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L^q-SPECTRUM OF BERNOULLI CONVOLUTIONS ASSOCIATED WITH P.V. NUMBERS

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1. Introduction

Let μ be a positive bounded Borel measure on $\mathbb{R}^d$ with bounded support and let supp(μ) denote the support of μ. For $\delta > 0$ and $q \in \mathbb{R}$, the *L^q-(moment) spectrum* of μ is defined as

$$\tau(q) = \varliminf_{\delta \to 0^+} \frac{\ln \sup \sum_i \mu(B_\delta(x_i))^q}{\ln \delta},$$

where $\{B_\delta(x_i)\}_i$ is a disjoint family of δ-balls with center $x_i \in \text{supp}(\mu)$ and the supremum is taken over all such families. For $q > 1$, the *L^q-dimension* (or *generalized Rényi dimension*, see e.g. [10], [22]) of μ is defined as

$$\underline{\dim}_q(\mu) = \frac{\tau(q)}{q-1}.$$

The spectra $\tau(q)$ and $\underline{\dim}_q(\mu)$ play a central role in studying the multifractal structure of the measure μ (e.g., the multifractal formalism [6], [9], [10]) and it is of great interest to compute them. There is a simple formula for $\tau(q)$ if μ is a self-similar measure defined by an iterated function system of contractive similitudes satisfying the *open set condition* (OSC) ([3], [4], [8], [18], [19]).

The OSC is a separation condition on the similitudes. In the absence of this condition, the dynamics of the iteration is not clear and very few results are known. In [14] the authors introduced a weak separation condition to study some interesting self-similar measures defined by similitudes that do not satisfy the OSC. An important class of examples comes from the self-similar measure μ satisfying the identity

$$\mu = \frac{1}{2}\mu \circ \psi_1^{-1} + \frac{1}{2}\mu \circ \psi_2^{-1}, \tag{1.1}$$

where $\psi_1 x = \rho x$, $\psi_2 x = \rho x + (1-\rho)$, $1/2 < \rho < 1$ ([12], [13], [14]). It is called an *infinitely convolved Bernoulli measure* (ICBM) because it can be identified (up to a scalar

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multiple) with the random variable $\boldsymbol{X} = \sum_{k=0}^{\infty} \rho^k \boldsymbol{X}_k$ where $\{\boldsymbol{X}_k\}$ is a sequence of i.i.d. random variables each taking values 0 or 1 with probability $1/2$. If ρ^{-1} is a P.V. number, then $\{\psi_1, \psi_2\}$ satisfies the above mentioned weak separation condition and μ is singular ([5], [20]). (Recall that an algebraic integer $\beta > 1$ is a *Pisot-Vijayaraghavan (P.V.) number* if all of its conjugates are in modulus strictly less than 1. The golden ratio $(\sqrt{5}+1)/2$ is such a number.)

It is interesting to calculate the exact values of $\tau(q)$, $q \in \mathbb{R}$, for the class of measures in (1.1) because the OSC fails. Only some partial results to this problem are known. For ρ^{-1} equal to a P.V. number, the value of $\tau(2)$ for the associated measure has been calculated in [12] and [13]. For the special case ρ^{-1} equal to the golden ratio, the entropy dimension (corresponding to the L^1-dimension) has been studied and estimated by a number of authors (e.g., [1], [2], [11], [17]). For this particular measure, an explicit formula defining $\tau(q)$ for $q > 0$ was given in [15] recently.

In this paper we continue our study of the ICBM associated with P.V. numbers. Our goal is to obtain a simple algorithm to calculate the L^q-spectrum $\tau(q)$ for such measures when $q \geq 2$ is an integer. Note that for any bounded positive Borel measure with bounded support, it is known that $\tau(1) = 0$ and $-\tau(0)$ is the box dimension of the support of the measure (see e.g. [14]). The basic idea to construct the algorithm can be summarized as follows: First, we observe that for $q > 0$,

$$\tau(q) = \inf\left\{\alpha : \varlimsup_{h\to 0^+} \frac{1}{h^{1+\alpha}} \int_{-\infty}^{\infty} \mu\big(B_h(x)\big)^q \, dx > 0\right\}, \tag{1.2}$$

where $B_h(x)$ is the interval $[x-h, x+h)$ (see [12], [15], [22]). For q equal to a positive integer we let $\boldsymbol{s} = (s_1, \ldots, s_q) \in \mathbb{R}^q$ and let

$$\Phi_{\boldsymbol{s}}^{(\alpha)}(h) = \frac{1}{h^{1+\alpha}} \int_{-\infty}^{\infty} \mu\big(B_h(t+s_1)\big) \cdots \mu\big(B_h(t+s_q)\big) \, dt. \tag{1.3}$$

By using the self-similar identity (1.1) we can introduce a dynamics on a suitable parameter set of the $\boldsymbol{s}$ including $\mathbf{0}$. When ρ^{-1} is a P.V. number, there are only finitely many $\boldsymbol{s}$'s involved (i.e., $\Phi_{\boldsymbol{s}}^{(\alpha)}(h) \neq 0$) and we can represent this dynamics in terms of a sub-Markov matrix. The maximal eigenvalue of the matrix will give the desired $\tau(q)$. This technique is a simplification of that used in [13]. For P.V. numbers that are solutions of the polynomials $x^n - x^{n-1} - \cdots - x - 1 = 0$, $n \geq 2$ (including the golden ratio), the matrix can be reduced to a very simple form.

We organize this paper as follows. In Section 2, we give some algebraic preliminaries and set up the involved matrix. The main result is proved in Section 3. In Section 4 we present techniques to reduce the size of the matrix and describe an algorithm to generate it. In Section 5, we derive an explicit expression for the matrix corresponding to the special class of P.V. numbers mentioned above. Finally in Section 6, we make some remarks on the more general case when ψ_1, ψ_2 are allowed to take different probability weights.

2. Algebraic preliminaries

Let $1/2 < \rho < 1$. For $q \geq 2$ an integer, we define a set of q-dimensional vectors by letting $\boldsymbol{s}_0 = (0, \ldots, 0)$, and for $n \geq 1$,

$$\boldsymbol{s}_n = \rho^{-1}\big(\boldsymbol{s}_{n-1} + (1-\rho)\boldsymbol{\epsilon}\big), \tag{2.1}$$

where $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_q)$, $\epsilon_i = 0$ or 1.

