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Note on Lattice-Isomorphisms between Abelian Groups and Non-Abelian Groups

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The purpose of this note is to settle the problem of determining the groups lattice-isomorphic to abelian groups. This question was first put and studied by R. Baer. K. Iwasawa determined completely those finite groups and infinite groups with elements of infinite order whose lattices of subgroups are modular (= *m-groups*), and determined the infinite m-group without elements of infinite order under the hypothesis that any m-group which has the lattice of subgroups of finite dimension is a finite group[1]. We shall call this *hypothesis* (A). So the only thing for us to do now is to find out whether non-abelian m-groups are lattice-isomorphic to abelian groups or not. In the case of finite groups this question was completely studied by A. W. Jones[2], and in the general case by R. E. Beaumont[3] to some extent.

We shall show in this note the following:

If G is a non-abelian infinite m-group and has no element of infinite order, under the hypothesis (A), similar theorems as those by Jones in the finite case hold, while if G has at least one element of infinite order, then there exists always an abelian group lattice-isomorphic to G.

We shall denote by $LC(G)$ and $L(G)$ the partially ordered set formed of all cyclic subgroups and the lattice formed of all subgroups of a group G respectively.

Definition. Let s, u and x be positive integers and α be an integral p-adic number such that $\alpha \equiv 1 \mod p^s (s \geq 2$ if $p=2)$, then we define

$$(1) \qquad \varphi(\alpha, u, x) = \sum_{i=0}^{x-1} \alpha^{iu}.$$

When the value of α remains fixed, we shall write $\varphi(u, x)$ for $\varphi(\alpha, u, x)$. Let $\alpha = 1 + p^s\beta$, then $(1+p^s\beta)\cdot\varphi(1, u) = \varphi(1, u) + (1+p^s\beta)^u - 1$, hence $(1+p^s\beta)^u = \varphi(1, u)\ p^s\beta + 1$, and so

$$\begin{aligned}\varphi(u, x) &= \sum_{i=0}^{x-1} \{1+\varphi(1, u)\ p^s\beta\}^i \\ &= x + \frac{x!}{2!\,(x-2)!}\varphi(1, u)\ p^s\beta + \dots + \frac{x!}{r!\,(x-r)!}(\varphi(1, u)\ p^s\beta)^{r-1} + \dots \\ &\quad + \frac{x!}{x!}(\varphi(1, u)\ p^s\beta)^{x-1}.\end{aligned}$$

1) Iwasawa [1], [2], [3], cf. also Sato [1].
2) Jones [1].
3) Beaumont [1]. I could not see this paper.

But $s(r-1)-1 \geqq b$[4], where $p^b \| r$, so we have

Lemma 1.[5] *$\varphi(u, x) = ax$ has a solution a of p-adic number for every pair of integers x and u, and $a \equiv 1 \bmod p$.*

We shall denote the solution a in Lemma 1 by $\varphi'(u, x)$: $\varphi(u, x) = x \cdot \varphi'(u, x)$. Then we have

Lemma 2. $\varphi'(u, x_1) \cdot \varphi'(ux_1, x_2) = \varphi'(u, x_1 x_2)$.

$$\textit{Proof.} \quad \varphi(u, x_1) \cdot \varphi(ux_1, x_2) = \Big(\sum_{i=0}^{x_1-1} \alpha^{iu}\Big)\Big(\sum_{j=0}^{x_2-1} \alpha^{jux_1}\Big)$$
$$= \sum_{i=0}^{x_1-1} \sum_{j=0}^{x_2-1} \alpha^{(ij+x_1)u} = \sum_{h=0}^{x_1x_2-1} \alpha^{hu} = \varphi(u, x_1 x_2).$$

Hence our lemma is the immediate consequence of the definition of φ'.

