The growth series of the n-extended affine Weyl group of type A_1

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Abstract: N-extended affine Weyl groups are Weyl groups associated to n-extended affine root systems introduced by K. Saito [1]. We calculate the growth series of the n-extended affine Weyl group of type A_1 with a generator system of an n-toroidal sense.

Key words: Growth series; n-extended affine Weyl group.

1. Introduction. Extended affine root systems and the associated Weyl groups were introduced and studied by K. Saito [1]. Especially 2-extended affine root systems are also called elliptic root systems from the point of view of the elliptic singularities. The defining relations of generators of the elliptic Weyl groups associated to the elliptic root systems were determined by K. Saito and the author [2]. The growth series $W(t)$ of the Weyl group W with respect to a fixed generator system is defined by $W(t) = \sum_{w\in W} t^{l(w)}$, where $l(w)$ is the minimal length of w, and t is an indeterminate. Here, we note that "growth series" is also called "Poincaré series", if it has geometric or representation theoretical meanings. In the case of the elliptic Weyl group W of type $A_1^{(1,1)}$, the growth series (Poincaré series) was calculated by Wakimoto [4], and in the case of type $A_2^{(1,1)}$, by the author [5]. In this paper, we calculate the growth series of the Weyl group associated to the n-extended affine root system of type A_1 with respect to a generator system of an n-toroidal sense, and the result is interesting in a combinatorial meaning.

2. The n-extended affine Weyl group of type A_1. The n-extended affine root system of type A_1 is defined by the set [1];

$$\Phi = \{\pm(\epsilon_1 - \epsilon_2) + k_1 b_1 + k_2 b_2 + \cdots \\ \cdots + k_n b_n \quad (k_1, \ldots, k_n \in \mathbf{Z})\},$$

with the inner product $\langle \epsilon_i, \epsilon_j \rangle = \delta_{ij}$, $\langle \epsilon_i, b_j \rangle = 0$, $\langle b_i, b_j \rangle = 0$, so that b_i $(1 \le i \le n)$ are generators of its radical. We choose a basis of Φ as follows:

$$\alpha_0 = \epsilon_2 - \epsilon_1 + b_1, \quad \alpha_1 = \epsilon_1 - \epsilon_2,$$
$$\alpha_i = \epsilon_2 - \epsilon_1 + b_i \quad (2 \le i \le n).$$

Let $w_i := w_{\alpha_i}$ be the reflection with respect to the root α_i $(0 \le i \le n)$. Let W be the finite Weyl group of type A_1, then the n-extended affine Weyl group of type A_1 is realized by $\widetilde{W} = W \ltimes (\underbrace{Q^\vee \times \cdots \times Q^\vee}_{n\text{-times}})$, where $Q^\vee = \mathbf{Z}\alpha_1^\vee$ $\big($in this case $\alpha_1^\vee := \frac{2\alpha_1}{\langle \alpha_1, \alpha_1 \rangle} = \alpha_1\big)$, and the action of each $\alpha_1^\vee \in Q^\vee$ on $V := \mathbf{R}(\epsilon_1 - \epsilon_2) \oplus \bigoplus_{i=1}^n \mathbf{R} b_i$ is given by

$$T_i := T_i(\alpha_1^\vee) : \lambda \longrightarrow \lambda - \langle \lambda, \alpha_1^\vee \rangle b_i \quad \text{for } \lambda \in V.$$

By using w_1 and T_i $(1 \le i \le n)$, each w_i $(i \ne 2)$ is expressed by $w_0 = T_1 w_1$, $w_i = T_i w_1$ $(2 \le i \le n)$.

Proposition 2.1. *The n-extended affine Weyl group $\widetilde{W}$ of type A_1 is presented as follows*:
generators: w_i $(0 \le i \le n)$,
relations: $w_i^2 = 1$ $(0 \le i \le n)$,
$(w_i w_1 w_j)^2 = 1$ $(i, j \ne 1,\ 0 \le i \ne j \le n)$.

Proof. If we choose, w_1, T_i $(1 \le i \le n)$ as generators of $\widetilde{W}$, then their relations are given by

$$w_1^2 = 1, \quad T_i w_1 T_i w_1 = 1 \ (1 \le i \le n),$$
$$T_i T_j = T_j T_i \ (1 \le i, j \le n).$$

The relation $T_i w_1 T_i w_1 = 1$ is rewritten as $w_i^2 = 1$, and $T_i T_j = T_j T_i$ is rewritten as $(w_i w_1 w_j)^2 = 1$, so the proof is completed. □

Theorem 2.2. *The growth series of the n-extended affine Weyl group $\widetilde{W}$ of type A_1 with the above generator system is given by*

$$\sum_{w\in\widetilde{W}} t^{l(w)} = \frac{1}{(1-t)^n(1+t)^n} \sum_{i=0}^{n} \left\{ \begin{bmatrix} n \\ i \end{bmatrix}^2 t^{2i} + \begin{bmatrix} n-1 \\ i \end{bmatrix} \begin{bmatrix} n+1 \\ i+1 \end{bmatrix} t^{2i+1} \right\}$$

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$$= \frac{1}{(1-t)^n(1+t)^{n-2}}\Big\{\sum_{i=0}^{n-1}\begin{bmatrix} n-1 \\ i \end{bmatrix}^2 t^{2i} + \sum_{i=0}^{n-2}\begin{bmatrix} n-1 \\ i \end{bmatrix}\begin{bmatrix} n-1 \\ i+1 \end{bmatrix} t^{2i+1}\Big\},$$

where

$$\begin{bmatrix} N \\ m \end{bmatrix} = \begin{cases} 1 & \text{if} \quad m=0 \quad \text{or} \quad m=N, \\ 0 & \text{if} \quad m<0 \quad \text{or} \quad m>N, \\ \frac{N(N-1)\cdots(N-m+1)}{m(m-1)\cdots 2\cdot 1} & \text{if} \quad 0<m<N. \end{cases}$$

Examples.

$$n=1, \quad \sum_{w\in\widetilde{W}} t^{l(w)} = \frac{1+t}{1-t}$$

$$n=2, \quad \sum_{w\in\widetilde{W}} t^{l(w)} = \frac{1+t+t^2}{(1-t)^2}$$

$$n=3, \quad \sum_{w\in\widetilde{W}} t^{l(w)} = \frac{1+2t+4t^2+2t^3+t^4}{(1-t)^3(1+t)}$$

$$n=4, \quad \sum_{w\in\widetilde{W}} t^{l(w)} = \frac{1+3t+9t^2+9t^3+9t^4+3t^5+t^6}{(1-t)^4(1+t)^2}.$$

In the sequel, we prove Theorem 2.2. For the purpose, we prepare the following.

