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5-COLORED KNOT DIAGRAM WITH FOUR COLORS

Dedicated to Professor Takao Matumoto on his sixtieth birthday

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Abstract

We study Fox 5-colorings for diagrams of 1- and 2-dimensional knots. We prove that any 5-colorable 1-knot has a non-trivially 5-colored diagram such that exactly four colors of five are assigned to the arcs of the diagram. Moreover, we prove that there is a 5-colorable 2-knot such that, for any non-trivially 5-colored diagram, all five colors are assigned to the sheets of the diagram.

1. Introduction

Let us observe the 5-colored diagrams of the knots 4_1, 5_1, and 7_4 as shown in Fig. 1, where the pallet $\mathbb{Z}_5 = \{0, 1, \dots, 4\}$ is used to provide a 5-coloring for each diagram. What is the common property of these 5-colorings?

Each 5-coloring in the figure uses exactly four colors $1, \dots, 4$ except 0. Hence, it is natural to ask the question: *Which* 5*-colorable knot has a* 5*-colored diagram with exactly four colors?* The first aim of this note is to give the answer to this question as follows:

Theorem 1.1. *Any* 5*-colorable knot has a non-trivially* 5*-colored diagram with exactly four colors.*

Harary and Kauffman [5] study the minimal number of colors assigned to the arcs for all non-trivially p-colored diagrams of a p-colorable knot K, which is denoted by $C_p(K)$. Refer to [7] also. Theorem 1.1 implies that $C_5(K) = 4$ for any 5-colorable knot K. We remark that, if p is a prime with $p > 3$, then any p-colorable knot K satisfies $C_p(K) \geq 4$ (Lemma 2.1).

On the other hand, a p-coloring is also defined for a diagram of a 2-dimensional knot (a 2-sphere in $\mathbb{R}^4$), which satisfies the property that any non-trivial p-coloring needs at least four colors for $p > 3$. Hence, we can ask a similar question to the 1-dimensional knot case concerning the minimal number of colors for all non-trivial 5-colorings. The second aim of this note is to prove the following:

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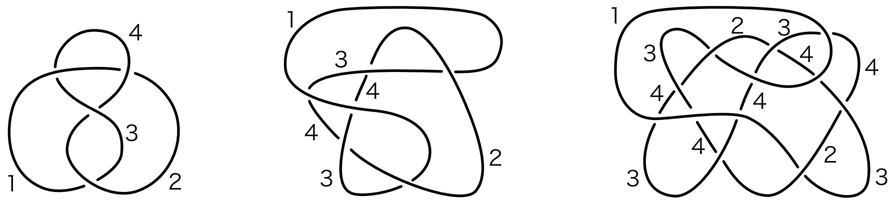

Fig. 1. 5-colored diagrams of 4_1, 5_1, and 7_4.

Theorem 1.2. *There is a* 2*-knot whose any non-trivially* 5*-colored diagram needs all of the five colors*.

For example, Theorem 1.2 holds for the 2-twist-spun figure-eight knot and (2, 5)-torus knot, which are both non-ribbon 2-knots. On the other hand, we have the following for the family of ribbon 2-knots:

Proposition 1.3. *Any* 5*-colorable ribbon* 2*-knot has a non-trivially* 5*-colored diagram with exactly four colors*.

2. 5-colored 1-knot diagrams

Throughout this section, a *knot* means a circle embedded in $\mathbb{R}^3$. Any knot diagram D is regarded as a disjoint union of arcs obtained from the projected planar curves by cutting the lower paths at crossings. For an odd prime p, we consider an assignment of an element of $\mathbb{Z}_p = \{0, 1, \dots, p-1\}$ to each arc of D. It is called a *p-coloring* if $a + c = 2b$ in $\mathbb{Z}_p$ holds near each crossing, where the lower arcs are colored by a and c and the upper is colored by b. The color of the crossing is denoted by $\{a \mid b \mid c\}$. We say that a p-coloring is *trivial* if all arcs of D have the same color, and otherwise *non-trivial*.