Corollary. *If $\lambda(0)$ is a p-adic number and $\lambda(j) = \varphi'(p^{j-1}, p) \cdot \lambda(j-1)$, $(j=1, 2, \ldots)$, then $\lambda(j) \cdot \varphi'(p^j, p^i m) = \lambda(j+i) \cdot \varphi'(p^{j+i}, m)$, where m is a natural number.*

Lemma 3. *Let $G = E \cdot \{z\}$ be a non-abelian m-group with elements of infinite order, where E is the abelian normal subgroup consisting of all elements of finite order from G and $\{z\}$ is a free cyclic subgroup generated by z. Then G is lattice-isomorphic to an abelian group $H = E' \times \{w\}$, where $E' \cong E$ and $\{w\} \cong \{z\}$.*

Proof. E, E' are the direct products of p_i-components; $E = P_1 \times P_2 \times \ldots$, $E' = P_1' \times P_2' \times \ldots$, and $P \cong P'$. There exist integral p_i-adic numbers α_i such that $\alpha_i \equiv 1 \bmod p_i^{s_i}$ for positive integers $s_i (s_i \geqq 2$ if $p_i = 2)$, and $zAz^{-1} = A^{\alpha_i}$ for any $A \in P_i$.[6] We shall denote by φ_i the function φ in (1) defined concerning α_i, and consider the following correspondence τ between $LC(G)$ and $LC(H)$.

$$(*) \quad \Big\{\Big(\Big(\prod_{i=1}^{l} B_i\Big) w^a\Big)^r\Big\} \overset{\tau}{\leftrightarrow} \Big\{\Big(\Big(\prod_{i=1}^{l} A_i^{\lambda_i(t_i) \cdot \varphi_i'(p_i^{t_i},\, a/p_i^{t_i})}\Big) z^a\Big)^r\Big\},$$

where $a = \prod_{i=1}^{l} p_i^{t_i}$, and A_i, B_i are elements from P_i, P_i' respectively which correspond to each other by a fixed isomorphism $E \cong E'$, r is a non-negative integer, and λ_i are p_i-adic numbers defined as follows:

$\lambda_i(0) \equiv 1 \bmod p_i$, $\lambda_i(j) = \varphi_i'(p_i^{j-1}, p_i) \cdot \lambda_i(j-1)$, $(j = 1, 2, \ldots)$, for every i. We shall fix those notations A, B and λ throughout this proof. Then τ is one to one and can be extended to a lattice-isomorphism between G and H. We shall show this by induction on the exponents of P_i and P_i'.

4), 5) For the detail of the proofs cf. Jones [1] 3. 10 Lemma.

6) Iwasawa [2], cf. also Sato [1].

Put $E^{(n)} = \Pi P_i^{(n)}$ and $E'^{(n)} = \Pi P_i'^{(n)}$, where $P_i^{(n)}$ and $P_i'^{(n)}$ are the greatest subgroups of exponent equal to or lower than p^n in P_i and P_i' respectively. We shall denote by τ_n the correspondence $(*)$ defined on the cyclic subgroups of $G^{(n)} = E^{(n)} \cdot \{z\}$ and those of $H^{(n)} = E'^{(n)} \times \{w\}$. Then we have $E^{(1)} \cdot \{z\} \cong E'^{(1)} \times \{w\}$. Since $\lambda_i(j) \equiv 1$ and $\varphi_i' \equiv 1 \bmod p_i$, we can suppose that τ_1 is induced by this isomorphism.

Now we assume that τ_{n-1} is a one to one index preserving correspondence between $LG(G^{(n-1)})$ and $LC(H^{(n-1)})$, and can be extended to a lattice-isomorphism between $G^{(n-1)}$ and $H^{(n-1)}$ if we let correspond to each other such two subgroups that every cyclic subgroup of each of them has the image of it by τ_{n-1} in the other. And we shall prove τ_n has the similar properties and $\tau_n > \tau_{n-1}$, i.e., τ_n induces τ_{n-1}. This will complete the proof of our lemma, for $\tau = \bigcup \tau_n$, $G = \bigcup G^{(n)}$ and $H = \bigcup H^{(n)}$.

I) τ_n is defined on every cyclic subgroup of $H^{(n)}$ and $G^{(n)}$.

For $H^{(n)}$ the proof is trivial. As for $G^{(n)}$, according to the fact that

$$r \cdot \lambda_i(t_i) \cdot \varphi_i(a, m) \cdot \varphi_i'(p_i^{t_i}, a/p_i^{t_i}) \equiv 1 \bmod p_i \quad (i = 1, \dots, l)$$

have always a solution r if $(m, p_i) = 1$ $(i = 1, \dots, l)$, there exists always $\{((\prod_{i=1}^{l} B_i)^r w^a)^m\}$ in $H^{(n)}$ that corresponds to a given cyclic subgroup $\{(\prod_{i=1}^{l} A_i)\, z^{am}\}$, where $a = \prod_{i=1}^{l} p_i^{t_i}$,

II) τ_n is a one-to-one correspondence between $LC(G^{(n)})$ and $LC(H^{(n)})$.