Lemma 2.3 ([3]). (1) *The number of compositions of n with exactly k parts is called k-composition of n and equal to* $\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$, *(see* [3]*) i.e.* $\sharp\{y_1,\dots,y_k \in \mathbf{Z}_{>0} \mid n = y_1+\cdots+y_k\} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$.

(2) *From* (1), *we have* $\sharp\{y_1,\dots,y_k \in \mathbf{Z}_{>0} \mid n \geq y_1+\cdots+y_k\} = \sum_{j=k}^{n}\begin{bmatrix} j-1 \\ k-1 \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}$.

Lemma 2.4.

$$(w_iw_1)^k(w_1w_j)^l = \begin{cases} (w_iw_1)^{k-l}(w_iw_j)^l & (k \geq l) \\ (w_iw_j)^k(w_1w_j)^{l-k} & (k<l). \end{cases}$$

Proof. If $k \geq l$, then

$$\begin{aligned}(w_iw_1)^k(w_1w_j)^l &= (w_iw_1)^{k-1}w_iw_j(w_1w_j)^{l-1} \\ &= (w_iw_1)^{k-2}w_iw_jw_iw_1(w_1w_j)^{l-1} \\ &\qquad (\text{by } w_1w_iw_j = w_jw_iw_1) \\ &= (w_iw_1)^{k-2}(w_iw_j)^2(w_1w_j)^{l-2} \\ &= (w_iw_1)^{k-l}(w_iw_j)^l \\ &\qquad (\text{by iterated the same procedure}).\end{aligned}$$

If $k<l$, similarly, we have $(w_iw_1)^k(w_1w_j)^l = (w_iw_j)^k(w_1w_j)^{l-k}$. □

To calculate the growth series, we regard W as the following set.

$$\begin{aligned} W = \{&T_1^{k_1}T_2^{k_2}\cdots T_n^{k_n},\ T_1^{k_1}T_2^{k_2}\cdots T_n^{k_n}w_1, \\ &T_1^{k_1}\cdots \check{T}_i^{k_i}\cdots T_n^{k_n},\ T_1^{k_1}\cdots \check{T}_i^{k_i}\cdots T_n^{k_n}w_1 \\ &(1\leq i\leq n),\ T_1^{k_1}\cdots \check{T}_i^{k_i}\cdots \check{T}_j^{k_j}\cdots T_n^{k_n}, \\ &T_1^{k_1}\cdots \check{T}_i^{k_i}\cdots \check{T}_j^{k_j}\cdots T_n^{k_n}w_1\ (1\leq i<j\leq n), \\ &\cdots\cdots, T_i^{k_i},\ T_i^{k_i}w_1\ (1\leq i\leq n),\cdots \\ &\cdots, w_1,\ id \ \text{ for all } k_i \neq 0 \in \mathbf{Z}\}, \end{aligned}$$

where $\check{T}_i$ means that T_i is omitted. For $1 \leq m \leq n$, at first we consider the elements $T_1^{k_1}\cdots T_m^{k_m}$ ($\forall\, k_i \neq 0 \in \mathbf{Z}$). In which, we assume that the number of indices j for the positive elements $k_j > 0$ equals to i, i.e. we consider the following elements.

(I) $T_1^{k_1}\cdots T_i^{k_i}T_{i+1}^{-k_{i+1}}\cdots T_m^{-k_m}$ ($\forall\, k_j \in \mathbf{Z}_{>0}$).

Then we have the expression;

$$\begin{aligned}&T_1^{k_1}\cdots T_i^{k_i}T_{i+1}^{-k_{i+1}}\cdots T_m^{-k_m} \\ &= (w_0w_1)^{k_1}\cdots(w_iw_1)^{k_i}(w_1w_{i+1})^{k_{i+1}}\cdots(w_1w_m)^{k_m}.\end{aligned}$$

Further, we divide (I) into the following two cases.

(a) If $k_1+\cdots+k_i \geq k_{i+1}+\cdots+k_m$ ($1 \leq i \leq m$), then by using Lemma 2.4, the length of the elements (I), $l(T_1^{k_1}\cdots T_i^{k_i}T_{i+1}^{-k_{i+1}}\cdots T_m^{-k_m}) = 2(k_1+\cdots+k_i)$, and the number of those elements is equal to $\sharp\{y_1,\dots,y_i \in \mathbf{Z}_{>0} \mid p = y_1+\cdots+y_i\} \times \sharp\{x_1,\dots,x_{m-i} \in \mathbf{Z}_{>0} \mid p \geq x_1+\cdots+x_{m-i}\}$, where $p = k_1+\cdots+k_i$, from Lemma 2.3, which is equal to $\begin{bmatrix} p-1 \\ i-1 \end{bmatrix} \times \begin{bmatrix} p \\ m-i \end{bmatrix}$.

(b) If $k_1+\cdots+k_i < k_{i+1}+\cdots+k_m$ ($0 \leq i \leq m-1$), then the length of the elements (I), $l(T_1^{k_1}\cdots T_i^{k_i}T_{i+1}^{-k_{i+1}}\cdots T_m^{-k_m}) = 2(k_{i+1}+\cdots+k_m)$, the number of those elements is equal to $\sharp\{y_1,\dots,y_{m-i} \in \mathbf{Z}_{>0} \mid p = y_1+\cdots+y_{m-i}\} \times \sharp\{x_1,\dots,x_i \in \mathbf{Z}_{>0} \mid p-1 \geq x_1+\cdots+x_i\}$, where $p = k_{i+1}+\cdots+k_m$, from Lemma 2.3, which is equal to $\begin{bmatrix} p-1 \\ m-i-1 \end{bmatrix} \times \begin{bmatrix} p-1 \\ i \end{bmatrix}$. From (a), (b), the growth series of the part $\{T_1^{k_1}T_2^{k_2}\cdots T_n^{k_n}$, $T_1^{k_1}\cdots \check{T}_i^{k_i}\cdots T_n^{k_n}$, $T_1^{k_1}\cdots \check{T}_i^{k_i}\cdots \check{T}_j^{k_j}\cdots T_n^{k_n}(1 \leq i<j\leq n),\cdots\cdots, T_i^{k_i}$ $(1\leq i\leq n)$ $(\forall\, k_i \neq 0 \in \mathbf{Z})\}$ is