Lemma 2.1. *If* $p > 3$, *then any non-trivial* p*-coloring for* D *needs at least four colors of* $0, 1, \dots, p-1$.

Proof. By definition, D has a crossing with the color $\{a \mid b \mid c\}$ which does not satisfy $a = b = c$. Since $a + c = 2b$, we see that a, b, c are mutually different. Hence, any non-trivial p-coloring needs at least three colors.

Assume that exactly three colors are assigned to the arcs of D. Then it is easy to see that D has a pair of crossings whose colors are $\{a \mid b \mid c\}$ and $\{a \mid c \mid b\}$ for some mutually different $a, b, c \in \mathbb{Z}_p$. By the equations $a + c = 2b$ and $a + b = 2c$, we have $3(b - c) = 0$. This is impossible for $p > 3$ and $b \neq c$. □

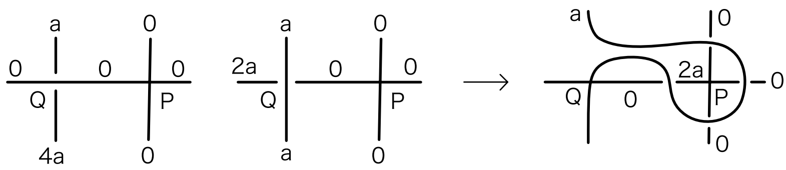

Fig. 2. Eliminating a crossing with the color $\{0 \mid 0 \mid 0\}$.

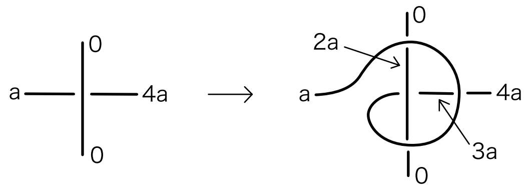

Fig. 3. Eliminating a crossing with the color $\{a \mid 0 \mid 4a\}$.

Lemma 2.2. *Any* 5*-colorable knot has a non-trivially* 5*-colored diagram D with no crossing whose color is* $\{0 \mid 0 \mid 0\}$.

Proof. Assume that D has a crossing of $\{0 \mid 0 \mid 0\}$. Then it is easy to see that D has an adjacent pair of crossings P and Q such that P is of $\{0 \mid 0 \mid 0\}$ and Q is of $\{a \mid 0 \mid 4a\}$ or $\{0 \mid a \mid 2a\}$ for some $a \neq 0$. See the left or middle of Fig. 2. We deform the arc with the color a near Q which detours around P passing over the arcs. Then the color of P changes into $\{2a \mid 2a \mid 2a\}$, and new crossings are of $\{0 \mid a \mid 2a\}$. See the right of the figure. We repeat the deformation above if the obtained diagram still has a crossing of $\{0 \mid 0 \mid 0\}$. □

Lemma 2.3. *Any* 5*-colorable knot has a non-trivially* 5*-colored diagram D with no crossing whose color is* $\{* \mid 0 \mid *\}$.

Proof. We may assume that D has no crossing whose color is $\{0 \mid 0 \mid 0\}$ by Lemma 2.2. Assume that D has a crossing of $\{a \mid 0 \mid 4a\}$ for some $a \neq 0$. Then we deform the arc with the color a which detour around the crossing. See Fig. 3. Then the color of the original crossing changes into $\{a \mid 2a \mid 3a\}$, and new crossings are of $\{0 \mid a \mid 2a\}$ and $\{3a \mid 2a \mid 4a\}$. We repeat the deformation above if the obtained diagram still has a crossing of $\{a \mid 0 \mid 4a\}$ for some $a \neq 0$. □

Proof of Theorem 1.1. Let D be a non-trivially 5-colored diagram of the knot. By Lemma 2.3, we may assume that the upper arc of any crossing of D have a non-zero color. Hence, each arc with the color 0 connects a pair of crossings directly whose colors are $\{0 \mid a \mid 2a\}$ and $\{0 \mid b \mid 2b\}$ for some $a, b \neq 0$. According to $b =$

