A certain expression of the first Painlevé hierarchy

By Shun SHIMOMURA

Department of Mathematics, Keio University
3-14-1, Hiyoshi, Yokohama, Kanagawa 223-8522

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Abstract: We show that each equation in the first Painlevé hierarchy is equivalent to a system of nonlinear equations determined by a kind of generating function, and that it admits the Painlevé property. Our results are derived from the fact that the first Painlevé hierarchy follows from isomonodromic deformation of certain linear systems with an irregular singular point.

Key words: Isomonodromic deformation; the first Painlevé hierarchy.

1. Introduction. Let $d_n[y]$ $(n = 0, 1, 2, \dots)$ be differential polynomials in y determined by the recursive relation

$$\begin{aligned} &d_0[y] = 1, \\ (1)\qquad &Dd_{n+1}[y] = (D^3 - 8yD - 4y')d_n[y], \\ &\qquad\qquad n \in \mathbf{N} \cup \{0\} \end{aligned}$$

$(' = D = d/dt)$. In what follows, we suppose that all the integration constants contained in $d_n[y]$ are zero. For example,

$$\begin{aligned} d_1[y]/4 &= -y, \\ d_2[y]/4 &= -y'' + 6y^2, \\ d_3[y]/4 &= -y^{(4)} + 20yy'' + 10(y')^2 - 40y^3, \\ d_4[y]/4 &= -y^{(6)} + 28yy^{(4)} + 56y'y^{(3)} + 42(y'')^2 \\ &\quad - 280(y^2y'' + y(y')^2 - y^4). \end{aligned}$$

The first Painlevé hierarchy is a sequence of $2n$-th order differential equations of the form

$$(\mathrm{PI}_{2n}) \qquad d_{n+1}[y] + 4t = 0, \quad n \in \mathbf{N}$$

(cf. e.g. [2], [3], [4]), which contains the first Painlevé equation (PI_2).

In this paper, we show that (PI_{2n}) is equivalent to a $2n$-dimensional system of nonlinear equations determined by a kind of generating function, and that it admits the Painlevé property. These results are derived from the fact that (PI_{2n}) follows from isomonodromic deformation of a certain linear system with an irregular singular point. The special case $n = 2$ was treated in [6], and see also [7].

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2. Main results. Consider the formal power series in ξ:

$$\begin{aligned} Q(\xi) &= \sum_{\nu \ge 1} Z_\nu \xi^\nu, \qquad R(\xi) = \sum_{\nu \ge 1} U_\nu \xi^\nu, \\ F(\xi) &= 2\xi^{-1}Q(\xi)(1 + Z_1\xi) \\ &\quad + \frac{\xi^{-1}Q(\xi)^2 - R(\xi)^2}{1 - Q(\xi)} - u_0^2, \end{aligned}$$

where u_0, Z_ν, U_ν $(\nu \in \mathbf{N})$ are variables depending on t. Then, $F(\xi)$ is written in the form

$$F(\xi) = \sum_{\nu \ge 0} F_\nu \xi^\nu$$

with

$$\begin{aligned} F_0 &= 2Z_1 - u_0^2, \\ F_\nu &= 2Z_{\nu+1} + G_\nu(Z_j, U_k; {}_{1\le j\le\nu,\, 1\le k\le\nu-1}) \quad (\nu \in \mathbf{N}). \end{aligned}$$

Here G_ν is a polynomial in Z_j and U_k $(1 \le j \le \nu,$ $1 \le k \le \nu - 1)$. For each $n \in \mathbf{N}$, the relations

$$\begin{aligned} \frac{d}{dt}(u_0 + R(\xi)) &\equiv F(\xi) + 2(t - Z_{n+1})\xi^n \pmod{\xi^{n+1}}, \\ \frac{d}{dt}Q(\xi) &\equiv 2R(\xi) \pmod{\xi^{n+1}} \end{aligned}$$

define the following system:

$$(\mathrm{S}_n) \qquad \begin{aligned} Z_\nu' &= 2U_\nu, \\ U_\nu' &= 2Z_{\nu+1} + G_\nu(Z_j, U_k; {}_{1\le j\le\nu,\, 1\le k\le\nu-1}) \\ &\qquad\qquad (1 \le \nu \le n - 1), \\ Z_n' &= 2U_n, \\ U_n' &= 2t + G_n(Z_j, U_k; {}_{1\le j\le n,\, 1\le k\le n-1}) \end{aligned}$$