Let $\mathfrak{a}$ be an arbitrary cyclic subgroup of infinite order of $G^{(n)}$ and $\mathfrak{a}' \supset \mathfrak{a}$ be a maximal cyclic subgroup of $G^{(n)}$. To prove our proposition, it is sufficient to show that, if $\mathfrak{a}' \overset{\tau_n}{\leftrightarrow} \mathfrak{b}'$, $\mathfrak{a} \overset{\tau_n}{\leftrightarrow} \mathfrak{b}$, then we have always $\mathfrak{b}' \supset \mathfrak{b}$ and the index $[\mathfrak{a}' : \mathfrak{a}]$ is equal to $[\mathfrak{b}' : \mathfrak{b}]$ and conversely, because, from the definition of τ_n, the mapping $\mathfrak{a}' \to \mathfrak{b}'$ and $\mathfrak{b}' \to \mathfrak{a}'$ are one-valued.

Let $\mathfrak{a} = \{(\prod_{i=1}^{l} A_i)\, z^{am}\}$, $\mathfrak{b} = \{((\prod_{i=1}^{l} B_i)^r w^a)^m\}$, $\mathfrak{a}' = \{(\prod_{j=1}^{l'} A_j')\, z^{a'}\}$, $\mathfrak{b}' = \{(\prod_{j=1}^{l'} B_j)^{r'} w^{a'}\}$ and

$$(2) \qquad \mathfrak{a}'^{\,bh} = \mathfrak{a},$$

where $a = \prod_{i=1}^{l} p_i^{t_i}$, $a' = \prod_{j=1}^{l'} p_j^{t_j'}$, $(p_i, m) = 1$ $(i = 1, \dots, l)$, $(p_j, h) = 1$ $(j = 1, \dots, l')$, b has only those prime factors p_j $(j = 1, \dots, l')$, $\mathfrak{a}'^{bh}$ is a cyclic subgroup consisting of those elements expressed as x^{bh} for $x \in \mathfrak{a}'$, and r, r' satisfy the following congruences,

$$(3) \qquad r \cdot \lambda_i(t_i) \cdot \varphi_i'(p_i^{t_i}, a/p_i^{t_i}) \cdot \varphi_i(a, m) \equiv 1 \ mod\ p_i^n \quad (i = 1, \dots, l),$$

(4) $r'\cdot\lambda_j(t_j')\cdot\varphi_j'(p_j^{t_j'}, a'/p_j^{t_j'})\equiv 1 \ mod \ p_j^n \qquad (j=1,\dots, l')$.

From (2) we have

(5) $$A_i = A_i'^{s_i},$$

where $s_i=\varphi_i(a', b)\cdot\varphi_i(a'b, h)$. Now we are only to prove $B_i^{rm}=B_i'^{r'bh}$ $(i=1,\dots, l)$. From (4) and (5) we have

$$s_i \equiv r'\cdot\lambda_i(t_i')\cdot\varphi_i'(p_i^{t_i'}, a'/p_i^{t_i'})\cdot\varphi_i(a', b)\cdot\varphi_i(a'b, h)$$
$$\equiv r'\cdot\lambda_i(t_i')\cdot\varphi_i'(p_i^{t_i'}, a'bh/p_i^{t_i'})\ bh$$
$$\equiv r'\cdot\lambda_i(t_i)\cdot\varphi_i'(p_i^{t_i}, am/p_i^{t_i}) \bmod p_i^n \ (i=1,\dots, l), \ (\text{cf. } a'bh=am).$$

Hence, from (3), we have $rs_i\equiv r'bhm \bmod p_i^n$, i.e.,

$$A_i^r = A_i'^{r'bhm^{-1}}.$$

But this is equivalent to $B_i^{rm}=B'^{r'bh}$. The converse is also obvious from the proof above.