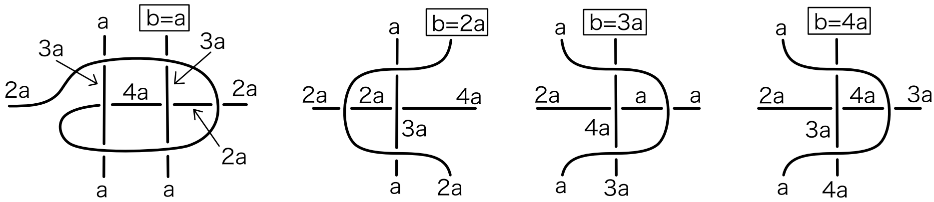

Fig. 4. Eliminating an arc colored by 0.

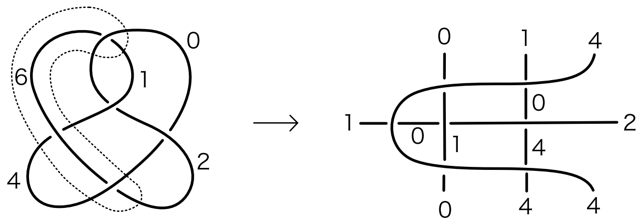

Fig. 5. A 7-colored diagram of 5_2 with four colors.

a, $2a$, $3a$, $4a$, we deform the arc as shown in Fig. 4 so that the arc with 0 is eliminated. We repeat the deformation above if the obtained diagram still has an arc whose color is 0. □

REMARK 2.4. (i) The argument as above can be easily applied to the families of 5-colored virtual knot diagrams [6] and virtual arc diagrams [9].
(ii) In the proof of Theorem 1.1, we eliminate the color 0 from a 5-colored diagram. By adding $a \neq 0$ to the color of each arc, we can easily eliminate the color a instead of 0.

For the cases of 7- and 11-colorings, we have just several examples as follows:

EXAMPLE 2.5. By Lemma 2.1, any non-trivial 7-coloring requires at least four colors assigned to the arcs of a diagram.

Consider the 7-colored diagram of the knot 5_2 with five colors $0, 1, 2, 4, 6$ as shown in the left of Fig. 5. We deform a neighborhood of the arc with 6 as in the right, so that we can eliminate the color 6 without introducing new colors except 0, 1, 2, 4. Hence, we have $C_7(5_2) = 4$.

Similarly, consider the 7-colored diagram of the $(2, 7)$-torus knot $T_{2,7}$ with five colors $0, 1, 2, 3, 4$ as shown in the left of Fig. 6. We deform neighborhoods of the arcs with the color 3 as in the right, so that we obtain a diagram colored by 0, 1, 2, 4. Hence, we have $C_7(T_{2,7}) = 4$. (Kauffman and Lopes [7] conjectured $C_p(T_{2,p}) = (p + 3)/2$.)

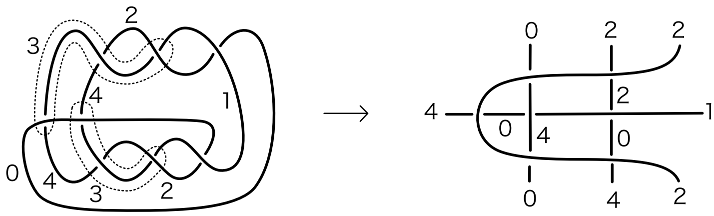

Fig. 6. A 7-colored diagram of $T_{2,7}$ with four colors.

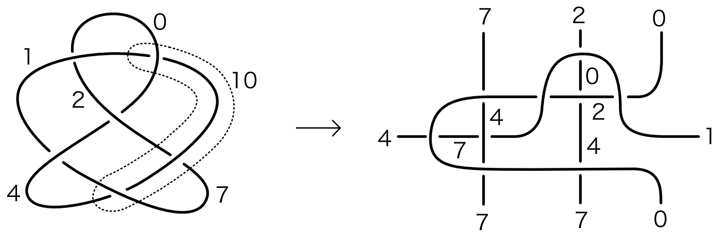

Fig. 7. A 11-colored diagram of 6_2 with five colors.