(if $n = 1$, skip the first two equations). For example,

$$(\mathrm{S}_1) \qquad \begin{aligned} Z_1' &= 2U_1, \\ U_1' &= 2t + 3Z_1^2 \end{aligned}$$

and

$$(\mathrm{S}_2)\qquad \begin{aligned} Z_1' &= 2U_1,\\ U_1' &= 2Z_2 + 3Z_1^2,\\ Z_2' &= 2U_2,\\ U_2' &= 2t + 4Z_1Z_2 + Z_1^3 - U_1^2. \end{aligned}$$

Theorem 2.1. *For each $n \in \mathbf{N}$, system* (S_n) *is essentially equivalent to* (PI_{2n}). *Namely,*

(1) *for every solution* $(Z_\nu(t), U_\nu(t))$ $(1 \le \nu \le n)$ *of* (S_n), *the function* $y = Z_1(t)$ *satisfies*

$$(\mathrm{PI}_{2n}^*)\qquad d_{n+1}[y] + 4^{n+1}t = 0;$$

(2) *for every solution* $y = Y(t)$ *of* (PI_{2n}^*), *there exists a solution* $(Z_\nu(t), U_\nu(t))$ $(1 \le \nu \le n)$ *of* (S_n) *such that* $Z_1(t) \equiv Y(t)$.

Remark 2.1. For the differential monomial $y^{\iota_0}(y')^{\iota_1}\cdots(y^{(p)})^{\iota_p}$, we define the weight by $\sum_{\kappa=0}^{p}(\kappa+2)\iota_\kappa$. Since all terms of $d_{n+1}[y]$ have the same weight $2+2n$ (cf. the proof of [8, Lemma 2.6]), by the change of variables $y = \lambda^2\eta$, $t = \lambda^{-1}\tau$ $(\lambda^{2n+3} = 4^n)$, equation (PI_{2n}^*) is reduced to (PI_{2n}).

Furthermore, we have

Theorem 2.2. *Every solution* $(Z_\nu(t), U_\nu(t))$ $(1 \le \nu \le n)$ *of* (S_n) *is meromorphic in* $\mathbf{C}$.

3. Linear systems and a Schlesinger transformation. Consider the matrix differential equation

$$(\mathrm{E})\qquad \frac{d\Xi}{dx} = A(x)\Xi,\quad A(x) = -\sum_{j=0}^{2n+2} A_{-j}x^j - A_1x^{-1}.$$

Here Ξ is a 2 by 2 unknown matrix and

$$\begin{aligned} &A_{-2n-2} = J,\quad A_{-2n-1} = -u_0L,\\ &A_{-2n-2+2i} = v_iK - w_iJ,\\ &A_{-2n-1+2i} = -u_iL \qquad (1 \le i \le n),\\ &A_0 = s(J+K),\quad A_1 = (I-L)/2 \end{aligned}$$

with

$$I = \begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix},\qquad J = \begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix},$$

$$K = \begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix},\qquad L = \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}.$$

Proposition 3.1. *Suppose that t and the entries u_0, u_i, v_i $(1 \le i \le n)$ are arbitrary parameters, and write $w_{n+1} := t - s$. System* (E) *admits a formal matrix solution of the form*

$$(2)\qquad \Xi = \Xi(x) = Y(x)\exp T(x),$$

$$T(x) = -\frac{J}{2n+3}x^{2n+3} - tJx - \frac{I}{2}\log x,$$

$$Y(x) = \sum_{j\ge 0} Y_j x^{-j},$$

if and only if the relations

$$(3)\qquad w_\nu = \frac{1}{2}\sum_{j=1}^{\nu-1}(w_jw_{\nu-j} - v_jv_{\nu-j}) + \frac{1}{2}\sum_{j=1}^{\nu} u_{j-1}u_{\nu-j}$$

(*in particular* $w_1 = u_0^2/2$) *hold for* $1 \le \nu \le n+1$.

Proof. Suppose that (E) admits formal solution (2). As was shown in [1, Proposition 2.2 and its proof], the series $Y(x)$ is decomposed into

$$Y(x) = F(x)D(x),\quad F(x) = \sum_{j\ge 0} F_jx^{-j},$$

$$D(x) = \sum_{j\ge 0} D_jx^{-j},\quad F_0 = D_0 = I,$$

$$F_j = f_jL + g_jK,\quad D_j = \mathrm{diag}(d_j^1, d_j^2)\quad (j \ge 1);$$

and hence $F'(x) + F(x)(D'(x)D(x)^{-1} + T'(x)) = A(x)F(x)$. Comparing the coefficients of x^k $(-1 \le k \le 2n+1)$ on both sides, we have

$$(4)\qquad A_{-2n-2+j} = [F_j, J] - \sum_{m=1}^{j-1} A_{-2n-2+m}F_{j-m}$$

for $1 \le j \le 2n+1$, and

$$(5)\qquad \begin{aligned} A_0 &= [F_{2n+2}, J]\\ &\quad - \sum_{m=1}^{2n+1} A_{-2n-2+m}F_{2n+2-m} + tJ,\\ A_1 &= [F_{2n+3}, J]\\ &\quad - \sum_{m=1}^{2n+2} A_{-2n-2+m}F_{2n+3-m} + tF_1J + \frac{I}{2}. \end{aligned}$$