$$\sum_{p=1}^{\infty}\sum_{m=1}^{n}\begin{bmatrix} n \\ m \end{bmatrix}\Big\{\sum_{i=1}^{m}\begin{bmatrix} m \\ i \end{bmatrix}\begin{bmatrix} p-1 \\ i-1 \end{bmatrix}\begin{bmatrix} p \\ m-i \end{bmatrix}$$

$$+\sum_{i=0}^{m-1}\begin{bmatrix} m\\ i\end{bmatrix}\begin{bmatrix} p-1\\ i\end{bmatrix}\begin{bmatrix} p-1\\ m-i-1\end{bmatrix}\Bigg\} t^{2p}$$
$$=\sum_{p=1}^{\infty}\sum_{m=1}^{n}\begin{bmatrix} n\\ m\end{bmatrix}\sum_{i=1}^{m}\Bigg\{\begin{bmatrix} m\\ i\end{bmatrix}\begin{bmatrix} p\\ m-i\end{bmatrix}$$
$$+\begin{bmatrix} m\\ i-1\end{bmatrix}\begin{bmatrix} p-1\\ m-i\end{bmatrix}\Bigg\}\begin{bmatrix} p-1\\ i-1\end{bmatrix} t^{2p}.$$

Next we consider the elements, $T_1^{k_1}\cdots T_i^{k_i}\cdots T_m^{k_m}w_1$ ($\forall\, k_i\neq 0\in\mathbf{Z}$). In which, we assume that the number of indices j for the positive elements $k_j>0$ equals to i, i.e. we consider the following elements.

(II) $T_1^{k_1}\cdots T_i^{k_i}T_{i+1}^{-k_{i+1}}\cdots T_m^{-k_m}w_1$ ($\forall\, k_j\in\mathbf{Z}_{>0}$). We have the expression;

$T_1^{k_1}\cdots T_i^{k_i}T_{i+1}^{-k_{i+1}}\cdots T_m^{-k_m}w_1=$
$(w_0w_1)^{k_1}\cdots(w_iw_1)^{k_i}(w_1w_{i+1})^{k_{i+1}}\cdots(w_1w_m)^{k_m}w_1.$

Further, we divide (II) into the following two cases.

(a) If $k_1+\cdots+k_i\geq k_{i+1}+\cdots+k_m+1$ ($1\leq i\leq m$), then its length $l(T_1^{k_1}\cdots T_i^{k_i}T_{i+1}^{-k_{i+1}}\cdots T_m^{-k_m}w_1)=2(k_1+\cdots+k_i)-1$, and the number of those elements is equal to $\sharp\{y_1,\ldots,y_i\in\mathbf{Z}_{>0}\mid p=y_1+\cdots+y_i\}\times\sharp\{x_1,\ldots,x_{m-i}\in\mathbf{Z}_{>0}\mid p-1\geq x_1+\cdots+x_{m-i}\}$, where $p=k_1+\cdots+k_i$, which is $\begin{bmatrix} p-1\\ i-1\end{bmatrix}\times\begin{bmatrix} p-1\\ m-i\end{bmatrix}$.

(b) If $k_1+\cdots+k_i<k_{i+1}+\cdots+k_m+1$ ($0\leq i\leq m-1$), then its length $l(T_1^{k_1}\cdots T_i^{k_i}T_{i+1}^{-k_{i+1}}\cdots T_m^{-k_m}w_1)=2(k_{i+1}+\cdots+k_m)+1$, and the number of those elements is equal to $\sharp\{y_1,\ldots,y_{m-i}\in\mathbf{Z}_{>0}\mid p=y_1+\cdots+y_{m-i}\}\times\{x_1,\ldots,x_i\in\mathbf{Z}_{>0}\mid p\geq x_1+\cdots+x_i\}$, where $p=k_{i+1}+\cdots+k_m$, which is $\begin{bmatrix} p-1\\ m-i-1\end{bmatrix}\times\begin{bmatrix} p\\ i\end{bmatrix}$. From (a), (b), the growth series of the part $\{T_1^{k_1}T_2^{k_2}\cdots T_n^{k_n}w_1,\ T_1^{k_1}\cdots\check{T_i}^{k_i}\cdots T_n^{k_n}w_1$ ($1\leq i\leq n$), $T_1^{k_1}\cdots\check{T_i}^{k_i}\cdots\check{T_j}^{k_j}\cdots T_n^{k_n}w_1$ ($1\leq i<j\leq n$), $\cdots\cdots, T_i^{k_i}w_1$ ($1\leq i\leq n$), ($\forall\, k_i\neq 0\in\mathbf{Z}$)$\}$ is

$$\sum_{p=1}^{\infty}\sum_{m=1}^{n}\begin{bmatrix} n\\ m\end{bmatrix}\Bigg\{\sum_{i=1}^{m}\begin{bmatrix} m\\ i\end{bmatrix}\begin{bmatrix} p-1\\ i-1\end{bmatrix}\begin{bmatrix} p-1\\ m-i\end{bmatrix} t^{2p-1}$$
$$+\sum_{i=0}^{m-1}\begin{bmatrix} m\\ i\end{bmatrix}\begin{bmatrix} p-1\\ m-i-1\end{bmatrix}\begin{bmatrix} p\\ i\end{bmatrix} t^{2p+1}\Bigg\}$$
$$=\sum_{p=1}^{\infty}\sum_{m=1}^{n}\begin{bmatrix} n\\ m\end{bmatrix}\Bigg\{\sum_{i=1}^{m}\begin{bmatrix} m\\ i\end{bmatrix}\begin{bmatrix} p\\ i-1\end{bmatrix}\begin{bmatrix} p\\ m-i\end{bmatrix}$$
$$+\sum_{i=0}^{m-1}\begin{bmatrix} m\\ i\end{bmatrix}\begin{bmatrix} p-1\\ m-i-1\end{bmatrix}\begin{bmatrix} p\\ i\end{bmatrix}\Bigg\} t^{2p+1}+nt$$
$$=\sum_{p=1}^{\infty}\sum_{m=1}^{n}\begin{bmatrix} n\\ m\end{bmatrix}\sum_{i=1}^{m}\Bigg\{\begin{bmatrix} m\\ i\end{bmatrix}\begin{bmatrix} p\\ m-i\end{bmatrix}$$
$$+\begin{bmatrix} m\\ i-1\end{bmatrix}\begin{bmatrix} p-1\\ m-i\end{bmatrix}\Bigg\}\begin{bmatrix} p\\ i-1\end{bmatrix} t^{2p+1}+nt.$$

From (I), (II), and considering the cases of w_1, id, we obtain the following.

