } j_1 \ne j_2 \\ n & \text{if } j_1 = j_2 \ne 0, 1 \\ 2n & \text{if } j_1 = j_2 = 0 \text{ or } 1. \end{cases}$$

Note, in particular, that if $j_1 \ne j_2$ (i.e. if C_1 and C_2 are not isomorphic), then $F^{(5)}$ and $F^{(6)}$ have no reducible fibres, and we have

$$r^{(n)} = \rho(F^{(n)}) - 2 = 16 + h \quad (n = 5, 6).$$

2. Reconstruction. Recall that the Kummer surface $X = \mathrm{Km}(A)$ is a smooth $K3$ surface obtained from the quotient surface A/ι_A (ι_A: the inversion map of A) by resolving the 16 singular points corresponding to the points of order 2 on A.

Suppose $A = C_1 \times C_2$ where C_i is defined by the

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Weierstrass equation

$$C_i : y_i^2 = f_i(x_i) = x_i^3 + \cdots \quad (i = 1, 2). \tag{5}$$

Then X is birational to the surface defined in the (x_1, x_2, t)-space by

$$f_1(x_1)t^2 = f_2(x_2). \tag{6}$$

The function $t = y_2/y_1$ on A is invariant under ι_A, and it defines an elliptic fibration on X whose generic fibre is the plane cubic curve defined by the above equation over $k(t)$. This yields the elliptic $K3$ surface $F^{(2)}$.

We consider the linear pencil of cubic curves (in the projective plane with affine coordinates x_1, x_2), associated with (6):

$$f_1(x_1)t = f_2(x_2) \quad (t \in \mathbf{P}^1). \tag{7}$$

It has the 9 base points corresponding to the 9 (out of 16) torsion points of order 2 on A. By choosing one of them $x^0 = (x_1^0, x_2^0)$, we perform a *Weierstrass transformation* (cf. [11, §2]). Namely, by transforming the cubic curve (7) into a Weierstrass cubic over $k(t)$ so that x^0 is mapped to the point at infinity O, we obtain

$$E^{(1)} : y^2 = x^3 + t^2 Ax + t^2 B(t) \tag{8}$$

where

$$A \in k, \; B(t) = B_0 t^2 + B_1 t + B_2 \in k[t] \quad (B_0, B_2 \neq 0) \tag{9}$$

are determined in terms of the coefficients of f_i. We then consider the twist of the elliptic curve $E^{(1)}$

$$F^{(1)} : y^2 = x^3 + t^4 Ax + t^5 B(t) \tag{10}$$

with respect to the quadratic extension $k(\sqrt{t})/k(t)$. This is essentially the same as Inose's equation (2) and that given by Kuwata [6].

Further let $E^{(n)}/k(t)$ (resp. $F^{(n)}/k(t)$) denote the elliptic curve obtained from (8) (resp. (10)) by replacing $t \to t^n$. In other words, fixing a compatible system of variables $\{t_n (1 \leq n \leq 6)\}$ such that $(t_n)^n = t_1$, $(t_n)^d = t_e$ for $n = de$, we define the elliptic curves over $k(t_n)$ by

$$\begin{aligned} E^{(n)} &= E^{(1)} \otimes_{k(t_1)} k(t_n), \\ F^{(n)} &= F^{(1)} \otimes_{k(t_1)} k(t_n) \; (1 \leq n \leq 6). \end{aligned} \tag{11}$$

We sometimes write t for t_n in dealing with $E^{(n)}/k(t_n)$.

In the following, given an elliptic curve $E/k(t)$, we use the same symbol E to denote the elliptic surface over $\mathbf{P}^1$ (the t-line). Note that we have

$$E^{(2)} \simeq F^{(2)} \simeq \mathrm{Km}(C_1 \times C_2). \tag{12}$$

Also we have $E^{(4)} \simeq F^{(4)}$ and $E^{(6)} \simeq F^{(6)}$. The elliptic surfaces $F^{(n)}$ are $K3$ surfaces for all $n \leq 6$, while $E^{(1)}$ and $E^{(3)}$ are rational elliptic surfaces. There is a rational map of degree $d = n/e$

$$\pi = \pi_{n,e} : F^{(n)} \longrightarrow F^{(e)}, \; (x, y, t) \mapsto (x, y, t^d). \tag{13}$$

3. Mordell-Weil lattices of rank 16,17,18.

Now we assume that C_1 and C_2 are ont isomorphic (i.e. $j_1 \neq j_2$), and let $h = \mathrm{rk}\,\mathrm{Hom}(C_1, C_2)$ as before. Also we denote by C_τ the elliptic curve isomorphic to $\mathbf{C}/\mathbf{Z}+\mathbf{Z}\tau$ as a complex torus, $\tau \in \mathbf{C}$ being a point in the upper half plane $\mathcal{H}$.

