The "abc" conjecture over function fields

By Pei-Chu HU*) and Chung-Chun YANG**)

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Abstract: This paper shows that the analogy of "abc" conjecture for non-Archimedean entire functions is true.

Key words: "abc" conjecture; non-Archimedean; entire functions.

1. Introduction. Mason (see [4], [5], [6]) started one recent trend of thoughts by discovering an entirely new relation among polynomials as follows. Let $f(z)$ be a polynomial with coefficients in an algebraically closed field of characteristic 0 and let $\overline{n}(1/f)$ be the number of distinct zeros of f.

Theorem 1.1 (Mason's theorem, cf. [3]). *Let $a(z)$, $b(z)$, $c(z)$ be relatively prime polynomials in κ and not all constants such that $a+b=c$. Then*

$$\max\{\deg(a), \deg(b), \deg(c)\} \le \overline{n}\left(\frac{1}{abc}\right) - 1.$$

Influenced by Mason's theorem, and considerations of Szpiro and Frey, Masser and Oesterlé formulated the "abc" conjecture for integers as follows:

Conjecture 1.1. *Let q be a non-zero integer. Define the radical of q to be*

$$\overline{N}\left(\frac{1}{q}\right) = \prod_{p|q} p$$

i.e. the product of the distinct primes dividing q. Given $\varepsilon > 0$, there exists a number $C(\varepsilon)$ having the following property. For any nonzero relatively prime integers a, b, c such that $a+b=c$, we have

$$\max\{|a|, |b|, |c|\} \le C(\varepsilon)\overline{N}\left(\frac{1}{abc}\right)^{1+\varepsilon}.$$

This conjecture also is a consequence of the Vojta's Conjecture (see Vojta [7]). Over the field of non-Archimedean meromorphic functions and its associated notations such as $T(r,a)$ and $\overline{N}(r,1/a)$ used in the value distribution theory (cf. §2). We can prove the following main theorem in this paper:

Theorem 1.2. *Let κ be an algebraically closed field of characteristic zero, complete for a non-trivial non-Archimedean absolute value $|\cdot|$. Let $a(z)$, $b(z)$, $c(z)$ be entire functions in κ without common zeros and not all constants such that $a+b=c$. Then*

$$\max\{T(r,a), T(r,b), T(r,c)\} \le \overline{N}\left(r, \frac{1}{abc}\right) - \log r + O(1).$$

If f is a polynomial, it is easy to show

$$\deg(f) = \lim_{r\to\infty} \frac{T(r,f)}{\log r}, \quad \overline{n}\left(\frac{1}{f}\right) = \lim_{r\to\infty} \frac{\overline{N}(r,1/f)}{\log r}.$$

Obviously, Mason's theorem can be deduced by Theorem 1.2 over the field in Theorem 1.2, and so Theorem 1.2 can be served as an analogue of "abc" conjecture over the field of non-Archimedean meromorphic functions. For complex variables, it is clear, by the second main theorem, that the term $\max\{T(r,a/c), T(r,b/c)\}$ but not $\max\{T(r,a), T(r,b), T(r,c)\}$ can be controlled by $\overline{N}(r, 1/(abc))$.

2. Basic facts. Let κ be an algebraically closed field of characteristic zero, complete for a non-trivial non-Archimedean absolute value $|\cdot|$. Define

$$\kappa[0;r] = \{z \in \kappa \mid |z| \le r\}.$$

Let $\mathcal{A}(\kappa)$ be the set of entire functions on κ. Then each $f \in \mathcal{A}(\kappa)$ can be given by a power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad (a_n \in \kappa), \tag{1}$$

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*) Department of Mathematics, Shandong University, Jinan 250100, Shandong, China.