(2.1)
$$\sum_{w\in\widetilde{W}} t^{l(w)}=\sum_{p=1}^{\infty}\sum_{m=1}^{n}\begin{bmatrix} n\\ m\end{bmatrix}\sum_{i=1}^{m}\Bigg\{\begin{bmatrix} m\\ i\end{bmatrix}\begin{bmatrix} p\\ m-i\end{bmatrix}$$
$$+\begin{bmatrix} m\\ i-1\end{bmatrix}\begin{bmatrix} p-1\\ m-i\end{bmatrix}\Bigg\}\Bigg\{\begin{bmatrix} p-1\\ i-1\end{bmatrix} t^{2p}$$
$$+\begin{bmatrix} p\\ i-1\end{bmatrix} t^{2p+1}\Bigg\}+(n+1)t+1.$$

In the sequel, we prove that the infinite series (2.1) turns out to be the expansion of the rational function given in Theorem 2.2. At first we show the following.

Proposition 2.5.

(1) $$\sum_{p=1}^{\infty}\sum_{m=1}^{n}\begin{bmatrix} n\\ m\end{bmatrix}\sum_{i=1}^{m}\Bigg\{\begin{bmatrix} m\\ i\end{bmatrix}\begin{bmatrix} p\\ m-i\end{bmatrix}$$
$$+\begin{bmatrix} m\\ i-1\end{bmatrix}\begin{bmatrix} p-1\\ m-i\end{bmatrix}\Bigg\}\begin{bmatrix} p-1\\ i-1\end{bmatrix} t^{2p}+1$$
$$=\frac{1}{(1-t)^n(1+t)^n}\sum_{i=0}^{n}\begin{bmatrix} n\\ i\end{bmatrix}^2 t^{2i},$$

(2) $$\sum_{p=1}^{\infty}\sum_{m=1}^{n}\begin{bmatrix} n\\ m\end{bmatrix}\sum_{i=1}^{m}\Bigg\{\begin{bmatrix} m\\ i\end{bmatrix}\begin{bmatrix} p\\ m-i\end{bmatrix}$$
$$+\begin{bmatrix} m\\ i-1\end{bmatrix}\begin{bmatrix} p-1\\ m-i\end{bmatrix}\Bigg\}\begin{bmatrix} p\\ i-1\end{bmatrix} t^{2p+1}+(n+1)t$$
$$=\frac{1}{(1-t)^n(1+t)^n}\sum_{i=0}^{n}\begin{bmatrix} n-1\\ i\end{bmatrix}\begin{bmatrix} n+1\\ i+1\end{bmatrix} t^{2i+1}.$$

For the proof, we prepare the following lemmas.

Lemma 2.6. *For a positive integer k,*

(1) $$\sum_{p=1}^{\infty}\sum_{m=2k}^{n+2k}\begin{bmatrix} n\\ m-2k\end{bmatrix}\sum_{i=1}^{m}\Bigg\{\begin{bmatrix} m\\ i\end{bmatrix}\begin{bmatrix} p\\ m-i\end{bmatrix}$$
$$+\begin{bmatrix} m\\ i-1\end{bmatrix}\begin{bmatrix} p-1\\ m-i\end{bmatrix}\Bigg\}\begin{bmatrix} p-1\\ i-1\end{bmatrix} t^{2p}$$
$$=\frac{2t^2}{(1-t^2)^{n+2k}}\sum_{i=0}^{n+k}\Bigg\{\sum_{s=1}^{k-1}\begin{bmatrix} 2k\\ s\end{bmatrix}\begin{bmatrix} n+s\\ i\end{bmatrix}$$

$$\begin{bmatrix} n+2k-s \\ i+2k-2s \end{bmatrix} t^{4k-2-2s} + \\ + \begin{bmatrix} 2k-1 \\ k-1 \end{bmatrix} \begin{bmatrix} n+k \\ i \end{bmatrix}^2 t^{2k-2} \\ + \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} n+2k \\ i+2k \end{bmatrix} t^{4k-2} \Big\} t^{2i},$$

(2)
$$\sum_{p=1}^{\infty} \sum_{m=2k-1}^{n+2k-1} \begin{bmatrix} n \\ m-2k+1 \end{bmatrix} \sum_{i=1}^{m} \Big\{ \begin{bmatrix} m \\ i \end{bmatrix} \begin{bmatrix} p \\ m-i \end{bmatrix} + \begin{bmatrix} m \\ i-1 \end{bmatrix} \begin{bmatrix} p-1 \\ m-i \end{bmatrix} \Big\} \begin{bmatrix} p-1 \\ i-1 \end{bmatrix} t^{2p}$$
$$= \frac{2t^2}{(1-t^2)^{n+2k-1}} \sum_{i=0}^{n+k-1} \Big\{ \sum_{s=1}^{k-1} \begin{bmatrix} 2k-1 \\ s \end{bmatrix} \begin{bmatrix} n+s \\ i \end{bmatrix} \begin{bmatrix} n+2k-1-s \\ i+2k-1-2s \end{bmatrix} t^{4k-4-2s} + \\ + \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} n+2k-1 \\ i+2k-1 \end{bmatrix} t^{4k-4} \Big\} t^{2i}.$$