Theorem 3.1. *The Mordell-Weil lattice $L^{(5)} = F^{(5)}(k(t))$ is a positive-definite even integral lattice of rank $r = 16 + h$ and minimal norm $\mu = 4$. Its determinant and the center density δ are given as follows*:

(i) *If $h = 0$ (i.e. C_1 and C_2 are not isogenous), we have $r = 16$ and*

$$\det L^{(5)} = 5^4/\nu^2, \quad \delta(L^{(5)}) = \nu/25$$

for some ν dividing 5^2. Actually we have $\nu = 1$ here (see below).

(ii) *If $h = 1$, let $m \geq 2$ be the minimal degree of isogenies $\phi : C_1 \to C_2$ (the assumption $j_1 \neq j_2$ implies that $m \neq 1$). Then $r = 17$ and*

$$\det L^{(5)} = 2m \cdot 5^3/\nu^2, \quad \delta(L^{(5)}) = \nu/5 \cdot \sqrt{10m}$$

for some ν dividing 5^3.

(iii) *If $h = 2$, we may assume that $C_1 = C_\tau$ and $C_2 = C_{m\tau}$ with $\tau \in \mathcal{H}$ satisfying $a\tau^2 + b\tau + c = 0$ for some integers a, b, c with $\gcd(a, b, c) = 1$, and some positive integer $m \geq 2$ ([7]). Then $r = 18$ and*

$$\det L^{(5)} = (4ac - b^2)m^2 5^2/\nu^2,$$
$$\delta(L^{(5)}) = \nu/5m\sqrt{4ac - b^2}$$

for some ν dividing 5^2.

Theorem 3.2. *The Mordell-Weil lattice $L^{(6)} = F^{(6)}(k(t))$ is a positive-definite even integral lattice of rank $r = 16 + h$ and minimal norm $\mu = 4$. Its determinant and the center density δ are given as follows*:

(i) *If $h = 0$ (i.e. C_1 and C_2 are not isogenous), we have $r = 16$ and*

$$\det L^{(6)} = 6^4/\nu^2, \quad \delta(L^{(6)}) = \nu/36$$

for some ν dividing 3^2.

(ii) *If $h = 1$, let $m \geq 2$ be the minimal degree of isogenies from C_1 to C_2 as before. Then $r = 17$ and*

$$\det L^{(6)} = 2m \cdot 6^3/\nu^2, \quad \delta(L^{(6)}) = \nu/12 \cdot \sqrt{3m}$$

for some ν dividing 3^3.

(iii) *If $h = 2$, we take $C_1 = C_\tau$ and $C_2 = C_{m\tau}$ with $\tau \in \mathcal{H}$ as before, with some $m \geq 2$. Then $r = 18$ and*

$$\det L^{(6)} = (4ac - b^2)m^2 6^2/\nu^2,$$
$$\delta(L^{(6)}) = \nu/6m\sqrt{4ac - b^2}$$

for some ν dividing 3^2.

Remark. According to [2, Table 1.2], the record lattice in dimension $r = 16, 17, 18$ is given by the lattice Λ_r, whose center density is as follows:

$$\delta(\Lambda_{16}) = 1/16, \ \delta(\Lambda_{17}) = 1/16, \ \delta(\Lambda_{18}) = 1/8\sqrt{3}.$$

Moreover an upper bound β_r for center density is known as follows:

$$\beta_{16} = 0.11774, \ \beta_{17} = 0.14624, \ \beta_{18} = 0.18629.$$

Let us compare the lattices in the above theorems with these data.