**) Department of Mathematics, The Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong.

such that for any $z \in \kappa$, one has $|a_n z^n| \to 0$ as $n \to \infty$. Define the *maximum term*:

$$\mu(r, f) = \max_{n \geq 0} |a_n| r^n$$

with the associated *central index*:

$$n\left(r, \frac{1}{f}\right) = \max_{n \geq 0}\{n \mid |a_n| r^n = \mu(r, f)\}.$$

Then $n(r, 1/f)$ just is the *counting function* of zeros of f, which denotes the number of zeros (counting multiplicity) of f with absolute value $\leq r$. Fix a real ρ_0 with $0 < \rho_0 < \rho$. Define the *valence function* of f for a by

$$\text{(2)} \qquad N\left(r, \frac{1}{f-a}\right) = \int_{\rho_0}^{r} \frac{n(t, 1/(f-a))}{t} dt \quad (\rho_0 < r < \rho).$$

Then the following *Jensen formula*

$$\text{(3)} \qquad N\left(r, \frac{1}{f}\right) = \log \mu(r, f) - \log \mu(\rho_0, f)$$

holds. We also denote the number of distinct zeros of $f - a$ on $\kappa[0; r]$ by $\overline{n}(r, 1/(f-a))$ and define

$$\overline{N}\left(r, \frac{1}{f-a}\right) = \int_{\rho_0}^{r} \frac{\overline{n}(t, 1/(f-a))}{t} dt \quad (\rho_0 < r < \rho).$$

The field of fractions of $\mathcal{A}(\kappa)$ will is denoted by $\mathcal{M}(\kappa)$. An element f in the set $\mathcal{M}(\kappa)$ will be called a *meromorphic function* on κ. Take $f \in \mathcal{M}(\kappa)$. Since greatest common divisors of any two elements in $\mathcal{A}(\kappa)$ exist, then there are $g, h \in \mathcal{A}(\kappa)$ with $f = g/h$ such that g and h have not any common factors in the ring $\mathcal{A}(\kappa)$. We can uniquely extend μ to meromorphic function $f = g/h$ by defining

$$\mu(r, f) = \frac{\mu(r, g)}{\mu(r, h)} \quad (0 \leq r < \infty).$$

Define the *compensation function* by

$$m(r, f) = \max\{0, \log \mu(r, f)\}.$$

As usual, we define the *characteristic function*:

$$T(r, f) = m(r, f) + N(r, f) \quad (\rho_0 < r < \infty),$$

where

$$N(r, f) = N\left(r, \frac{1}{h}\right).$$

Lemma 2.1 (cf. [1]). *Let $f \in \mathcal{A}_r(\kappa)$ have q zeros in $\kappa[0; r]$ with $q \geq 1$ (taking multiplicities into account) and let $b \in f(\kappa[0; r])$. Then $f - b$ also admits q zeros in $\kappa[0; r]$ (counting multiplicity).*

Corollary 2.1. *Assume that f is a non-constant entire function. Then for any $b \in \kappa$, we have*

$$N\left(r, \frac{1}{f-b}\right) = O(1) + N\left(r, \frac{1}{f}\right).$$

Proof. Note that f and $f - b$ all have at least one zero since $f - b$ also is a non-constant entire function. Thus there is a $r' \in \mathbf{R}^+$ such that f has at least one zero in $\kappa[0; r']$ and such that $b \in f(\kappa[0; r'])$. By Lemma 2.1, one obtains

$$n\left(r, \frac{1}{f-b}\right) = n\left(r, \frac{1}{f}\right) \quad (r \geq r').$$

Therefore, when $r \geq r'$, we have

$$\begin{aligned} & N\left(r, \frac{1}{f-b}\right) \\ &= N\left(r', \frac{1}{f-b}\right) + \int_{r'}^{r} \frac{n(t, 1/(f-b))}{t} dt \\ &= N\left(r', \frac{1}{f-b}\right) + \int_{r'}^{r} \frac{n(t, 1/f)}{t} dt \\ &= N\left(r', \frac{1}{f-b}\right) - N\left(r', \frac{1}{f}\right) + N\left(r, \frac{1}{f}\right), \end{aligned}$$

and the corollary follows. □

Let f be a non-constant entire function in κ. Then

$$N\left(r, \frac{1}{f}\right) = \log \mu(r, f) - \log \mu(\rho_0, f) \to +\infty$$

as $r \to \infty$, and hence $\mu(r, f) > 1$ when r is sufficiently large. Therefore

$$N\left(r, \frac{1}{f}\right) = T(r, f) + O(1),$$

and hence Corollary 2.1 implies

$$\text{(4)} \qquad N\left(r, \frac{1}{f-a}\right) = T(r, f) + O(1)$$

for all $a \in \kappa$.