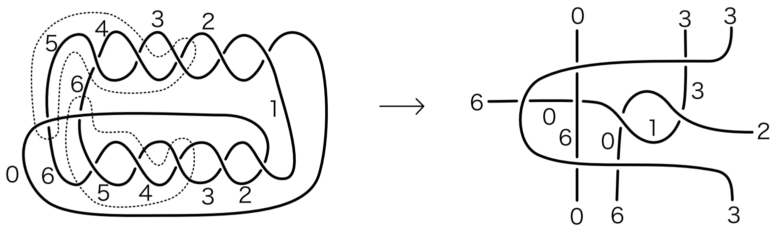

Fig. 8. A 11-colored diagram of $T_{2,11}$ with five colors.

QUESTION 2.6. Does it hold $C_7(K) = 4$ for any 7-colorable knot K?

EXAMPLE 2.7. Similarly to Lemma 2.1, it is easy to see that if $p > 7$, then any non-trivial p-coloring for a knot diagram needs at least five colors of $0, 1, \ldots, p-1$ assigned to the arcs of a diagram, that is, $C_p(K) \geq 5$.

Consider the 11-colored diagram of the knot 6_2 with six colors 0, 1, 2, 4, 7, 10 as shown in the left of Fig. 7. We deform a neighborhood of the arc with 10 as in the right, so that we obtain a diagram colored by 0, 1, 2, 4, 7. Hence, we have $C_{11}(6_2) = 5$.

Similarly, consider the 11-colored diagram of the (2, 11)-torus knot $T_{2,11}$ with seven colors $0, 1, 2, 3, 4, 5, 6$ as shown in the left of Fig. 8. We deform neighborhoods of the arcs with 5 and 6 as shown in the right, so that we obtain a diagram colored by 0, 1, 2, 3, 6. Hence, we have $C_{11}(T_{2,11}) = 5$.

QUESTION 2.8. Does it hold $C_{11}(K) = 5$ for any 11-colorable knot K?

3. 5-colored 2-knot diagrams

Throughout this section, a 2-*knot* means a 2-dimensional sphere embedded in $\mathbb{R}^4$ smoothly. A *diagram* of a 2-knot K is a projection image $\pi(K)$ under a projection $p\colon \mathbb{R}^4 \to \mathbb{R}^3$ equipped with crossing information. Refer to [3] for more details.

Any 2-knot diagram is regarded as a disjoint union of compact, connected surfaces, each of which is called a *sheet*. For an odd prime p, an assignment of an element of $\mathbb{Z}_p$ to each sheet of the diagram is called a *p-coloring* if $a + c = 2b$ in $\mathbb{Z}_p$ holds near each double point, where the lower sheets are colored by a and c and the upper is colored by b.

Let D be a p-colored diagram of a 2-knot. Consider a triple point of D, where the top sheet is colored by a, the middle sheets are colored by b_1 and b_2, and the bottom sheets on both sides of the middle sheet with b_i are colored by c_{i1} and c_{i2} ($i = 1, 2$). We may assume that the bottom sheets colored by c_{1j} and c_{2j} are adjacent along the top sheet ($j = 1, 2$). See Fig. 9. We say that a triple point is *degenerated* with respect to the p-coloring if $a = b_i$ or $b_i = c_{ij}$ holds for some $i, j \in \{1, 2\}$, and otherwise *non-degenerated*. Hence, a triple point is non-degenerated if and only if $a \neq b_i \neq c_{ij}$ holds for any $i, j \in \{1, 2\}$. The notion of non-degeneracy was used in [10].

Lemma 3.1. *For a non-degenerated triple point with the colors as above, we have the following.*
(i) *It holds that $c_{11} \neq c_{12}$ and $c_{21} \neq c_{22}$.*
(ii) *It holds that $c_{11} \neq c_{22}$ and $c_{12} \neq c_{21}$.*
(iii) *It holds that $c_{11} \neq c_{21}$ or $c_{12} \neq c_{22}$.*

Proof. We first remark that, since $b_1+b_2 = 2a$ and $a \neq b_1, b_2$, it holds that $b_1 \neq b_2$.