First, for Theorem 3.1:

1. In (i), $\nu = 1$ must hold, since otherwise the center density of $L^{(5)}$ will exceed the bound:

$$\delta(L^{(5)})|_{\nu=1} = 1/25 < \delta(\Lambda_{16})$$
$$= 1/16 < \beta_{16} < \delta(L^{(5)})|_{\nu=5} = 1/5.$$

2. In (ii), consider the case $m = 2$. Then $\nu = 1$ must hold again, because

$$\delta(L^{(5)})|_{\nu=1} = 1/10\sqrt{5} < \delta(\Lambda_{17})$$
$$= 1/16 < \beta_{17} < \delta(L^{(5)})|_{\nu=5} = 1/2\sqrt{5}.$$

3. In (iii), consider the case $a = b = c = 1$ and $m = 2$. Then $\nu = 1$ must hold again, because

$$\delta(L^{(5)})|_{\nu=1} = 1/10\sqrt{3} < \delta(\Lambda_{18})$$
$$= 1/8\sqrt{3} < \beta_{18} < \delta(L^{(5)})|_{\nu=5} = 1/2\sqrt{3}.$$

Next, for Theorem 3.2:

4. In (i), we have

$$\delta(L^{(6)})|_{\nu=1} = 1/36 < \delta(\Lambda_{16})$$
$$= 1/16 < \delta(L^{(6)})|_{\nu=3} = 1/12 < \beta_{16}.$$

Therefore, if $\nu = 3$, the lattice $L^{(6)}$ would break the record in dimension 16, but this is unlikely.

5. In (ii), consider the case $m = 2$. Then we have

$$\delta(L^{(6)})|_{\nu=1} = 1/12\sqrt{6} < \delta(\Lambda_{17})$$
$$= 1/16 < \delta(L^{(6)})|_{\nu=3} = 1/4\sqrt{6} < \beta_{17}.$$

Hence, if $\nu = 3$, the lattice $L^{(6)}$ would break the record in dimension 17.

6. In (iii), consider the case $a = b = c = 1$ and $m = 2$. Then

$$\delta(L^{(6)})|_{\nu=1} = 1/12\sqrt{3} < \delta(\Lambda_{18})$$
$$= 1/8\sqrt{3} < \delta(L^{(6)})|_{\nu=3} = 1/4\sqrt{3} < \beta_{18}.$$

Here again, if $\nu = 3$, the lattice $L^{(6)}$ would break the record in dimension 18.

4. Outline of the proof. The elliptic $K3$ surface $X = F^{(n)}$ $(n = 5, 6)$ has no reducible fibres under the assumption $j_1 \neq j_2$.

As for the minimal norm, this implies that the Mordell-Weil lattice coincides with the narrow Mordell-Weil lattice (see [9], [10] for this and other facts on MWL). Then the height formula says that

$$\langle P, P \rangle = 2\chi + 2(PO) \geq 4$$

for any $P \in F^{(n)}(k(t))$, $p \neq O$, which shows that the minimal norm μ of $L^{(n)}$ is at least 4 (cf. [9, Th.8.7]). That $\mu = 4$ follows from the following:

Proposition 4.1. *The lattice $L^{(n)}$ $(n = 5, 6)$ contains a sublattice isomorphic to $E_8[2]$ whose minimal norm is equal to* 4.

N. B. Given a lattice L, we denote by $L[n]$ the lattice whose pairing is n-times the original pairing on L. The letters $A_n, D_n, \ldots, E_s$ stand for the root lattices as usual.

The proof of Proposition 4.1 for $n = 6$ reduces to (ii) of the next proposition, by using the natural map of degree 2 from $F^{(6)}$ to $E^{(3)}$ in view of [9, Prop.8.12]. The case $n = 5$ in similar.

Proposition 4.2.

(i) *The Mordell-Weil lattice of the rational elliptic surafce $E^{(1)}$ is equal to $(A_2^*)^{\oplus 2}$, where A_2^* is the dual lattice of the root lattice A_2.*

(ii) *The Mordell-Weil lattice of the rational elliptic surface $E^{(3)}$ is isomorphic to the root lattice E_8 (cf. [8]).*

On the other hand, for the determinant, we have (ignoring the sign)

$$\det L^{(n)} = \det NS(F^{(n)}) = \det T(F^{(n)}), \tag{14}$$

where $T(X)$ denotes the lattice of transcendental cycles on a complex algebraic surface X. The first equality holds because there are no reducible fibres, while the second equality results from the following: by definition, $T(X)$ is the orthogonal complement of $NS(X)$ in $H^2(X, \mathbf{Z})$, which is (modulo torsion) a unimodular lattice by the Poincaré duality. Let $\lambda(X) = \mathrm{rk}\, T(X) = b_2(X) - \rho(X)$.