We first make an important identification for the set of $\boldsymbol{s}_n$'s generated by this iteration process. We identify $\boldsymbol{s}_n$ and $\boldsymbol{s}_m$ whenever $\boldsymbol{s}_n = \boldsymbol{s}_m + \boldsymbol{c}$ for some $\boldsymbol{c} = c(1, \ldots, 1)$ (i.e., whenever $\boldsymbol{s}_n$ and $\boldsymbol{s}_m$ both lie on the same straight line in $\mathbb{R}^q$ parameterized by $x_i = t + a_i$, $1 \leq i \leq q$). Let S be the quotient set under such identification. Intuitively, we think of each $\boldsymbol{s} \in S$ as a "line" in $\mathbb{R}^q$. It follows from (2.1) that each element in S has a representation of the form

$$\frac{1-\rho}{\rho} \sum_{k=0}^{n} \rho^{-(n-k)} \boldsymbol{\epsilon}_k.$$

Let

$$S_1 = \big\{(s_1, \ldots, s_q) \in S : \ |s_i - s_j| \leq 1 \ \text{ for all } \ 1 \leq i, j \leq q\big\}.$$

Geometrically, S_1 consists of those lines in S that intersect the unit cube $[0,1]^q$.

We consider S to be a set of states that spans some vector space $\langle S \rangle$ (i.e., S is a basis for the vector space). Define a Markov matrix T on S by

$$T(\boldsymbol{s}) = \frac{1}{2^q} \cdot {\sum_{\boldsymbol{\epsilon}}}' \boldsymbol{s}^{\boldsymbol{\epsilon}}, \tag{2.2}$$

where $(\boldsymbol{s}^{\boldsymbol{\epsilon}})_i = \rho^{-1}(s_i + (1-\rho)\epsilon_i)$, $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_q)$, $\epsilon_i = 0$ or 1, and the summation is taken over all such $\boldsymbol{\epsilon}$. We emphasize that the operations $\cdot$ and $\sum'$ in (2.2) are respectively scalar multiplication and addition in the vector space $\langle S \rangle$. They should be distinguished from the linear combinations in $\mathbb{R}^d$ by regarding the $\boldsymbol{s}$ as usual vectors, as those in (2.1). We also note that the sum of the entries of each column of T is 1.

Proposition 2.1. *T is invariant on the subspace of $\langle S \rangle$ spanned by $S \setminus S_1$.*

Proof. Let $\boldsymbol{s} \in S \setminus S_1$ with $|s_i - s_j| > 1$ and let $\boldsymbol{t} = \boldsymbol{s}^{\boldsymbol{\epsilon}}$. Then

$$|t_i - t_j| = \rho^{-1}\big|(s_i - s_j) + (1-\rho)(\epsilon_i - \epsilon_j)\big| > \rho^{-1}(1 - (1-\rho)) = 1.$$

This implies that $\boldsymbol{t} \in S \setminus S_1$. The assertion follows since $T(\boldsymbol{s})$ is a linear combination, in $\langle S \rangle$, of the $\boldsymbol{s}^{\boldsymbol{\epsilon}}$. □

Proposition 2.2. *If ρ^{-1} is a P.V. number, then S_1 is a finite set.*

Proof. The proof of this result can be found in [13]. We give another proof using a lemma of Garsia ([7, Lemma 1.51]): Let $\beta > 1$ be an algebraic integer, let $\beta_1, \ldots, \beta_\ell$ be the algebraic conjugates of β and let σ be the number of β_i such that $|\beta_i| = 1$. For an n-th degree polynomial L with integer coefficients a_i and height $M := \max\{|a_i| : i = 1, \ldots, n\}$, if $L(\beta) \neq 0$, then

$$|L(\beta)| \geq \frac{\prod_{|\beta_i|\neq 1} ||\beta_i| - 1|}{(n+1)^\sigma (\prod_{\beta_i > 1} |\beta_i|)^{n+1} M^\ell}.$$

Now if $\rho^{-1} = \beta$ is a P.V. number, then the above reduces to

$$|L(\beta)| \geq M^{-\ell} \prod_{|\beta_i|\neq 1} ||\beta_i| - 1| := C. \tag{2.3}$$

We observe that for $s \in S$, $s_j = \frac{1-\rho}{\rho} \sum_{k=0}^n \beta^{n-k} \epsilon_k^{(j)}$. Hence $s \in S_1$ if and only if

$$\left| \sum_{k=0}^n \beta^{n-k} (\epsilon_k^{(i)} - \epsilon_k^{(j)}) \right| \leq \frac{1}{\beta - 1}, \quad \text{for all } 1 \leq i, j \leq q.$$

Therefore to show that S_1 is finite, it suffices to show that the set B defined below is finite:

$$B = \bigcup_{n=0}^{\infty} \left\{ y_n = \sum_{k=0}^n \beta^{n-k} \eta_k : \eta_k = 0 \text{ or } \pm 1, \ |y_n| \leq \frac{1}{\beta - 1} \right\}.$$

If $y_n \neq y_m$ are two elements in B with $n \geq m$, then

$$y_n - y_m = \sum_{k=0}^n \beta^{n-k} \eta_k - \sum_{k=0}^m \beta^{m-k} \eta'_k.$$

We use this to define a polynomial L with coefficients $\eta_k - \eta'_k$ (letting $\eta'_k = 0$ for $m < k \leq n$). In this case, L has height at most 2. It follows from (2.3) that $|y_n - y_m| \geq C$. Since all elements of B are bounded in between $\pm 1/(\beta - 1)$, B must be a finite set and hence S_1 must also be finite. □

It follows from Propositions 2.1 and 2.2 that if ρ^{-1} is a P.V. number, then the matrix T is of the form

$$T = \begin{bmatrix} T_1 & \mathbf{0} \\ Q & T_2 \end{bmatrix} \tag{2.4}$$

where T_1 corresponds to the states S_1 and is a finite sub-Markov matrix. The matrix T_1 is the one we need to calculate the spectrum $\tau(q)$.

We will make a further identification for the states in S_1. For $\boldsymbol{s} = (s_1, \dots, s_q) \in S_1$, we let σ (depending on $\boldsymbol{s}$) be a particular permutation on $\{1, 2, \dots, q\}$ such that the state $\boldsymbol{s}_\sigma = (s_{\sigma(1)}, \dots, s_{\sigma(q)}) \in S_1$ satisfies $s_{\sigma(1)} \geq s_{\sigma(2)} \geq \cdots \geq s_{\sigma(q)}$. Let $S_1^\sigma = \{\boldsymbol{s}_\sigma : \boldsymbol{s} \in S_1\}$ and define

$$\pi : S_1 \to S_1^\sigma \text{ by } \pi(\boldsymbol{s}) = \boldsymbol{s}_\sigma \quad \text{and} \quad T_1^\sigma : S_1^\sigma \to S_1^\sigma \text{ by } T_1^\sigma = \pi \circ T_1.$$

Extend π and T_1^σ linearly to $\langle S_1 \rangle$ and $\langle S_1^\sigma \rangle$ respectively. (Note that we have used T_1 both as a matrix and an operator. This slight abuse of notation will also apply to other matrices and operators throughout the paper.)

It is easy to see that the entry $(\boldsymbol{s}, \boldsymbol{t}) \in S_1^\sigma \times S_1^\sigma$ of the matrix T_1^σ is given by

$$\frac{1}{2^q} \#\{\boldsymbol{s}^\epsilon \in S_1 : (\boldsymbol{s}^\epsilon)_\sigma = \boldsymbol{t}, \ \boldsymbol{\epsilon} = (\epsilon_1, \dots, \epsilon_q), \ \epsilon_i = 0, \text{ or } 1\},$$

where $\boldsymbol{s}^\epsilon$ is defined as in (2.2) and we use the notation $\#E$ to denote the cardinality of a set E.