Proof. We prove by induction on n and k. We assume the cases of less than or equal to n and k in (1), (2) and set $P := \sum_{i=1}^{m} \Big\{ \begin{bmatrix} m \\ i \end{bmatrix} \begin{bmatrix} p \\ m-i \end{bmatrix} + \begin{bmatrix} m \\ i-1 \end{bmatrix} \begin{bmatrix} p-1 \\ m-i \end{bmatrix} \Big\} \begin{bmatrix} p-1 \\ i-1 \end{bmatrix} t^{2p}$, then

$$\sum_{p=1}^{\infty} \sum_{m=2k+1}^{n+2k+1} \begin{bmatrix} n \\ m-2k-1 \end{bmatrix} P \\ = \sum_{p=1}^{\infty} \sum_{m=2k+1}^{n-1+2k+1} \begin{bmatrix} n-1 \\ m-2k-1 \end{bmatrix} P \\ + \sum_{p=1}^{\infty} \sum_{m=2k+2}^{n-1+2k+2} \begin{bmatrix} n-1 \\ m-2k-2 \end{bmatrix} P \\ = \frac{2t^2}{(1-t^2)^{n+2k}} \sum_{i=0}^{n+k-1} \Big\{ \sum_{s=1}^{k} \begin{bmatrix} 2k+1 \\ s \end{bmatrix} \begin{bmatrix} n-1+s \\ i \end{bmatrix} \begin{bmatrix} n+2k-s \\ i+2k+1-2s \end{bmatrix} t^{4k-2s} + \\ + \begin{bmatrix} n-1 \\ i \end{bmatrix} \begin{bmatrix} n+2k \\ i+2k+1 \end{bmatrix} t^{4k} \Big\} t^{2i} \\ + \frac{2t^2}{(1-t^2)^{n+2k+1}} \sum_{i=0}^{n+k} \Big\{ \sum_{s=1}^{k} \begin{bmatrix} 2k+2 \\ s \end{bmatrix} \begin{bmatrix} n-1+s \\ i \end{bmatrix} \begin{bmatrix} n+2k+1-s \\ i+2k+2-2s \end{bmatrix} t^{4k+2-2s} + \begin{bmatrix} 2k+1 \\ k \end{bmatrix} \begin{bmatrix} n+k \\ i \end{bmatrix}^2 t^{2k} \\ + \begin{bmatrix} n-1 \\ i \end{bmatrix} \begin{bmatrix} n+2k+1 \\ i+2k+2 \end{bmatrix} t^{4k+2} \Big\} t^{2i}$$

$$= \frac{2t^2}{(1-t^2)^{n+2k+1}} \sum_{i=0}^{n+k} \Big\{ \sum_{s=1}^{k} \begin{bmatrix} 2k+1 \\ s \end{bmatrix} \begin{bmatrix} n-1+s \\ i \end{bmatrix} \begin{bmatrix} n+2k-s \\ i+2k+1-2s \end{bmatrix} t^{4k-2s} - \\ - \sum_{s=1}^{k} \begin{bmatrix} 2k+1 \\ s \end{bmatrix} \begin{bmatrix} n-1+s \\ i-1 \end{bmatrix} \begin{bmatrix} n+2k-s \\ i+2k-2s \end{bmatrix} t^{4k-2s} \\ + \sum_{s=1}^{k} \begin{bmatrix} 2k+2 \\ s \end{bmatrix} \begin{bmatrix} n-1+s \\ i-1 \end{bmatrix} \begin{bmatrix} n+2k+1-s \\ i+2k+1-2s \end{bmatrix} t^{4k-2s} \\ + \begin{bmatrix} 2k+1 \\ k \end{bmatrix} \begin{bmatrix} n+k \\ i \end{bmatrix}^2 t^{2k} + \begin{bmatrix} n-1 \\ k \end{bmatrix} \begin{bmatrix} n+2k \\ i+2k+1 \end{bmatrix} t^{4k} \\ - \begin{bmatrix} n-1 \\ i-1 \end{bmatrix} \begin{bmatrix} n+2k \\ i+2k \end{bmatrix} t^{4k} \\ + \begin{bmatrix} n-1 \\ i-1 \end{bmatrix} \begin{bmatrix} n+2k+1 \\ i+2k+1 \end{bmatrix} t^{4k} \Big\} t^{2i}$$

$$= \frac{2t^2}{(1-t^2)^{n+2k+1}} \sum_{i=0}^{n+k} \Big\{ \sum_{s=1}^{k} \begin{bmatrix} 2k+1 \\ s \end{bmatrix} \begin{bmatrix} n-1+s \\ i \end{bmatrix} \begin{bmatrix} n+2k-s \\ i+2k+1-2s \end{bmatrix} t^{4k-2s} - \\ - \sum_{s=1}^{k} \begin{bmatrix} 2k+1 \\ s \end{bmatrix} \begin{bmatrix} n-1+s \\ i-1 \end{bmatrix} \begin{bmatrix} n+2k-s \\ i+2k-2s \end{bmatrix} t^{4k-2s} \\ + \sum_{s=1}^{k} \Big(\begin{bmatrix} 2k+1 \\ s \end{bmatrix} + \begin{bmatrix} 2k+1 \\ s-1 \end{bmatrix} \Big) \begin{bmatrix} n-1+s \\ i-1 \end{bmatrix} \begin{bmatrix} n+2k+1-s \\ i+2k+1-2s \end{bmatrix} t^{4k-2s} + \begin{bmatrix} 2k+1 \\ k \end{bmatrix} \begin{bmatrix} n+k \\ i \end{bmatrix}^2 t^{2k} \\ + \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} n+2k \\ i+2k+1 \end{bmatrix} t^{4k} \Big\} t^{2i}$$

$$= \frac{2t^2}{(1-t^2)^{n+2k+1}} \sum_{i=0}^{n+k} \Big\{ \sum_{s=1}^{k} \begin{bmatrix} 2k+1 \\ s \end{bmatrix} \begin{bmatrix} n-1+s \\ i \end{bmatrix} \begin{bmatrix} n+2k-s \\ i+2k+1-2s \end{bmatrix} t^{4k-2s} - \\ - \sum_{s=1}^{k} \begin{bmatrix} 2k+1 \\ s \end{bmatrix} \begin{bmatrix} n-1+s \\ i-1 \end{bmatrix} \begin{bmatrix} n+2k-s \\ i+2k-2s \end{bmatrix} t^{4k-2s} \\ + \sum_{s=1}^{k} \begin{bmatrix} 2k+1 \\ s \end{bmatrix} \begin{bmatrix} n-1+s \\ i-1 \end{bmatrix} \begin{bmatrix} n+2k+1-s \\ i+2k+1-2s \end{bmatrix} t^{4k-2s} \\ + \sum_{s=1}^{k} \begin{bmatrix} 2k+1 \\ k \end{bmatrix} \begin{bmatrix} n+s \\ i \end{bmatrix} \begin{bmatrix} n+2k-s \\ i+2k-2s \end{bmatrix} t^{4k-2s} \\ + \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} n+2k \\ i+2k \end{bmatrix} t^{4k} + \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} n+2k \\ i+2k+1 \end{bmatrix} t^{4k} \Big\} t^{2i}$$