Note the relations $J^2 = -K^2 = L^2 = I$, $JK = -KJ = L$, $KL = -LK = J$, $LJ = -JL = -K$. Using (4) and (5), we can verify, by induction on i,

$$\begin{aligned} f_{2i-1} &= 0\quad (1 \le i \le n+2),\\ g_{2i} &= 0\quad (1 \le i \le n+1). \end{aligned}$$

Then, by (4) with $j = 2i$ $(1 \le i \le n)$,

$$\begin{aligned} v_iK - w_iJ &= \Big(-2f_{2i} + \sum_{l=1}^{i-1} w_lf_{2i-2l}\Big)K\\ &\quad - \Big(u_0g_{2i-1} + \sum_{l=1}^{i-1}(u_lg_{2i-1-2l} + v_lf_{2i-2l})\Big)J, \end{aligned}$$

which yields

$$f_{2i} = \frac{1}{2}\biggl(-v_i + \sum_{l=1}^{i-1} w_l f_{2i-2l}\biggr), \tag{6}$$

$$w_i = u_0 g_{2i-1} + \sum_{l=1}^{i-1}(u_l g_{2i-1-2l} + v_l f_{2i-2l}) \tag{7}$$

$(1 \le i \le n)$, in particular $f_2 = -v_1/2$, $w_1 = u_0 g_1$. Moreover, by (4) with $j = 2i-1$ $(1 \le i \le n+1)$,

$$-u_{i-1}L = \biggl(-2g_{2i-1} + \sum_{l=1}^{i-1} w_l g_{2i-1-2l}\biggr)L + (\cdots)I,$$

and hence

$$g_{2i-1} = \frac{1}{2}\biggl(u_{i-1} + \sum_{l=1}^{i-1} w_l g_{2i-1-2l}\biggr) \tag{8}$$

$(1 \le i \le n+1)$, in particular $g_1 = u_0/2$. Analogously, from (5), we have

$$\begin{aligned} f_{2n+2} &= \frac{1}{2}\biggl(-s + \sum_{l=1}^{n} w_l f_{2n+2-2l}\biggr), \\ s &= t - u_0 g_{2n+1} - \sum_{l=1}^{n}(u_l g_{2n+1-2l} + v_l f_{2n+2-2l}) \end{aligned} \tag{9}$$

and

$$g_{2n+3} = \frac{1}{2}\biggl(\frac{1}{2} - (s+t)g_1 + \sum_{l=1}^{n} w_l g_{2n+3-2l}\biggr). \tag{10}$$

Using (6) through (10), we can check (3). Conversely suppose that w_ν are given by (3). By (6), (8), (9) and (10), we can recursively determine f_j, g_j $(1 \le j \le 2n+3)$ and s. Then, tracing the computation above, we see that

$$F_*(x) = \sum_{j=0}^{2n+3} F_j x^{-j}, \quad F_0 = I$$

satisfies

$$\begin{aligned} &F_*(x)T'(x) \\ &= \biggl(A(x) + \Bigl(\sum_{j=-1}^{2n+1}\delta_j x^j\Bigr)I\biggr)F_*(x) + \sum_{j\ge 2} E_j x^{-j}. \end{aligned}$$

Observing that $\operatorname{tr}(F_*(x)T'(x)F_*(x)^{-1}) = \operatorname{tr} T'(x) = -x^{-1}I$, and that $\operatorname{tr} A(x) = -\operatorname{tr}(A_1 x^{-1}) = -x^{-1}I$, we have $\delta_j = 0$ for $-1 \le j \le 2n+1$. This fact implies the existence of formal solution (2). $\square$

System (E) possesses an apparent singularity at $x = 0$. To remove it, we employ the Schlesinger transformation

$$W = \Psi(x)\Xi, \quad \Psi(x) = \begin{pmatrix} 1 & 1 \\ u_0/2 & u_0/2 + x \end{pmatrix}. \tag{11}$$

Then (E) is changed into

$$\frac{dW}{dx} = B(x)W, \quad B(x) = -\sum_{j=0}^{2n+2} B_{-j}x^j. \tag{E$_0$}$$