Proposition 2.3. *Let $1/2 < \rho < 1$ such that ρ^{-1} is a P.V. number. Then the maximal eigenvalues of T_1 and T_1^σ are equal.*

Proof. It follows directly from (2.2) that $T_1^\sigma \circ \pi = \pi \circ T_1$ on $\langle S_1 \rangle$. Let λ and λ^σ be the maximal eigenvalues of T_1 and T_1^σ respectively. Suppose $\boldsymbol{s}$ is a nonnegative λ-eigenvector of T_1. Then $\boldsymbol{s} = \sum_i' c_i \cdot \boldsymbol{s}_i$, where $c_i \geq 0$ and not all c_i are zero. This implies that $\pi(\boldsymbol{s}) \neq 0$. Since

$$T_1^\sigma\big(\pi(\boldsymbol{s})\big) = \pi\big(T_1(\boldsymbol{s})\big) = \pi(\lambda \cdot \boldsymbol{s}) = \lambda \cdot \pi(\boldsymbol{s}),$$

we conclude that λ is an eigenvalue of T_1^σ, and $\lambda \leq \lambda^\sigma$.

On the other hand taking the adjoint of the identity $T_1^\sigma \circ \pi = \pi \circ T_1$, we have $\pi^* \circ (T_1^\sigma)^* = T_1^* \circ \pi^*$. The eigenvalues of $T_1{}^*$ and $(T_1^\sigma)^*$ are unchanged, and the same argument as above implies that $\lambda^\sigma \leq \lambda$. (We can also consider the left eigenvector instead of using the adjoints.) □

3. The basic theorem.

Let μ be the ICBM as defined in (1.1). For each $\boldsymbol{s} \in S$, $\alpha \geq 0$ and $h > 0$, we define

$$\Phi_{\boldsymbol{s}}^{(\alpha)}(h) = \frac{1}{h^{1+\alpha}} \int_{-\infty}^{\infty} \mu\big(B_h(t + s_1)\big) \cdots \mu\big(B_h(t + s_q)\big)\, dt. \tag{3.1}$$

It is straight forward to verify the following proposition, which justifies the identifications made in the previous section.

Proposition 3.1. *Let $\Phi_{\boldsymbol{s}}^{(\alpha)}(h)$ be defined as in (3.1). Then the following hold:*

(a)*If $\boldsymbol{s}'$ is another representation of $\boldsymbol{s}$, i.e., $\boldsymbol{s}' = \boldsymbol{s} + c(1,\dots,1)$ for some $c \in \mathbb{R}$, then $\Phi_{\boldsymbol{s}'}^{(\alpha)}(h) = \Phi_{\boldsymbol{s}}^{(\alpha)}(h)$.*

(b)*Let $\boldsymbol{s}_\sigma$ be the decreasing rearrangement of the coordinates of $\boldsymbol{s}$ as defined in the previous section. Then $\Phi_{\boldsymbol{s}_\sigma}^{(\alpha)}(h) = \Phi_{\boldsymbol{s}}^{(\alpha)}(h)$.*

Proposition 3.2. *$\Phi_{\boldsymbol{s}}^{(\alpha)}(h) \neq 0$ for all $h > 0$ if and only if $\boldsymbol{s} \in S_1$.*

Proof. The proposition is a simple consequence of the following observation: $\boldsymbol{s} \in S_1$ if and only if the line $\boldsymbol{t} + \boldsymbol{s}$ with $\boldsymbol{t} = (t,\dots,t), -\infty < t < \infty$ has nonvoid intersection with $[0,1]^q$. Using the fact that $\operatorname{supp}(\mu) = [0,1]$, it is easy to show that this is equivalent to

$$\int_{-\infty}^{\infty} \mu\big(B_h(t+s_1)\big)\cdots\mu\big(B_h(t+s_q)\big)dt \neq 0 \qquad \text{for all } h > 0,$$

i.e., $\Phi_{\boldsymbol{s}}^{(\alpha)}(h) \neq 0$ for all $h > 0$. □

For $\boldsymbol{s} \in \langle S\rangle$ with $\boldsymbol{s} = \sum' c_i \cdot \boldsymbol{s}_i$, $\boldsymbol{s}_i \in S$, we define

$$\Phi_{\boldsymbol{s}}^{(\alpha)}(h) = \sum c_i \Phi_{\boldsymbol{s}_i}^{(\alpha)}(h).$$

Hence $\Phi_{T\boldsymbol{s}}^{(\alpha)}(h) = 2^{-q}\sum_{\boldsymbol{\epsilon}} \Phi_{\boldsymbol{s}^{\boldsymbol{\epsilon}}}^{(\alpha)}(h)$. The Markov matrix T has the following important invariance property.

Proposition 3.3. *Let $\boldsymbol{s} \in \langle S\rangle$. Then for any $\alpha \geq 0$ and any $h > 0$,*

$$\Phi_{\boldsymbol{s}}^{(\alpha)}(h) = \frac{1}{\rho^\alpha}\Phi_{T\boldsymbol{s}}^{(\alpha)}(\frac{h}{\rho}). \tag{3.2}$$

Proof. By linearity, it suffices to show that this holds for all $\boldsymbol{s} \in S$. Using the self-similar identity (1.1) followed by a change of variables, we have

$$\begin{aligned}\Phi_{\boldsymbol{s}}^{(\alpha)}(h) &= \frac{1}{2^q h^{1+\alpha}}\int_{-\infty}^{\infty}\prod_{i=1}^{q}\Big(\mu\Big(B_{\frac{h}{\rho}}(\frac{t}{\rho}+\frac{s_i}{\rho})\Big)+\mu\Big(B_{\frac{h}{\rho}}(\frac{t}{\rho}+\frac{s_i}{\rho}-\frac{1-\rho}{\rho})\Big)\Big)\,dt\\ &= \frac{1}{2^q h^{1+\alpha}}\sum_{\boldsymbol{\epsilon}}\int_{-\infty}^{\infty}\prod_{i=1}^{q}\mu\Big(B_{\frac{h}{\rho}}(\frac{t}{\rho}+\frac{s_i}{\rho}-\epsilon_i\frac{1-\rho}{\rho})\Big)\,dt\end{aligned}$$

$$= \frac{1}{2^q \rho^\alpha (\frac{h}{\rho})^{1+\alpha}} \sum_{\epsilon} \int_{-\infty}^{\infty} \prod_{i=1}^{q} \mu\Big(B_{\frac{h}{\rho}}(t + \frac{s_i}{\rho} + \epsilon_i \frac{1-\rho}{\rho})\Big)\, dt$$
$$= \frac{1}{\rho^\alpha} \Phi_{T\boldsymbol{s}}^{(\alpha)}(\frac{h}{\rho}),$$

where the summation $\sum_\epsilon$ is over all $\epsilon = (\epsilon_1, \dots, \epsilon_q)$, $\epsilon_i = 0$ or 1. □

Proposition 3.4. *Let μ be the self-similar measure defined by* (1.1) *with ρ^{-1} equal to a P.V. number, let T_1 be defined as in* (2.4) *and let λ be its maximal eigenvalue. Then $\rho^{q-1} < \lambda < 1$.*

Proof. In view of (2.4) and the fact that $\langle S_1 \rangle$ contains no invariant subspaces of T, it is easy to show that the maximal eigenvalue of T_1 is strictly less than 1 (see e.g. [21]).