$$= \frac{2t^2}{(1-t^2)^{n+2k+1}} \sum_{i=0}^{n+k} \Big[\sum_{s=1}^{k} \Big\{ \begin{bmatrix} 2k+1 \\ s \end{bmatrix} \begin{bmatrix} n-1+s \\ i \end{bmatrix} \begin{bmatrix} n+2k-s \\ i+2k+1-2s \end{bmatrix} + \\ + \begin{bmatrix} 2k+1 \\ s \end{bmatrix} \begin{bmatrix} n-1+s \\ i-1 \end{bmatrix} \begin{bmatrix} n+2k-s \\ i+2k+1-2s \end{bmatrix}$$

$$+\begin{bmatrix}2k+1\\s\end{bmatrix}\begin{bmatrix}n+s\\i\end{bmatrix}\begin{bmatrix}n+2k-s\\i+2k-2s\end{bmatrix}\Big\} t^{4k-2s}$$

$$+\begin{bmatrix}n\\i\end{bmatrix}\begin{bmatrix}n+2k+1\\i+2k+1\end{bmatrix}t^{4k}\Big]t^{2i}$$

$$=\frac{2t^2}{(1-t^2)^{n+2k+1}}\sum_{i=0}^{n+k}\sum_{s=1}^{k}\Big\{\begin{bmatrix}2k+1\\s\end{bmatrix}\begin{bmatrix}n+s\\i\end{bmatrix}$$

$$\begin{bmatrix}n+2k+1-s\\i+2k+1-2s\end{bmatrix}t^{4k-2s}+\begin{bmatrix}n\\i\end{bmatrix}\begin{bmatrix}n+2k+1\\i+2k+1\end{bmatrix}t^{4k}\Big\}t^{2i}.$$

Nextly,

$$\sum_{p=1}^{\infty}\sum_{m=2k+2}^{n+2k+2}\begin{bmatrix}n\\m-2k-2\end{bmatrix}P$$

$$=\sum_{p=1}^{\infty}\sum_{m=2k+2}^{n-1+2k+2}\begin{bmatrix}n-1\\m-2k-2\end{bmatrix}P$$

$$+\sum_{p=1}^{\infty}\sum_{m=2k+3}^{n-1+2k+3}\begin{bmatrix}n-1\\m-2k-3\end{bmatrix}P$$

$$=\frac{2t^2}{(1-t^2)^{n+2k+1}}\sum_{i=0}^{n+k}\Big\{\sum_{s=1}^{k}\begin{bmatrix}2k+2\\s\end{bmatrix}\begin{bmatrix}n-1+s\\i\end{bmatrix}$$

$$\begin{bmatrix}n+2k+1-s\\i+2k+2-2s\end{bmatrix}t^{4k+2-2s}+$$

$$+\begin{bmatrix}2k+1\\k\end{bmatrix}\begin{bmatrix}n+k\\i\end{bmatrix}^2t^{2k}$$

$$+\begin{bmatrix}n-1\\i\end{bmatrix}\begin{bmatrix}n+2k+1\\i+2k+2\end{bmatrix}t^{4k+2}\Big\}t^{2i}$$

$$+\frac{2t^2}{(1-t^2)^{n+2k+2}}\sum_{i=0}^{n+k}\Big\{\sum_{s=1}^{k+1}\begin{bmatrix}2k+3\\s\end{bmatrix}\begin{bmatrix}n-1+s\\i\end{bmatrix}$$

$$\begin{bmatrix}n+2k+2-s\\i+2k+3-2s\end{bmatrix}t^{4k+4-2s}+$$

$$+\begin{bmatrix}n-1\\i\end{bmatrix}\begin{bmatrix}n+2k+2\\i+2k+3\end{bmatrix}t^{4k+4}\Big\}t^{2i}$$

$$=\frac{2t^2}{(1-t^2)^{n+2k+2}}\sum_{i=0}^{n+k+1}\Big\{\sum_{s=1}^{k}\begin{bmatrix}2k+2\\s\end{bmatrix}\begin{bmatrix}n-1+s\\i\end{bmatrix}$$

$$\begin{bmatrix}n+2k+1-s\\i+2k+2-2s\end{bmatrix}t^{4k+2-2s}-$$

$$-\sum_{s=1}^{k}\begin{bmatrix}2k+2\\s\end{bmatrix}\begin{bmatrix}n-1+s\\i-1\end{bmatrix}\begin{bmatrix}n+2k+1-s\\i+2k+2-2s\end{bmatrix}t^{4k+2-2s}$$

$$+\sum_{s=1}^{k+1}\Big(\begin{bmatrix}2k+2\\s\end{bmatrix}+\begin{bmatrix}2k+2\\s-1\end{bmatrix}\Big)$$

$$\begin{bmatrix}n-1+s\\i\end{bmatrix}\begin{bmatrix}n+2k+2-s\\i+2k+3-2s\end{bmatrix}t^{4k+4-2s}+$$

$$+\begin{bmatrix}2k+1\\k\end{bmatrix}\begin{bmatrix}n+k\\i\end{bmatrix}^2t^{2k}-\begin{bmatrix}2k+1\\k\end{bmatrix}\begin{bmatrix}n+k\\i-1\end{bmatrix}^2t^{2k}$$

$$+\begin{bmatrix}n-1\\i\end{bmatrix}\begin{bmatrix}n+2k+1\\i+2k+2\end{bmatrix}t^{4k+2}$$

$$-\begin{bmatrix}n-1\\i-1\end{bmatrix}\begin{bmatrix}n+2k+1\\i+2k+1\end{bmatrix}t^{4k+2}$$