Take $f \in \mathcal{M}(\kappa)$ again and write $f = f_1/f_0$, where $f_0, f_1 \in \mathcal{A}(\kappa)$ have no common factors. Then

$$\tilde{f} = (f_0, f_1) : \kappa \longrightarrow \kappa^2$$

is called a *reduced representation* of f. Write

$$\mu(r, \tilde{f}) = \max\{\mu(r, f_0), \mu(r, f_1)\}.$$

Noting that

$$\begin{aligned}\log \mu(r,\tilde{f}) &= \max\{\log \mu(r,f_0), \log \mu(r,f_1)\}\\ &= \max\left\{0, \log \frac{\mu(r,f_1)}{\mu(r,f_0)}\right\} + \log \mu(r,f_0)\\ &= \max\{0, \log \mu(r,f)\} + \log \mu(r,f_0)\\ &= m(r,f) + \log \mu(r,f_0),\end{aligned}$$

and by the Jensen formula

$$N(r,f) = N\left(r, \frac{1}{f_0}\right) = \log \mu(r,f_0) - \log \mu(\rho_0, f_0),$$

we obtain

$$T(r,f) = \log \mu(r,\tilde{f}) - \log \mu(\rho_0, f_0), \tag{5}$$

or equivalently

$$T(r,f) = \log \mu(r,\tilde{f}) - \log \mu(\rho_0, \tilde{f}) + m(\rho_0, f). \tag{6}$$

By (5) and the Jensen formula, the following formula

$$T(r,f) = \max\left\{N(r,f), N\left(r, \frac{1}{f}\right)\right\} + O(1)$$

holds for a non-constant meromorphic function f in κ. Thus it is easy to prove that the following formula

$$T(r,f) = \max\left\{N\left(r, \frac{1}{f-a}\right), N\left(r, \frac{1}{f-b}\right)\right\} + O(1) \tag{7}$$

holds for any two distinct elements $a, b \in \kappa \cup \{\infty\}$.

3. Proof of the main theorem. Write

$$f = \frac{a}{c}, \quad g = \frac{b}{c}.$$

Then f and g all are not constants by our assumptions, and satisfy $f + g = 1$. By the second main theorem (see [2]), and noting that

$$\overline{N}\left(r, \frac{1}{f-1}\right) = \overline{N}\left(r, \frac{1}{g}\right) = \overline{N}\left(r, \frac{1}{b}\right),$$

we obtain

$$\begin{aligned}&T(r,f)\\ &\le \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f-1}\right) - \log r + O(1)\\ &= \overline{N}\left(r, \frac{1}{c}\right) + \overline{N}\left(r, \frac{1}{a}\right) + \overline{N}\left(r, \frac{1}{b}\right) - \log r + O(1)\\ &= \overline{N}\left(r, \frac{1}{abc}\right) - \log r + O(1).\end{aligned}$$

Similarly, we have

$$T(r,g) \le \overline{N}\left(r, \frac{1}{abc}\right) - \log r + O(1).$$

By (7) and (4),

$$\begin{aligned}T(r,f) &= \max\left\{N(r,f), N\left(r, \frac{1}{f}\right)\right\} + O(1)\\ &= \max\left\{N\left(r, \frac{1}{c}\right), N\left(r, \frac{1}{a}\right)\right\} + O(1)\\ &= \max\{T(r,c), T(r,a)\} + O(1),\end{aligned}$$

similarly,

$$T(r,g) = \max\{T(r,c), T(r,b)\} + O(1),$$

and hence the main theorem follows from these estimates above. □

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