To prove the lower bound estimate for λ, we first claim that if α is such that $\rho^\alpha = \lambda$, then for any $\eta < \alpha$,

$$\overline{\lim_{h\to 0+}} \frac{1}{h^{1+\eta}} \int_{-\infty}^{\infty} \mu(B_h(t))^q \, dt = 0. \tag{3.3}$$

For this we first consider the case T_1 is irreducible. Let $\boldsymbol{s} = \sum_i' c_i \cdot \boldsymbol{s}_i$ be a positive eigenvector associated with λ. Then Propositions 3.2 and 3.3 imply that for $h > 0$ sufficiently small,

$$\Phi_{\boldsymbol{s}}^{(\eta)}(h) = \frac{1}{\rho^\eta} \Phi_{T_1 \boldsymbol{s}}^{(\eta)}(\frac{h}{\rho}) = \frac{\lambda}{\rho^\eta} \Phi_{\boldsymbol{s}}^{(\eta)}(\frac{h}{\rho}).$$

Inductively, for all $m \in \mathbb{N}$ and for all $h > 0$ sufficiently small,

$$\Phi_{\boldsymbol{s}}^{(\eta)}(\rho^m h) = (\frac{\lambda}{\rho^\eta})^m \Phi_{\boldsymbol{s}}^{(\eta)}(h). \tag{3.4}$$

Since $\lambda/\rho^\eta < 1$ by assumption, we have $\lim_{h\to 0+} \Phi_{\boldsymbol{s}}^{(\eta)}(h) = 0$. The irreducibility of T_1 implies that each c_i is positive (see e.g. [21]). Hence, for $\boldsymbol{s}_0 = (0, \dots, 0)$, $\lim_{h\to 0+} \Phi_{\boldsymbol{s}_0}^{(\eta)}(h) = 0$, which proves (3.3). In the case T_1 is reducible, by re-arranging the basis elements, we can assume that

$$T_1 = \begin{bmatrix} E_\ell & 0 & 0 & \dots & 0 \\ \times & E_{\ell-1} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \times & & & & 0 \\ \times & \times & \dots & \times & E_1 \end{bmatrix},$$

where E_i, $1 \leq i \leq \ell$ are irreducible. An inductive argument will yield the same conclusion [13, Lemma 4.4]. This proves the claim.

Now suppose $\lambda < \rho^{q-1}$. Then $\lambda = \rho^\alpha$ for some $\alpha > q-1$. By taking $\eta = q-1$ in the above claim, we have

$$\varlimsup_{h\to 0+} \frac{1}{h^q} \int_{-\infty}^{\infty} \mu(B_h(t))^q \, dt = 0, \tag{3.5}$$

which implies that

$$\sup_{h>0} \frac{1}{h^q} \int_{-\infty}^{\infty} \mu(B_h(t))^q \, dt < \infty.$$

By using the same argument as in Corollary 4.5 in [13], we have

$$\varlimsup_{h\to 0+} \frac{1}{h^q} \int_{-\infty}^{\infty} \mu(B_h(t))^q \, dt > 0,$$

contradicting (3.5). Hence $\lambda \geq \rho^{q-1}$.

It remains to show that $\lambda \neq \rho^{q-1}$. Let $q(x)$ be the characteristic polynomial of T_1. Suppose, on the contrary, $\lambda = \rho^{q-1}$. Let $\beta = \rho^{-1}$ and let $p(x) = \sum_{k=0}^n a_k x^k$ be the minimal polynomial of β^{q-1}, which is also a P.V. number (see e.g. [20, p.4, Theorem A]). Let β_1 be a conjugate of β^{q-1}. Then both ρ^{q-1} and β_1^{-1} are roots of the polynomial $\tilde{p}(x) = \sum_{k=0}^n a_{n-k} x^k$. This implies that β_1^{-1} is also a root of $q(x)$. But $|\beta_1^{-1}| > 1 > \rho^{q-1}$, contradicting the maximality of λ. □

Theorem 3.5. *Let $q \geq 2$ be a positive integer, let $1/2 < \rho < 1$ such that $\beta = \rho^{-1}$ is a P.V. number and let μ be the self-similar measure defined by* (1.1). *Then* $\tau(q) = \ln\lambda/\ln\rho$ *where λ is the the maximal eigenvalue of T_1.*

Proof. Let $\alpha = \ln\lambda/\ln\rho$. For $\boldsymbol{s} \in \langle S_1 \rangle$ a λ-eigenvector of T_1, a similar derivation as that for (3.4) yields

$$\Phi_{\boldsymbol{s}}^{(\alpha)}(\rho^m h) = \Phi_{\boldsymbol{s}}^{(\alpha)}(h), \quad \text{for } m \in \mathbb{N} \text{ and for all } h > 0 \text{ sufficiently small},$$

i.e., $\Phi_{\boldsymbol{s}}^{(\alpha)}(h)$ is multiplicatively periodic on h. Observe also that $\Phi_{\boldsymbol{s}}^{(\alpha)}(h)$ is strictly positive because $\boldsymbol{s} = \sum_i' c_i \cdot \boldsymbol{s}_i$, where $c_i \geq 0$ and $\boldsymbol{s}_i \in S_1$ (Proposition 3.2). Hence there exists $\boldsymbol{s}_i$ such that $\varlimsup_{h\to 0+} \Phi_{\boldsymbol{s}_i}^{(\alpha)}(h) > 0$. By using Hölder's inequality we have

$$\varlimsup_{h\to 0+} \frac{1}{h^{1+\alpha}} \int_{-\infty}^{\infty} \mu(B_h(t))^q \, dt > 0.$$

It follows that $\tau(q) \le \alpha = \ln\lambda/\ln\rho$. On the other hand, for any $\eta < \alpha$, the claim in the proof of Proposition 3.4 implies that

$$\overline{\lim_{h\to 0^+}} \frac{1}{h^{1+\eta}} \int_{-\infty}^{\infty} \mu(B_h(t))^q \, dt = 0.$$

Hence $\tau(q) \ge \alpha$ and the proof is complete. □

4. Simplification of the matrix T_1.

To actually compute the maximal eigenvalue of T_1, it is desirable to replace T_1 by a matrix of smaller order but has the same maximal eigenvalue. First, by making use of Proposition 2.3, we can reduce T_1 to T_1^σ. Recall that T_1^σ is defined on the span of the set of states S_1^σ, which consists only of those $\boldsymbol{s} \in S_1$ with $s_1 \ge s_2 \ge \cdots \ge s_q$.