As τ_n is an index preserving mapping, we have

III) τ_n is an isomorphism between the partially ordered sets $LC(G^{(n)})$ and $LC(H^{(n)})$.

IV) $\tau_n > \tau_{n-1}$. This is also evident from the prosf of II) if we consider the case $\mathfrak{a}\leq G^{(n-1)}$ but $\mathfrak{a}'\nleq G^{(n-1)}$.

V) τ_n can be extended to a lattice-isomorphism between $L(G^{(n)})$ and $L(H^{(n)})$.

When we prove the fact that, for any pair of cyclic subgroups $\mathfrak{a}_1$, $\mathfrak{a}_2\leq G^{(n)}$ and corresponding $\mathfrak{b}_1$, $\mathfrak{b}_2\leq H^{(n)}$, any cyclic subgroup of $\mathfrak{a}_1\cup\mathfrak{a}_2$ has its image by τ_n in $\mathfrak{b}_1\cup\mathfrak{b}_2$ and conversely, the validity of our proposition will follow immediately. If $\mathfrak{a}_1'\geq\mathfrak{a}_1$, $\mathfrak{a}_2'\geq\mathfrak{a}_2$ are maximal cyclic in $\mathfrak{a}_1\cup\mathfrak{a}_2$, then, from III) and the modularity of $L(G^{(n)})$ and $L(H^{(n)})$, we can conclude by simple calculations that $\mathfrak{b}_1'$, $\mathfrak{b}_2'$ are contained and maximal cyclic in $\mathfrak{b}_1\cup\mathfrak{b}_2$. The converse is also true. Hence we can assume that $\mathfrak{a}_1$, $\mathfrak{a}_2$ are maximal cyclic in $\mathfrak{a}_1\cup\mathfrak{a}_2$.

If x, y are minimal positive integers such that $\mathfrak{b}_1^x=\mathfrak{b}_2^y$[7], then it can easily be seen that x, y have no other prime factor than those of the orders of X and Y, where $X, Y\in E'$, and $\mathfrak{b}_1=\{Xw^{am}\}$, $\mathfrak{b}_2=\{Yw^{bh}\}$. Put $x=x'(x, y)$, $y=y'(x, y)$. Then, considering $m'=h'$, we can easily conclude that the unique maximal cyclic subgroup of finite order in $\mathfrak{b}_1\cup\mathfrak{b}_2$ is $\{X^{x'}Y^{-y'}\}$, where m' and h' are the greatest factors of m and

7) $\mathfrak{h}^x$ means the cyclic group generated by the x-th power of the generator of $\mathfrak{h}$. This notation will be fixed throughout this paper.

h respectively which have no common factor with both orders of X and Y. We can also see by calculations that the unique maximal cyclic subgroup of finite order of $\mathfrak{a}_1 \bigvee \mathfrak{a}_1$ is $\{R^{x'}S^{-y'}\}$, where $R, S \in E$, correspond to X, Y respectively. The converse is also true. Thus we can suppose without loss of generality that $\mathfrak{a}_1$ is a finite subgroup.

Then we can see without much difficulty that, if f, h are integers and $h \geqq 0$, the image of $\mathfrak{b}_1^f \mathfrak{b}_2^h$ by τ_n is $\mathfrak{a}_1^r \mathfrak{a}_2^h$ for some integer r. The converse is also true. This completes our proof.

Theorem 1. *Let* $G = \{E, z_1, z_2, \dots\}$ *be a non-abelian m-group.*[8] *Then there exists an abelian group* $H = \{E', w_1, w_2, \dots\}$ *which is lattice-isomorphic to G, where* z_i, w_i *are generators of infinite order and* E, E' *have the same significances as in Lemma* 3.