Here

$$\begin{aligned} B_{-2n-2} &= J, \\ B_{-2i-1} &= \begin{pmatrix} -U_{n-i} & 2Z_{n-i} \\ -V_{n-i} - v_{n-i+1} & U_{n-i} \end{pmatrix}, \\ B_{-2i} &= \begin{pmatrix} -Z_{n-i+1} & 0 \\ -U_{n-i+1} & Z_{n-i+1} \end{pmatrix} \quad (1 \le i \le n), \\ B_{-1} &= \begin{pmatrix} -U_n & 2Z_n \\ -V_n - s & U_n \end{pmatrix}, \quad B_0 = \begin{pmatrix} 0 & 0 \\ 1/2 & 0 \end{pmatrix} \end{aligned}$$

with

$$\begin{aligned} &Z_\nu = v_\nu + w_\nu, \quad U_\nu = u_\nu + u_0 Z_\nu, \\ &V_\nu = u_0 u_\nu + u_0^2 Z_\nu/2 \end{aligned} \tag{12}$$

$(0 \le \nu \le n)$, $v_0 = w_0 = 0$.

4. Isomonodromic deformation of (E) and ($\mathbf{E_0}$). Suppose that u_ν, v_ν, w_ν are functions of t. The isomonodromic deformation of (E) with respect to the deformation parameter t is governed by the completely integrable system

$$\begin{aligned} &dA(x) = \frac{\partial}{\partial x}\Omega(x,t) + [\Omega(x,t), A(x)], \\ &\Omega(x,t) = \Phi_{-1}(t)x + \Phi_0(t), \end{aligned} \tag{13}$$

where $\Phi_{-1}(t)$ and $\Phi_0(t)$ are one-forms with respect to t defined by

$$\sum_{k=-\infty}^{1} \Phi_{-k}(t)x^k = Y(x)(-x\,dt)JY(x)^{-1}$$

(cf. [1, Theorem 1 or 3.3]). It is easy to see that

$$\begin{aligned} &\Phi_{-1}(t) = -J\,dt, \\ &\Phi_0(t) = -(Y_1 J - JY_1)dt = -A_{-2n-1}dt; \end{aligned} \tag{14}$$

the latter equality follows from the relation $A(x) = Y(x)T'(x)Y(x)^{-1} + Y'(x)Y(x)^{-1}$. By (14), equation (13) is written in the form

$$\begin{aligned} &-dA_{-2n-2+j} \\ &\qquad = ([J, A_{-2n-1+j}] + [A_{-2n-1}, A_{-2n-2+j}])dt \\ &\qquad\qquad (1 \le j \le 2n+1), \\ &-dA_0 = (-J + [J, A_1] + [A_{-2n-1}, A_0])dt, \\ &-dA_1 = -[A_{-2n-1}, A_1]dt. \end{aligned}$$

These relations imply the following:

Proposition 4.1. *The isomonodromic deformation of* (E) *is governed by the system of equations* (*with respect to* u_0, u_ν, v_ν)

$$\begin{aligned} &u'_{\nu-1} = 2v_\nu, \quad v'_\nu = 2u_\nu + 2u_0 w_\nu, \\ &w'_\nu = 2u_0 v_\nu \qquad (1 \le \nu \le n), \\ &u'_n = 2s, \quad s' = 1 - 2u_0 s, \end{aligned} \tag{15}$$

where w_ν *and* $s = t - w_{n+1}$ *are variables defined by* (3).

Remark 4.1. The equations $w'_i = 2u_0 v_i$ ($1 \le i \le n$) and $s' = 1 - 2u_0 s$ are obtained from the others. Indeed, by (3), $w'_1 = u_0 u'_0 = 2u_0 v_1$; and supposing them for $i \le \nu - 1$, we have

$$w'_\nu = \sum_{j=1}^{\nu-1} (w'_j w_{\nu-j} - v'_j v_{\nu-j}) + \sum_{j=1}^{\nu} u'_{j-1} u_{\nu-j} = 2u_0 v_\nu.$$

Note that the isomonodromic property remains invariant under the Schlesinger transformation (11). Using (12), from Proposition 4.1 and Remark 4.1, we derive the following:

Proposition 4.2. *The isomonodromic deformation of* (E_0) *is governed by the system of equations* (*with respect to* u_0, Z_ν, U_ν)

$$\begin{aligned} &u'_0 = 2Z_1 - u_0^2, \\ &Z'_\nu = 2U_\nu, \\ &U'_\nu = 2(Z_{\nu+1} - w_{\nu+1}) \\ &\qquad + (2Z_1 - u_0^2)Z_\nu + 2u_0 U_\nu \\ &\qquad\qquad (1 \le \nu \le n-1), \\ &Z'_n = 2U_n, \\ &U'_n = 2(t - w_{n+1}) + (2Z_1 - u_0^2)Z_n + 2u_0 U_n. \end{aligned} \tag{16}$$

5. Proof of Theorem 2.2. By Miwa's theorem [5], every solution (u_0, Z_ν, U_ν) of (16) is meromorphic in $\mathbf{C}$. It is sufficient to show that system (S_n) coincides with a series of equations for Z_ν, U_ν in (16). In addition to $Q(\xi)$, $R(\xi)$, $F(\xi)$ in Section 2, set

$$p(\xi) = \sum_{\nu \ge 1} w_\nu \xi^\nu, \qquad q(\xi) = \sum_{\nu \ge 1} v_\nu \xi^\nu,$$
$$r(\xi) = \sum_{\nu \ge 0} u_\nu \xi^\nu.$$

For convenience, suppose that, by (12) and (3), the variables Z_ν, U_ν and w_ν are defined for all $\nu \in \mathbf{N}$. Then,

$$p(\xi) = \frac{1}{2}\big(p(\xi)^2 - q(\xi)^2 + \xi r(\xi)^2\big)$$
$$= \frac{1}{2}\Big(2Q(\xi)p(\xi) - Q(\xi)^2 + \xi(u_0(1 - Q(\xi)) + R(\xi))^2\Big)$$

and hence

$$p(\xi) = -\frac{Q(\xi)^2 - \xi(u_0(1 - Q(\xi)) + R(\xi))^2}{2(1 - Q(\xi))},$$

which expresses w_ν in terms of Z_i, U_i, u_0. The generating function for the right-hand side of the third equation in (16) is given by

$$2\xi^{-1}(Q(\xi) - p(\xi)) + (2Z_1 - u_0^2)Q(\xi) + 2u_0 R(\xi).$$

Substituting $p(\xi)$ into this, we obtain $F(\xi)$, which yields system (S_n).

6. Proof of Theorem 2.1. By the definition of (S_n), the pairs (Z_ν, U_ν) are recursively determined by

$$\begin{aligned} &Z_{\nu+1} = \frac{1}{2}\big(U'_\nu - G_\nu(Z_j, U_k;\, _{1 \le j \le \nu,\, 1 \le k \le \nu-1})\big), \\ &U_\nu = Z'_\nu/2 \end{aligned} \tag{17}$$

($\nu = 1, \dots, n$), with $Z_{n+1} = t$. By this fact, it is sufficient to show the following:

Lemma 6.1. *For* $0 \le \nu \le n$,

$$d_{\nu+1}[Z_1] = -4^{\nu+1} Z_{\nu+1}. \tag{18}$$

Proof. We show (18) by induction on ν. Since $d_1[y] = -4y$, (18) is valid for $\nu = 0$. Suppose that (18) is valid for $0 \le \nu \le k$. Then

$$\begin{aligned} Dd_{k+2}[Z_1] &= (D^3 - 8Z_1 D - 4Z'_1) d_{k+1}[Z_1] \\ &= -4^{k+1}(Z^{(3)}_{k+1} - 8Z_1 Z'_{k+1} - 4Z'_1 Z_{k+1}). \end{aligned} \tag{19}$$

By (15) and the first equation of (16),

$$\begin{aligned} v''_{k+1} &= 2u'_{k+1} + 2u'_0 w_{k+1} + 2u_0 w'_{k+1} \\ &= 4v_{k+2} + 4Z_1 w_{k+1} - 2u_0^2 w_{k+1} + 4u_0^2 v_{k+1}, \\ w''_{k+1} &= 2u'_0 v_{k+1} + 2u_0 v'_{k+1} \\ &= 4Z_1 v_{k+1} - 2u_0^2 v_{k+1} + 4u_0^2 w_{k+1} + 4u_0 u_{k+1}; \end{aligned}$$

and hence

$$Z''_{k+1} = 4Z_1 Z_{k+1} + 2u_0^2 Z_{k+1} + 4v_{k+2} + 4u_0 u_{k+1}.$$

Substituting this into (19) and using (15), we have

$$Dd_{k+2}[Z_1] = -4^{k+2} Z'_{k+2}.$$

By (17) together with the definition of G_ν, we have $d_{k+2}[Z_1] = -4^{k+2} Z_{k+2}$, which implies that (18) is valid for $\nu = k+1$. This completes the proof. □

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