In this section we will identify each state $\boldsymbol{s} = (s_1, \dots, s_q) \in S_1^\sigma$ uniquely with the point $(s_1 - s_q, \dots, s_{q-1} - s_q) \in \mathbb{R}^{q-1}$. Geometrically this corresponds to identifying the "line" $\boldsymbol{s} \in \mathbb{R}^q$ with its "point" of intersection with the hyperplane $\mathbb{R}^{q-1}$. We then describe an algorithm to generate the set W_1^σ of all such points and construct the matrix A_1 induced by T_1^σ and such identification. A_1 and T_1^σ have the same maximal eigenvalue (Proposition 4.1). In Proposition 4.2 we will further simplify the matrix A_1.

Let τ be the projection of $\mathbb{R}^q$ onto $\mathbb{R}^{q-1}$ defined by $\tau(\boldsymbol{s}) = (s_1 - s_q, \dots, s_{q-1} - s_q)$ and define $W_1^\sigma = \tau(S_1^\sigma)$. Let A_1 be the matrix that is defined on the states W_1^σ and is induced by T_1^σ and τ, i.e., A_1 is defined by the identity

$$A_1 \circ \tau = \tau \circ T_1^\sigma \quad \text{on} \quad S_1^\sigma. \tag{4.1}$$

Proposition 4.1. *Assume the same hypotheses of Theorem* 3.5. *Then*

$$W_1^\sigma = \Big\{ \boldsymbol{t} \in \mathbb{R}^{q-1} : t_i = (\beta - 1) \sum_{k=0}^{n} \beta^{n-k} (\epsilon_k^{(i)} - \epsilon_k^{(q)}),\ \epsilon_k^{(j)} = 0 \textit{ or } 1 \textit{ for } 1 \le j \le q$$
$$\textit{and } 0 \le k \le n,\ 1 \ge t_1 \ge \cdots \ge t_{q-1} \ge 0,\ \textit{and}\ \ n \in \mathbb{N} \Big\}.$$

Moreover, A_1 and T_1^σ have the same maximal eigenvalue.

Proof. The first part is a direct consequence of the explicit form of the states in S_1^σ and the definition of τ. The second part follows by using the identity in (4.1) and the same argument as that in the proof of Proposition 2.3. □

Proposition 4.1 provides us with a convenient algebraic criterion to determine whether a state in $\tau(S^\sigma)$ belongs to W_1^σ. Summarizing the previous arguments, we have the following

Algorithm to construct A_1:

(I) Starting from $\mathbf{0} \in \mathbb{R}^q$, suppose we have constructed $\boldsymbol{t} \in W_1^\sigma$ in the $(n-1)$-th step. Let $\boldsymbol{s} = \rho^{-1}((\boldsymbol{t}, 0) + (1-\rho)\boldsymbol{\epsilon})$, $\epsilon_i = 0$ or 1, $1 \le i \le q$. Rearrange $\boldsymbol{s}$ to $\boldsymbol{s}_\sigma$ so that $s_{\sigma(1)} \ge s_{\sigma(2)} \ge \cdots \ge s_{\sigma(q)}$ and let

$$\boldsymbol{t}' = \left(s_{\sigma(1)} - s_{\sigma(q)}, \ldots, s_{\sigma(q-1)} - s_{\sigma(q)}\right).$$

Keep those $\boldsymbol{t}'$ in W_1^σ that are distinct from those previously chosen. (The process terminates when no more new members are generated.)

(II) For the column of the matrix A_1 corresponding to $\boldsymbol{t}$, the entry corresponding to $\boldsymbol{t}' \in W_1^\sigma$ is given by

$$\frac{1}{2^q}(\text{number of appearances of the } \boldsymbol{s} \text{ that gives } \boldsymbol{t}').$$

We can further reduce the set of states in W_1^σ by discarding those states of the form $(1, \ldots, 1, t_{i+1}, \ldots, t_{q-1})$. These states correspond to those lines $\boldsymbol{s} \in S_1^\sigma$ that intersect only the boundary of the unit cube $[0,1]^q$.

Proposition 4.2. *Assume the same hypotheses of Theorem* 3.5. *Let*

$$W_0^\sigma = \{(t_1, t_2, \ldots, t_{q-1}) \in W_1^\sigma : \ t_1 < 1\},$$

and let A_0 be the restriction of A_1 on W_0^σ. Then A_0 and A_1 have the same maximal eigenvalue.

Proof. We can decompose $W_1^\sigma \setminus W_0^\sigma$ into the following disjoint sets

$$U_i = \left\{(1, \ldots, 1, t_{i+1}, \ldots, t_{q-1}) \in W_1^\sigma : \ t_{i+1} < 1\right\}, \quad 1 \le i \le q-2.$$

Let $A_{1,i}$ be the matrix obtained by restricting A_1 on U_i. It is easy to check that $A_1(U_i) \subseteq \langle U_i \bigcup \cdots \bigcup U_{q-2}\rangle$. Hence we can represent the matrix A_1 on W_1^σ as

$$A_1 = \begin{bmatrix} A_0 & 0 & 0 & \cdots & 0 \\ \times & A_{1,1} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ & & & & 0 \\ \times & \times & \cdots & \times & A_{1,q-2} \end{bmatrix}.$$

To compare the maximal eigenvalues of A_0 and $A_{1,1}$, we observe that the action of the iteration $\boldsymbol{s} = \rho^{-1}((\boldsymbol{t}, 0) + (1-\rho)\boldsymbol{\epsilon})$ (as in part (I) of the above algorithm) on

$$\left\{\boldsymbol{t} = (t_1, t_2, \ldots, t_{q-2}, 0) : \ \boldsymbol{t} \in W_0^\sigma\right\} \subseteq W_0^\sigma$$

is the same as that on $U_1 = \{\boldsymbol{t} = (1, t_2, \ldots, t_{q-1}) : \ t_2 < 1\}$. This implies that $A_{1,1}$ is a principal submatrix of A_0, so the maximal eigenvalue λ_0 of A_0 is larger. Inductively we can use the same method to compare A_i and A_{i+1} and conclude that λ_0 is the maximal eigenvalue of the matrix A_1. □

By using this proposition we can modify the above algorithm by replacing W_1^σ and A_1 with W_0^σ and A_0 respectively. We end this section by giving the following commutative diagram which shows the relationships among the different vector spaces and the associated operators. (Here $\mathrm{Id}|_E$ denotes the restriction of the identity map to the space E.)