(i) Since $c_{i1} + c_{i2} = 2b_i$ and $b_i \neq c_{ij}$ for $i, j = 1, 2$, we see that b_i, c_{i1}, c_{i2} are mutually different.

(ii) Assume that $c_{11} = c_{22}$ (the case $c_{12} \neq c_{21}$ is similarly proved). Then it holds that $c_{12} = 2a - c_{22} = 2a - c_{11} = c_{21}$, and hence, $b_1 = (c_{11} + c_{12})/2 = (c_{22} + c_{21})/2 = b_2$. This contradicts to $b_1 \neq b_2$.

(iii) Assume that $c_{11} = c_{21}$ and $c_{12} = c_{22}$. Then it holds that $b_1 = (c_{11} + c_{12})/2 = (c_{21} + c_{22})/2 = b_2$, which contradicts to $b_1 \neq b_2$. □

Lemma 3.2. *Let p be a prime with $p > 3$. If a p-colored diagram of a 2-knot has a non-degenerated triple point, then the p-coloring needs at least five colors assigned to the sheets of the diagram.*

Proof. It is sufficient to prove that there are at least five different colors in the set $\{a, b_i, c_{ij} \mid i, j = 1, 2\}$ near a non-degenerated triple point. By Lemma 3.1, we have

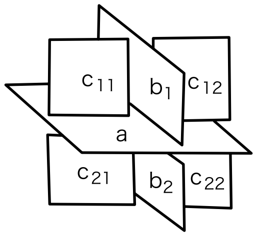

Fig. 9. Colors of sheets near a triple point.

two cases with respect to the colors of the bottom sheets (by changing the indices if necessary):

(i) c_{11}, c_{12}, c_{21}, and c_{22} are four different colors.

(ii) $c_{11} = c_{21}$, c_{12}, and c_{22} are three different colors.

For the case (i), since $a \neq c_{ij}$ holds for any $i, j = 1, 2$, we have five different colors a, c_{11}, c_{12}, c_{21}, and c_{22}.

Consider the case (ii). Since $c_{11} = c_{21} = a$, each of the triplets

$$\{a, b_1, b_2\},\ \{a, c_{12}, c_{22}\},\ \{a, b_1, c_{12}\}, \quad \text{and} \quad \{a, b_2, c_{22}\}$$

consists of mutually different colors. Hence, to prove the lemma, it is sufficient to prove $b_1 \neq c_{22}$ and $b_2 \neq c_{12}$. Assume that $b_1 = c_{22}$ (the case $b_2 = c_{12}$ is similarly proved). Since $b_1 + b_2 = 2a$ and $c_{12} + c_{22} = c_{12} + b_1 = 2a$, we have $b_2 = c_{12}$. Hence, it holds that

$$c_{11} + c_{12} = a + b_2 = 2b_1$$

and

$$c_{21} + c_{22} = a + b_1 = 2b_2,$$

which induces $3(b_1 - b_2) = 0$. This is impossible for $p > 3$ and $b_1 \neq b_2$. □

Let D be a diagram of a 2-knot K, and γ a (possibly trivial) p-coloring for D. By using the Mochizuki's 3-cocycle [8] of the dihedral quandle of order p, we can define a weight $W_p(t, \gamma) \in \mathbb{Z}_p$ for a triple point t of D in an appropriate manner. Take the sum $W_p(\gamma) = \sum_t W_p(t, \gamma)$ for all triple points of D. The cocycle invariant of the 2-knot K is defined by

$$\Phi_p(K) = \{W_p(\gamma) \mid \gamma\colon \text{ any } p\text{-coloring for } D\}$$

as a multi-set [2]. The weight $W_p(t, \gamma)$ has the property that, if t is degenerated with respect to γ, then it holds that $W_p(t, \gamma) = 0$. In particular, if γ is a trivial p-coloring, then any t is degenerated, and hence, we have $W_p(\gamma) = 0$. In other words, if a p-coloring

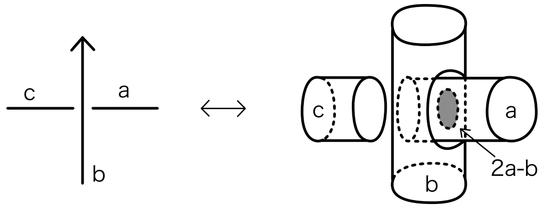

Fig. 10. Virtual arc presentation.