Based on Inose's work [3], [4], we can prove the following.

Lemma 4.3. *If X, Y are complex surfaces with the same geometric genus $p_g(X) = p_g(Y)$ and if $\pi : X \to Y$ is a rational map of degree n, then we have $\lambda(X) = \lambda(Y)$, and*

$$nT(X) \subset \pi^* T(Y) \subset T(X), \quad \pi^* T(Y) \simeq T(Y)[n].$$

Denoting by $\nu = \nu(X/Y)$ the index of $\pi^* T(Y)$ in $T(X)$, we have

$$\det T(X) = \det T(Y) \cdot n^{\lambda(X)}/\nu^2. \tag{15}$$

Theorem 4.4. *Let $A = C_1 \times C_2$ and let $F^{(n)}$ be the elliptic $K3$ surfaces defined in §2. Then, for any $n \leq 6$, $T(F^{(n)})$ has rank $\lambda = 4 - h$ and it contains $T(A)[n]$ as a sublattice of finite index ν which divides a power of n. Thus we have*

$$\det T(F^{(n)}) = \det T(A) \cdot n^{4-h}/\nu^2. \tag{16}$$

Moreover ν is equal to 1 if $n = 1$ or 2.

Thus we have outlined the proof of Theorems 3.1 and 3.2.

Of course, Mordell-Weil lattices contain more information on the rational points, not just the minimal norm or determinant. Here are some examples.

Example. (1) Take $C_1 : y^2 = x^3 - 1$, $C_2 : y^2 = x^3 - x$ so that $j_1 = 0$, $j_2 = 1$ and $h = 0$. Then $\mathrm{Km}(C_1 \times C_2)$ is isomorphic to the elliptic surface $F^{(2)} : y^2 = x^3 + t^4(t^4 + 1)$. Its Mordell-Weil lattice is $F^{(2)}(k(t)) \simeq (A_2^*[2])^{\oplus 2}$. It has 12 rational points $(x, y) = (at^2, \pm t^2(t^2 \pm 1)^2)$ $(a^3 = \pm 2)$ corresponding to the minimal vectors.

The Mordell-Weil lattice of $F^{(6)} : y^2 = x^3 + t^{12} + 1$ has rank 16. We have

$$\begin{aligned} F^{(6)}(k(t)) &\supset E^{(3)}(k(t^2)) \oplus F^{(3)}(k(t^2)) \\ &\supset E_8[2] \oplus (D_4^*[6])^{\oplus 2}, \end{aligned}$$

where the root lattices come from the Mordell-Weil lattices of rational elliptic surfaces. The generators (even basis) for this sublattice can be explicitly described.

In this case, $\det F^{(6)}(k(t)) = 36^2$ has been directly shown in Usui [12, II].

(2) Next take $C_1 : y^2 = x^3 - 1$ and let $\phi : C_1 \to C_2$ be an isogeny of degree 2. We have $h = 2$ in this case. Then $\mathrm{Km}(C_1 \times C_2)$ is isomorphic to the elliptic surface $F^{(2)} : y^2 = x^3 + t^4(t^4 - 11t^2 - 1)$, as given by Kuwata.

The Mordell-Weil lattice of $F^{(6)} : y^2 = x^3 + t^{12} - 11t^6 - 1$ has rank $16 + h = 18$. It has a sublattice of rank 16 which is similar to the above. The missing rank comes from $F^{(1)} : y^2 = x^3 + t^5(t^2 - 11t - 1)$. In [1], the authors have found rational points (sections) of height 4 on the latter.

Let $\Gamma \subset A = C_1 \times C_2$ be the graph of the isogeny $\phi : C_1 \to C_2$. By keeping track of the image of Γ under the rational map $A \to \mathrm{Km}(A) \to F^{(1)}$, these rational points can be found in a slightly more conceptual way.

More generally, $F^{(1)}$ is an interesting $K3$ surface related to $\mathrm{Hom}(C_1, C_2)$.

Proposition 4.5. *$\mathrm{Hom}(C_1, C_2)$ has the structure of an even lattice, with norm defined by $\phi \mapsto 2\deg(\phi)$. The Mordell-Weil lattice of the elliptic $K3$ surface $F^{(1)}$ has the same rank and determinant as the lattice $\mathrm{Hom}(C_1, C_2)$, provided that $j_1 \neq j_2$.*

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