$$\begin{array}{ccccccccc}
\langle S\rangle & \xrightarrow{\mathrm{Id}|_{\langle S_1\rangle}} & \langle S_1\rangle & \xrightarrow{\pi} & \langle S_1^\sigma\rangle & \xrightarrow{\tau} & \langle W_1^\sigma\rangle & \xrightarrow{\mathrm{Id}|_{\langle W_0^\sigma\rangle}} & \langle W_0^\sigma\rangle \\
T\big\downarrow & & \big\downarrow T_1 & & \big\downarrow T_1^\sigma & & \big\downarrow A_1 & & \big\downarrow A_0 \\
\langle S\rangle & \xrightarrow{\mathrm{Id}|_{\langle S_1\rangle}} & \langle S_1\rangle & \xrightarrow{\pi} & \langle S_1^\sigma\rangle & \xrightarrow{\tau} & \langle W_1^\sigma\rangle & \xrightarrow{\mathrm{Id}|_{\langle W_0^\sigma\rangle}} & \langle W_0^\sigma\rangle
\end{array}$$

5. A special family of P.V. numbers.

In this section we will consider the measure associated with the special family of P.V. numbers $\rho^{-1} = \beta_n$ defined by the algebraic equations

$$x^n - x^{n-1} - \cdots - x - 1 = 0\,, \quad n \geq 2. \tag{5.1}$$

This family includes the golden ratio $\beta_2 = (\sqrt{5}+1)/2$. For this class of numbers, we will derive a formula for the matrix A_0 for each integer $q \geq 2$. For completeness we include a proof of the following known result:

Proposition 5.1. *The $\beta_n > 1$ satisfying* (5.1) *is a P.V. number.*

Proof. We can write (5.1) as $x^n = (x^n - 1)/(x-1)$ and get

$$x^{n+1} - 2x^n + 1 = 0. \tag{5.2}$$

Except for the extra root $x = 1$, equation (5.2) has exactly the same set of roots as (5.1). Let $f(z) = z^{n+1} - 2z^n + 1$ and $g(z) = -2z^n + 1$. For $\epsilon > 0$ let C_ϵ denote the circle $\{z : \ |z| = 1+\epsilon\}$. Then for $\epsilon > 0$ sufficiently small we have

$$|f(z) - g(z)| = |z^{n+1}| < |-2z^n + 1| = |g(z)|, \quad z \in C_\epsilon.$$

Hence by Rouché's theorem $f(z)$ and $g(z)$ have the same number of zeros inside C_ϵ. Clearly $g(z)$ has n zeros inside C_ϵ, and therefore so does $f(z)$. By letting $\epsilon \to 0$, it follows that $f(z)$ must have n zeros in $\{z : |z| \leq 1\}$. It is easy to see that 1 is the only zero of $f(z)$ on the unit circle and we conclude that (5.1) has $n-1$ roots of modulus less than one. Lastly by writing (5.1) as $x = 1 + x^{-1} + \cdots + x^{-(n-1)}$, it is obvious that it has a root $\beta_n > 1$. This completes the proof. □

Proposition 5.2. *Let $n \geq 2$ and let β_n be the P.V. number defined by* (5.1). *Then (a) $1 < \beta_n < 2$; (b) $\{\beta_n\}_{n=2}^\infty$ is an increasing sequence and $\lim_{n\to\infty} \beta_n = 2$.*

Proof. Write $\beta = \beta_n$. (5.1) implies that

$$\beta = 1 + \beta^{-1} + \cdots + \beta^{-(n-1)} = \frac{1-\beta^{-n}}{1-\beta^{-1}}. \tag{5.3}$$

That $\beta > 1$ is obvious. Multiplying both sides by $1 - \beta^{-1}$, we get $\beta = 2 - \beta^{-n} < 2$ and (a) follows.

To prove (b) we let $g_n(x) = 1 + x^{-1} + \cdots + x^{-(n-1)}$. Then β_n is a solution of $x = g_n(x)$. Observe that since $g_n(x) < g_{n+1}(x)$ for all $x > 0$, we have

$$0 = g_n(\beta_n) - \beta_n < g_{n+1}(\beta_n) - \beta_n.$$

Moreover, $g_{n+1}(x) - x$ is a strictly decreasing function of x on $[0, \infty)$ with a unique zero at $x = \beta_{n+1}$. It follows that $\beta_n < \beta_{n+1}$. That $\lim_{n\to\infty} \beta_n = 2$ now follows from (5.3). □

For fixed $\beta = \beta_n$ with $n \geq 2$, we let $v_0 = 0$, $v_1 = 1$, and

$$v_m = \beta^{m-1} - \beta^{m-2} - \cdots - \beta - 1 \quad \text{for} \quad 2 \leq m \leq n.$$

Lemma 5.3. *The finite sequence $\{v_m\}_{m=1}^n$ is strictly decreasing and*

$$\frac{1}{\beta-1} - 1 \leq v_m \leq 1, \quad \textit{for} \quad 1 \leq m \leq n. \tag{5.4}$$

Proof. Since $v_{m+1} - v_m = \beta^m - 2\beta^{m-1} = \beta^{m-1}(\beta - 2) < 0$, $\{v_m\}_{m=1}^n$ is strictly decreasing and the upper bound in (5.4) follows. For the lower bound, we notice that by (5.1), $v_n = 1/\beta$. Hence for $1 \leq m \leq n$,

$$v_m - (\frac{1}{\beta-1} - 1) \geq v_n - (\frac{1}{\beta-1} - 1) = \frac{\beta(\beta-1)-1}{\beta(\beta-1)} \geq 0.$$

(The last inequality is because $\beta(\beta-1)-1 \geq \beta_2(\beta_2-1)-1=0$.) This completes the proof. □

Let $q \geq 2$ be a positive integer and $n \geq 2$ be the degree of the polynomial in (5.1). For $0 \leq k \leq q-1$ and $0 \leq m \leq n$, let

$$\boldsymbol{v}_{m,k} = (\beta-1)(\underbrace{v_m, \ldots, v_m}_{k}, 0, \ldots, 0) \in \mathbb{R}^{q-1}.$$

Note that for $m=0$ or $k=0$, all the $\boldsymbol{v}_{m,k}$ equal $(0,\ldots,0)$; we will simply denote them by $\boldsymbol{v}_0$.

Theorem 5.4. *Fix some $\beta = \beta_n$, $n \geq 2$, and a positive integer $q \geq 2$. Let W_0^σ be the set of states as defined in Proposition 4.2. Then $W_0^\sigma = \{\boldsymbol{v}_{m,k} : 0 \leq k \leq q-1$ and $0 \leq m \leq n\}$, which has $n(q-1)+1$ elements.*

Proof. We will make use of the modified iteration algorithm described at the end of last section:

$$\boldsymbol{s} = \rho^{-1}\big((\boldsymbol{t},0)+(1-\rho)\boldsymbol{\epsilon}\big) = \beta(\boldsymbol{t},0)+(\beta-1)\boldsymbol{\epsilon}, \quad \boldsymbol{t} \in W_0^\sigma,\ \epsilon_i = 0 \text{ or } 1.$$

We divide our proof into the following cases:

(i) $\boldsymbol{t} = \boldsymbol{v}_0 = (0,\ldots,0)$. Then $\boldsymbol{s} = (\beta-1)\boldsymbol{\epsilon} \in \mathbb{R}^q$. Denoting by $\boldsymbol{t}'$ the projection of the decreasing rearrangement of $\boldsymbol{s}$ onto $\mathbb{R}^{q-1}$ by τ (i.e., $\boldsymbol{t}' = \tau(\boldsymbol{s}_\sigma)$), we see that $\boldsymbol{t}'$ is of the form $\boldsymbol{v}_{1,k}$, $0 \leq k \leq q-1$.