Proof. Let $\alpha_j^{(i)}$, $\varphi_j^{(i)}$ have the corresponding significances for z_i to α_j and φ_j respectively in the proof of Lemma 3. We shall denote by σ_1 the lattice-isomorphism between $G_1 = \{E, z_1\}$ and $H_1 = \{E', w_1\}$ that is defined as in Lemma 3. If $z_2^q = Rz_1$ for an $R \in E$, we set $w_2^q = Xw_1$ for such an $X \in E'$ as $\sigma_1\{Rz_1\} = \{Xw_1\}$. Now we define the lattice-isomorphism σ_2 between $G_2 = \{E, z_1, z_2\} = \{E, z_2\}$ and $H_2 = \{E, w_1, w_2\} = \{E, w_2\}$, using $\mu_j(t_j) = \lambda_j(t_i) \cdot \varphi_{j'}^{(2)}(p_j^{t_j}, q)^{-1}$ for $\lambda_j(t_j)$. Then

$$\sigma_2 > \sigma_1.$$

To prove this, let $\mathfrak{b} = \{Yw_2^a\}^m \overset{\sigma_2}{\leftrightarrow} \mathfrak{a} = \{Sz_2^a\}^m$, where $a = \prod_{i=1}^{l} p_i^{t_i}$, $Y = \prod_{i=1}^{l} B_i$, $S = \prod_{i=1}^{i} A_i$ (some of B_j, A_i may be 1). Then $\mathfrak{b}^q \overset{\sigma_2}{\leftrightarrow} \mathfrak{a}^q$ and the exponent of A_i in $\mathfrak{a}^q$ is

$$\begin{aligned} &\mu_i(t_i) \cdot \varphi_i^{(2)'}(p_i^{t_i}, a/p_i^{t_i}) \cdot \varphi_i^{(2)}(a, q) \cdot \varphi_i^{(2)}(aq, m) \\ &= q \cdot \mu_i(t_i) \cdot \varphi_i^{(2)'}(p_i^{t_i}, q) \cdot \varphi_i^{(2)'}(p_i^{t_i}q, a/p_i^{t_i}) \cdot \varphi^{(2)}(aq, m) \\ &= q \cdot \lambda_i(t_i) \cdot \varphi_i^{(1)'}(p_i^{t_i}, a/p_i^{t_i}) \cdot \varphi_i^{(1)}(a, m) \end{aligned}$$

Hence $\mathfrak{b}^q \leftrightarrow \mathfrak{a}^q$ by σ_1. This shows $\sigma_2 > \sigma_1$. Repeating this process, we can construct an abelian group H which is lattice isomorphic to G by the correspondence $\sigma = \bigvee \sigma_i$, q. e. d.

As the infinite non-abelian m-group with elements of infinite order has always the same type as G in Theorem 1 or in Lemma 3[9], our problem is now solved for this type of m-groups.

Under the hypothesis (A), the directly indecomposable non-abelian infinite m-group M is either a p-group or of the following type[10];

8) Iwasawa [2], cf. also Sato [1].
9) Iwasawa [2].
10) Iwasawa [3].

$M = P \cdot \{Q\}$, where P is an elementary abelian normal p-subgroup of infinite order, $\{Q\}$ is a cyclic q-group generated by Q of order q^m for some natural number m, and there is a natural number r such that $r \not\equiv 1 \bmod p$, $r^q \equiv 1 \bmod p$ and $QAQ^{-1} = A^r$ for any $A \in P$.

If M is a p-group, it is isomorphic to some factor group $G/\{z^{p^h}\}$ of a non-abelian m-group $G = P \cdot \{z\}$, where P is an abelian normal subgroup of exponent p^n for a natural number n and z has the same significance as in Lemma 3. Hence any group of this type is also lattice-isomorphic to an abelian group.

If M is of the second type, it has non-nilpotent finite subgroups, so it cannot be lattice-isomorphic to any abelian group, unless $m = 1$, according to the results by Jones. Furthermore, under the hypothesis (A) we can prove quite analogously as Jones did in the case of finite groups, that, if $m \neq 1$, M is lattice-isomorphic only to such a m-group N as $N = P' \cdot \{T\}$, where $P' \cong P$ and normal in N, and $\{T\}$ is a cyclic t-group generated by T of order t^m, but if $m = 1$, M is lattice-isomorphic to an elementary abelian p-group. We shall omit the proof.

Finally, the only case to be considered is the 2-group that is the direct product of the quaternion group and an infinite elementary abelian 2-group.[11] In this case also, it is not difficult to prove that any group lattice-isomorphic to one of this type, under the hypothesis (A), is always isomorphic to it, as in the case of finite groups.

As any m-group without elements of infinite order is the direct product of subgroups of the types above, the study of our problem is completed.

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11) Iwasawa [3].