$$+\begin{bmatrix}n-1\\i-1\end{bmatrix}\begin{bmatrix}n+2k+2\\i+2k+2\end{bmatrix}t^{4k+2}\Big\}t^{2i}$$

$$=\frac{2t^2}{(1-t^2)^{n+2k+2}}\sum_{i=0}^{n+k+1}\Big\{\sum_{s=1}^{k}\begin{bmatrix}2k+2\\s\end{bmatrix}\begin{bmatrix}n-1+s\\i\end{bmatrix}$$

$$\begin{bmatrix}n+2k+1-s\\i+2k+2-2s\end{bmatrix}t^{4k+2-2s}-$$

$$-\sum_{s=1}^{k}\begin{bmatrix}2k+2\\s\end{bmatrix}\begin{bmatrix}n-1+s\\i-1\end{bmatrix}\begin{bmatrix}n+2k+1-s\\i+2k+1-2s\end{bmatrix}t^{4k+2-2s}$$

$$+\sum_{s=1}^{k}\begin{bmatrix}2k+2\\s\end{bmatrix}\begin{bmatrix}n-1+s\\i-1\end{bmatrix}\begin{bmatrix}n+2k+2-s\\i+2k+2-2s\end{bmatrix}t^{4k+2-2s}$$

$$+\sum_{s=1}^{k}\begin{bmatrix}2k+2\\s\end{bmatrix}\begin{bmatrix}n+s\\i\end{bmatrix}\begin{bmatrix}n+2k+1-s\\i+2k+1-2s\end{bmatrix}t^{4k+2-2s}$$

$$+\begin{bmatrix}n\\i\end{bmatrix}\begin{bmatrix}n+2k+1\\i+2k+1\end{bmatrix}t^{4k+2}$$

$$+\begin{bmatrix}2k+2\\k+1\end{bmatrix}\begin{bmatrix}n+k\\i-1\end{bmatrix}\begin{bmatrix}n+k+1\\i\end{bmatrix}t^{2k}$$

$$+\begin{bmatrix}2k+1\\k\end{bmatrix}\begin{bmatrix}n+k\\i\end{bmatrix}^2t^{2k}-\begin{bmatrix}2k+1\\k\end{bmatrix}\begin{bmatrix}n+k\\i-1\end{bmatrix}^2t^{2k}$$

$$+\begin{bmatrix}n-1\\i\end{bmatrix}\begin{bmatrix}n+2k+1\\i+2k+2\end{bmatrix}t^{4k+2}$$

$$-\begin{bmatrix}n-1\\i-1\end{bmatrix}\begin{bmatrix}n+2k+1\\i+2k+1\end{bmatrix}t^{4k+2}$$

$$+\begin{bmatrix}n-1\\i-1\end{bmatrix}\begin{bmatrix}n+2k+2\\i+2k+2\end{bmatrix}t^{4k+2}\Big\}t^{2i}$$

$$=\frac{2t^2}{(1-t^2)^{n+2k+2}}\sum_{i=0}^{n+k+1}\Big\{\sum_{s=1}^{k}\begin{bmatrix}2k+2\\s\end{bmatrix}\begin{bmatrix}n-1+s\\i\end{bmatrix}$$

$$\begin{bmatrix}n+2k+1-s\\i+2k+2-2s\end{bmatrix}t^{4k+2-2s}+$$

$$+\sum_{s=1}^{k}\begin{bmatrix}2k+2\\s\end{bmatrix}\begin{bmatrix}n-1+s\\i-1\end{bmatrix}\begin{bmatrix}n+2k+1-s\\i+2k+2-2s\end{bmatrix}t^{4k+2-2s}$$

$$+\sum_{s=1}^{k}\begin{bmatrix}2k+2\\s\end{bmatrix}\begin{bmatrix}n+s\\i\end{bmatrix}\begin{bmatrix}n+2k+1-s\\i+2k+1-2s\end{bmatrix}t^{4k+2-2s}$$

$$+\begin{bmatrix}2k+1\\k+1\end{bmatrix}\begin{bmatrix}n+k\\i-1\end{bmatrix}\begin{bmatrix}n+k+1\\i\end{bmatrix}t^{2k}$$

$$+\begin{bmatrix}2k+1\\k\end{bmatrix}\begin{bmatrix}n+k\\i-1\end{bmatrix}\begin{bmatrix}n+k+1\\i\end{bmatrix}t^{2k}$$

$$+\begin{bmatrix}2k+1\\k\end{bmatrix}\begin{bmatrix}n+k\\i\end{bmatrix}^2t^{2k}-\begin{bmatrix}2k+1\\k\end{bmatrix}\begin{bmatrix}n+k\\i-1\end{bmatrix}^2t^{2k}$$

$$+\begin{bmatrix}n\\i\end{bmatrix}\begin{bmatrix}n+2k+1\\i+2k+1\end{bmatrix}t^{4k+2}$$

$$+\begin{bmatrix}n\\i\end{bmatrix}\begin{bmatrix}n+2k+1\\i+2k+2\end{bmatrix}t^{4k+2}\Big\}t^{2i}$$

$$=\frac{2t^2}{(1-t^2)^{n+2k+2}}\sum_{i=0}^{n+k+1}\Big\{\sum_{s=1}^{k}\begin{bmatrix}2k+2\\s\end{bmatrix}\begin{bmatrix}n+s\\i\end{bmatrix}$$

$$\begin{bmatrix} n+2k+2-s \\ i+2k+2-2s \end{bmatrix} t^{4k+2-2s} + $$
$$+ \begin{bmatrix} 2k+1 \\ k \end{bmatrix} \begin{bmatrix} n+k+1 \\ i \end{bmatrix}^2 t^{2k}$$
$$+ \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} n+2k+2 \\ i+2k+2 \end{bmatrix} t^{4k+2} \Big\} t^{2i}.$$

And similarly to the above, we can show the following.