(ii) $\boldsymbol{t} = \boldsymbol{v}_{m,k}$, where $1 \leq m \leq n-1$ and $1 \leq k \leq q-1$. Then

$$\boldsymbol{s} = (\beta-1)(\beta v_m+\epsilon_1, \ldots, \beta v_m+\epsilon_k, \epsilon_{k+1}, \ldots, \epsilon_q). \tag{5.5}$$

By Lemma 5.3,

$$\beta v_m = v_{m+1}+1 \geq \frac{1}{\beta-1},$$

and therefore $(\beta-1)\beta v_m \geq 1$. Note that for $\tau(\boldsymbol{s}_\sigma)$ to belong to W_0^σ, the condition $|s_i - s_j| < 1$ must be satisfied. This forces $\epsilon_1 = \cdots = \epsilon_k = 0$ and $\epsilon_{k+1} = \cdots = \epsilon_q = 1$. Hence $\boldsymbol{s} = (\beta-1)(\underbrace{\beta v_m, \ldots, \beta v_m}_{k}, 1, \ldots, 1)$, and by projecting it to $\mathbb{R}^{q-1}$ by τ, we have $\boldsymbol{t}' = \boldsymbol{v}_{m+1,k} \in W_0^\sigma$.

(iii) $\boldsymbol{t} = \boldsymbol{v}_{n,k}$, where $1 \leq k \leq q-1$. Then by (5.1), $\beta v_n = 1$ and the analogue of expression (5.5) is

$$\boldsymbol{s} = (\beta-1)(1+\epsilon_1, \ldots, 1+\epsilon_k, \epsilon_{k+1}, \cdots, \epsilon_q), \quad 1 \leq k \leq q-1.$$

Consider the following two subcases.

Case 1. $\epsilon_i = 1$ for some $1 \le i \le k$. Then for $k+1 \le j \le q$, the condition $|s_i - s_j| < 1$ becomes $(\beta-1)(2-\epsilon_j) < 1$. This implies that all $\epsilon_j = 1$ for $k+1 \le j \le q$ and $\boldsymbol{s} = (\beta-1)(1+\epsilon_1, \dots, 1+\epsilon_k, 1, \dots, 1)$. If ℓ $(1 \le \ell \le k)$ of the ϵ_i are equal to 1, then the corresponding $\boldsymbol{t}' \in W_0^\sigma$ is of the form

$$\boldsymbol{t}' = (\beta-1)(\underbrace{1,\dots,1}_{\ell},0,\dots,0) = \boldsymbol{v}_{1,\ell}.$$

Case 2. $\epsilon_i = 0$ for all $1 \le i \le k$. Then

$$\boldsymbol{s} = (\beta-1)(1,\dots,1,\epsilon_{k+1},\dots,\epsilon_q)$$

where $\epsilon_j = 0$ or 1 for $k+1 \le j \le q$. If ℓ $(0 \le \ell \le q-k-1)$ of the ϵ_j are equal to 1, then $\boldsymbol{t}' = \boldsymbol{v}_{1,k+\ell}$. If $\epsilon_j = 1$ for all $k+1 \le j \le q$, then $\boldsymbol{t}' = \boldsymbol{v}_0$.

The above enumerates all the possible iterations, and hence W_0^σ is as described. It is direct to see that there are $n(q-1)+1$ distinct $\boldsymbol{v}_{m,k}$. □

We now describe the construction of the matrix A_0 based on the proof of Theorem 5.4. For an integer $q \ge 2$, we define

$$M_q^{(0)} = \begin{bmatrix} \binom{q}{1} & \binom{q-1}{1} & \cdots & \binom{2}{1} & \binom{1}{1} \\ \binom{q}{2} & \binom{q-1}{2} & \cdots & \binom{2}{2} & 0 \\ \vdots & & \vdots & & \vdots \\ \binom{q}{q-1} & \binom{q-1}{q-1} & 0 & \cdots & 0 \\ \binom{q}{q} & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad M_q^{(1)} = \begin{bmatrix} 0 & 0 & 0 & \cdots & \binom{q-1}{0} \\ 0 & 0 & \cdots & \binom{q-2}{0} & \binom{q-1}{1} \\ \vdots & & \vdots & & \vdots \\ 0 & \binom{1}{0} & \cdots & \binom{q-2}{q-3} & \binom{q-1}{q-2} \\ 1 & \binom{1}{1} & \cdots & \binom{q-2}{q-2} & \binom{q-1}{q-1} \end{bmatrix},$$

and let $M_q = M_q^{(0)} + M_q^{(1)}$. Also, we let I_m be the $m \times m$ identity matrix and let D_m be the $m \times m$ matrix of the form

$$\begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots \\ 1 & \cdots & 0 & 0 \end{bmatrix}.$$

(i.e., the ij-entry of D_m is 1 if $i+j = m+1$ and 0 otherwise.)

Theorem 5.5. *Suppose we arrange the above $\boldsymbol{v}_{m,k}$ in the order*

$$\{\boldsymbol{v}_{1,1},\dots,\boldsymbol{v}_{1,q-1},\boldsymbol{v}_0,\boldsymbol{v}_{n,q-1},\dots,\boldsymbol{v}_{n,1},\dots,\boldsymbol{v}_{2,q-1},\dots,\boldsymbol{v}_{2,1}\}.$$

Then the matrix A_0 *in Proposition 4.2 is given by*

$$A_0 = \frac{1}{2^q}\begin{bmatrix} \mathbf{0} & M_q & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{(n-2)(q-1)} \\ D_{q-1} & \mathbf{0} & \mathbf{0} \end{bmatrix}. \tag{5.6}$$

Proof. The submatrices D_{q-1} and $I_{(n-2)(q-1)}$ correspond to case (ii) in the proof of Theorem 5.4. To see the construction of M_q we re-examine the proofs in cases (i) and (iii). Fix a $\boldsymbol{v}_{n,k}$, $1 \le k \le q-1$. In the first subcase of (iii), for each $1 \le \ell \le k$, there are $\binom{k}{l}$ of the ϵ_i $(1 \le i \le k)$ equal to 1, and hence $\binom{k}{l}$ of the $\boldsymbol{s}$ can be rearranged and projected by τ to $\boldsymbol{t}' = \boldsymbol{v}_{1,\ell}$. This gives rise to the corresponding columns of $M_q^{(0)}$. Similarly the second subcase of (iii) determines the corresponding columns of $M_q^{(1)}$. Lastly, it is easy to see that the first columns of $M_q^{(0)}$ and $M_q^{(1)}$ are determined by case (i). This completes the proof. □

For the golden ratio β_2 and for $q = 2, 3$, A_0 equals respectively

$$\frac{1}{4}\begin{bmatrix} 0 & 2 & 2 \\ 0 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \qquad \frac{1}{8}\begin{bmatrix} 0 & 0 & 3 & 2 & 2 \\ 0 & 0 & 3 & 2 & 2 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We remark that the algorithm to compute $\tau(q)$, $q \ge 2$ an integer by using the matrix in Theorem 5.5 is much faster than the one used in Theorem 4.1 of [15], which requires computing the inverse of a $q \times q$ matrix with each of its entries containing the unknown defining $\tau(q)$. A complete description of $\tau(q)$ for $0 < q < \infty$ is given in [15].