γ satisfies $W_p(\gamma) \neq 0$, then there is a triple point t with $W_p(t, \gamma) \neq 0$ which implies that t is non-degenerated with respect to γ.

Proof of Theorem 1.2. Let K be the the 2-twist-spun figure-eight knot, and D a diagram of K. The cocycle invariant of K is calculated in [4] such that

$$\Phi_5(K) = \{0, \dots, 0 \ (5 \text{ times}),\ 2, \dots, 2 \ (10 \text{ times}),\ 3, \dots, 3 \ (10 \text{ times})\}.$$

The number of 5-colorings for D is 25 which includes 5 trivial ones. Hence, for any non-trivial 5-coloring γ, it holds that $W_5(\gamma) = 2$ or 3. Since $W_5(\gamma) \neq 0$, D has a non-degenerated triple point with respect to γ. The proof is completed by Lemma 3.2. □

REMARK 3.3. (i) Let K_{2n} be the $2n$-twist-spun figure-eight knot. The cocycle invariant of K_{2n} is given by $\Phi_5(K_{2n}) = n \cdot \Phi_5(K_2) = \{0, \dots, 2n, \dots, 3n, \dots\}$ (cf. [1]). Hence, if n is not divisible by 5, then K_{2n} has the same property as in Theorem 1.2.
(ii) Since the cocycle invariant of the 2-twist-spun (2, 5)-torus knot K is

$$\Phi_5(K) = \{0, \dots, 0 \ (5 \text{ times}),\ 1, \dots, 1 \ (10 \text{ times}),\ 4, \dots, 4 \ (10 \text{ times})\},$$

K has the same property as in Theorem 1.2.

A ribbon 2-knot is obtained by adding 1-handles to a trivial 2-link [11]. It is known that any ribbon 2-knot is presented by a virtual arc diagram [9]. Given an oriented virtual arc diagram A, we construct a diagram D of a ribbon 2-knot Tube(A). In Fig. 10, we shows a part of D corresponding to a classical crossing of A. Moreover, it is easy to see that there is a one-to-one correspondence between the set of the 5-colorings for A and that for D.

Proof of Proposition 1.3. Let K be a 5-colorable ribbon 2-knot. We may assume that $K = \text{Tube}(A)$ for some virtual arc diagram A. Since K is 5-colorable, so is A. As mentioned in Remark 2.4 (i), we may assume that A has a non-trivial 5-coloring with exactly four colors 1, 2, 3, 4.

Consider the 5-colored diagram D of $K = \text{Tube}(A)$ corresponding to A. By the assumption for A, if D has a sheet colored by 0, then the sheet is the small one colored

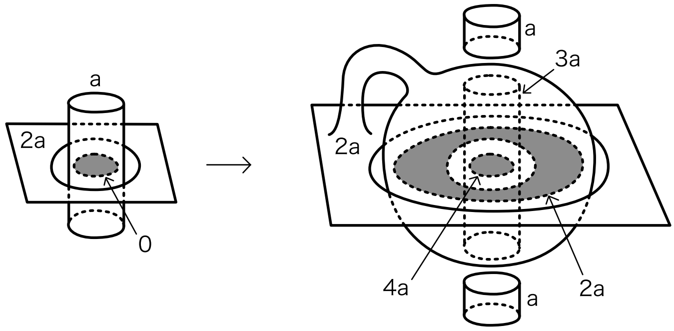

Fig. 11. Eliminating a sheet colored by 0.

by $2a - b$ (= 0) in Fig. 10. In a neighborhood of the sheet with 0, we deform the sheet with $2a$ (= b) as shown in Fig. 11 so that the color 0 is eliminated. The deformation is similar to the one in the most left of Fig. 4. This completes the proof. □

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