$$\sum_{p=1}^{\infty} \sum_{m=2k-1}^{n+1+2k-1} \begin{bmatrix} n+1 \\ m-2k+1 \end{bmatrix} P$$
$$= \frac{2t^2}{(1-t^2)^{n+2k}} \sum_{i=0}^{n+k} \Big\{ \sum_{s=1}^{k-1} \begin{bmatrix} 2k-1 \\ s \end{bmatrix} \begin{bmatrix} n+1+s \\ i \end{bmatrix}$$
$$\begin{bmatrix} n+2k-s \\ i+2k-1-2s \end{bmatrix} t^{4k-4-2s}$$
$$+ \begin{bmatrix} n+1 \\ i \end{bmatrix} \begin{bmatrix} n+2k \\ i+2k-1 \end{bmatrix} t^{4k-4} \Big\} t^{2i}, \quad \text{and}$$

$$\sum_{p=1}^{\infty} \sum_{m=2k}^{n+1+2k} \begin{bmatrix} n+1 \\ m-2k \end{bmatrix} P$$
$$= \frac{2t^2}{(1-t^2)^{n+2k+1}} \sum_{i=0}^{n+k+1} \Big\{ \sum_{s=1}^{k-1} \begin{bmatrix} 2k \\ s \end{bmatrix} \begin{bmatrix} n+1+s \\ i \end{bmatrix}$$
$$\begin{bmatrix} n+1+2k-s \\ i+2k-2s \end{bmatrix} t^{4k-2-2s} +$$
$$+ \begin{bmatrix} 2k-1 \\ k-1 \end{bmatrix} \begin{bmatrix} n+1+k \\ i \end{bmatrix}^2 t^{2k-2}$$
$$+ \begin{bmatrix} n+1 \\ i \end{bmatrix} \begin{bmatrix} n+1+2k \\ i+2k \end{bmatrix} t^{4k-2} \Big\} t^{2i},$$

therefore the proof is completed. □

Lemma 2.7. *For a positive integer k,*

$$\sum_{p=1}^{\infty} \sum_{m=k}^{n+k} \begin{bmatrix} n \\ m-k \end{bmatrix} \sum_{i=1}^{m} \Big\{ \begin{bmatrix} m \\ i \end{bmatrix} \begin{bmatrix} p \\ m-i \end{bmatrix} + \begin{bmatrix} m \\ i-1 \end{bmatrix} \begin{bmatrix} p-1 \\ m-i \end{bmatrix} \Big\} \begin{bmatrix} p \\ i-1 \end{bmatrix} t^{2p+1}$$
$$= \begin{cases} \dfrac{1}{(1-t^2)^{n+1}} e \displaystyle\sum_{i=0}^{n} \Big\{ \begin{bmatrix} n-1 \\ i-1 \end{bmatrix} \begin{bmatrix} n+2 \\ i+1 \end{bmatrix} + \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} n+1 \\ i \end{bmatrix} \Big\} t^{2i+1} - t \qquad (k=1), \\ \dfrac{1}{(1-t^2)^{n+k}} \displaystyle\sum_{i=0}^{n+k-1} \begin{bmatrix} k \\ j \end{bmatrix} \begin{bmatrix} n-1+j \\ i+1+j-k \end{bmatrix} \begin{bmatrix} n+k+1-j \\ i+2-j \end{bmatrix} t^{2i+3} \qquad (k \geq 2). \end{cases}$$

Proof. It is similarly proved as Lemma 2.6. □

Proof of Proposition 2.5. (1) We prove by induction on n by using Lemma 2.6.

$$\sum_{p=1}^{\infty} \sum_{m=1}^{n+1} \begin{bmatrix} n+1 \\ m \end{bmatrix} P + 1$$
$$= \sum_{p=1}^{\infty} \sum_{m=1}^{n} \begin{bmatrix} n \\ m \end{bmatrix} P + 1 + \sum_{p=1}^{\infty} \sum_{m=1}^{n+1} \begin{bmatrix} n \\ m-1 \end{bmatrix} P$$
$$= \frac{1}{(1-t^2)^n} \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}^2 t^{2i}$$
$$+ \frac{2t^2}{(1-t^2)^{n+1}} \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} n+1 \\ i+1 \end{bmatrix} t^{2i}$$
$$= \frac{1}{(1-t^2)^{n+1}} \sum_{i=0}^{n} \Big\{ \begin{bmatrix} n \\ i \end{bmatrix}^2 - \begin{bmatrix} n \\ i-1 \end{bmatrix}^2$$
$$+ 2 \begin{bmatrix} n \\ i-1 \end{bmatrix} \Big(\begin{bmatrix} n \\ i \end{bmatrix} + \begin{bmatrix} n \\ i-1 \end{bmatrix} \Big) \Big\} t^{2i}$$
$$= \frac{1}{(1-t^2)^{n+1}} \sum_{i=0}^{n+1} \Big(\begin{bmatrix} n \\ i \end{bmatrix} + \begin{bmatrix} n \\ i-1 \end{bmatrix} \Big)^2 t^{2i}$$
$$= \frac{1}{(1-t^2)^{n+1}} \sum_{i=0}^{n+1} \begin{bmatrix} n+1 \\ i \end{bmatrix}^2 t^{2i},$$

therefore (1) is proved and (2) is similaly shown by using Lemma 2.7. □

Proof of Theorem 2.2. By using Proposition 2.5,

L.H.S. of (2.1)

$$= \sum_{i=0}^{n} \Big\{ \begin{bmatrix} n \\ i \end{bmatrix}^2 t^{2i} + \begin{bmatrix} n-1 \\ i \end{bmatrix} \begin{bmatrix} n+1 \\ i+1 \end{bmatrix} t^{2i+1} \Big\}$$
$$= \sum_{i=0}^{n} \Big\{ \Big(\begin{bmatrix} n-1 \\ i \end{bmatrix} + \begin{bmatrix} n-1 \\ i-1 \end{bmatrix} \Big)^2 t^{2i}$$
$$+ \begin{bmatrix} n-1 \\ i \end{bmatrix} \Big(\begin{bmatrix} n-1 \\ i+1 \end{bmatrix} + 2 \begin{bmatrix} n-1 \\ i \end{bmatrix} + \begin{bmatrix} n-1 \\ i-1 \end{bmatrix} \Big) t^{2i+1} \Big\}$$
$$= (1+t)^2 \Big\{ \sum_{i=0}^{n-1} \begin{bmatrix} n-1 \\ i \end{bmatrix}^2 t^{2i} + \sum_{i=0}^{n-2} \begin{bmatrix} n-1 \\ i \end{bmatrix} \begin{bmatrix} n-1 \\ i+1 \end{bmatrix} t^{2i+1} \Big\}$$

= R.H.S. of (2.1)

therefore the proof is completed. □

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