As a simple application of Theorem 5.5 we have

Corollary 5.6. *Let* μ_{ρ_n} *be the ICBM corresponding to* $\beta_n = \rho_n^{-1}$ *as defined in* (1.1) *and let* $\tau_n(q)$ *be its* L^q*-spectrum. Then*

$$\lim_{n\to\infty} \tau_n(2) = \lim_{n\to\infty} \underline{\dim}_2(\mu_{\rho_n}) = 1.$$

Proof. For $q = 2$ the matrix A_n in Theorem 5.5 is

$$A_n = \frac{1}{4}\begin{bmatrix} 0 & 2 & 2 & 0 & \cdots & 0 \\ 0 & 2 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} := \frac{1}{4}B_n.$$

A direct calculation shows that det $(\lambda I - B_n) = \lambda^n(\lambda - 2) - 2(\lambda - 1)$. If we let λ_n denote the maximal eigenvalue of B_n, then

$$\lambda_n^n(\lambda_n - 2) - 2(\lambda_n - 1) = 0. \tag{5.7}$$

Observe that $\lambda_n > 1$ because the column sums of the irreducible matrix B_n are at least 1 and not all equal (see e.g. [21]). Moreover, (5.7) forces $\underline{\lim}_{n\to\infty}\lambda_n \geq c > 1$ for some constant c. By rewriting (5.7) as $(\lambda_n - 2) - 2(\lambda_n - 1)/\lambda_n^n = 0$, we conclude that $\lim_{n\to\infty} 2(\lambda_n - 1)/\lambda_n^n = 0$ and hence $\lim_{n\to\infty} \lambda_n = 2$. Consequently, using Theorem 3.5, we have

$$\lim_{n\to\infty} \tau_n(2) = \lim_{n\to\infty} \frac{\ln(\lambda_n/4)}{\ln \rho_n} = \frac{\ln(1/2)}{\ln(1/2)} = 1.$$

□

The following is a list of $\underline{\dim}_2(\mu_{\rho_n})$, rounded off to the 10th decimal place. Note that the smallest value of $\underline{\dim}_2(\mu_{\rho_n})$ occurs at $n = 3$.

n	β_n	$\underline{\dim}_2(\mu_{\rho_n})$
2	1.6180338997	0.9923994336
3	1.8392867552	0.9642200274
4	1.9275619755	0.9733294764
5	1.9659482366	0.9835653645
6	1.9835828434	0.9906789642
7	1.9919641966	0.9949638696
8	1.9960311797	0.9973606068
9	1.9980294703	0.9986428460
10	1.9990186327	0.9993102630
	⋮	⋮
15	1.9999694754	0.9999780091
	⋮	⋮
20	1.9999990463	0.9999993121

6. A remark

All the results in the previous sections can be generalized to allow arbitrary probability weights on the contractive similitudes ψ_1 and ψ_2. More precisely, for $1/2 < \rho < 1$, $0 < a < 1$ we can consider the self-similar measure μ_a defined by

$$\mu_a = a\mu_a \circ \psi_1^{-1} + (1 - a)\mu_a \circ \psi_2^{-1}. \tag{6.1}$$

For $\epsilon = (\epsilon_1, \ldots, \epsilon_q)$, $\epsilon_i = 0$ or 1, we let $|\epsilon| := \sum_{i=1}^q \epsilon_i$. We modify the Markov matrix T in (2.2) by

$$T(s) := \sum_{\epsilon}{}' a^{|\epsilon|}(1-a)^{q-|\epsilon|} \cdot s^{\epsilon},$$

and define T_1 and T_1^{σ} exactly the same way as in Section 2. Then the theory goes through without change.

We conclude this section with the following proposition:

Proposition 6.1. *Let μ_a be the self-similar measure defined as in* (6.1) *and let $D(a) = \underline{\dim}_2(\mu_a)$. Then $D(a)$ attains its maximum at $a = 1/2$.*

Proof. We will use the random variable setup of the measure μ_a, i.e., μ_a is the distribution measure of the random variable $X = \sum_{k=0}^{\infty} \rho^k X_k$ where X_k takes values 1 and -1 with probabilities a and $1-a$. Let $\mu_{a,k}$ be the distribution measure of $\rho^k X_k$. A direct calculation shows that its Fourier transform is

$$|\hat{\mu}_{a,k}(\xi)|^2 = (2a-1)^2 + 4a(1-a)\cos^2(\rho^k \xi),$$

which is minimum when $a = 1/2$ (as a function of a). It follows that for each ξ, $|\hat{\mu}_a(\xi)|^2$ is minimum when $a = 1/2$. It is known that

$$\overline{\lim_{h\to 0^+}} \frac{1}{h^{d+\alpha}} \int_{\mathbb{R}^d} \mu(B_h(x))^2\, dx \approx \overline{\lim_{M\to\infty}} \frac{1}{M^{d-\alpha}} \int_{|\xi|<M} |\hat{\mu}(\xi)|^2 d\xi$$

for any bounded Borel measure μ on $\mathbb{R}^d$ (see [16]). (Here $\approx$ means each quantity dominates the other by a positive constant.) Using this and the definition of $\tau(2)$ given in (1.2), we conclude that $D(a)$ attains its maximum at $a = 1/2$. In fact, the above proof shows that $D(a)$ is symmetric about $a = 1/2$ and is increasing from $a = 0$ to $a = 1/2$. □

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References

[1] J.C. Alexander and J.A. Yorke: *Fat baker's transformations*, Ergodic Theory Dynamical Systems **4** (1984), 1–23.

[2] J.C. Alexander and D. Zagier: *The entropy of a certain infinitely convolved Bernoulli measure*, J. London Math. Soc. **44** (1991), 121–134.

[3] R. Cawley and R.D. Mauldin: *Multifractal decompositions of Moran fractals*, Adv. Math. **92** (1992), 196–236.

[4] G.A. Edgar and R.D. Mauldin: *Multifractal decompositions of digraph recursive fractals*, Proc. London Math. Soc. **65** (1992), 604–628.

[5] P. Erdös: *On a family of symmetric Bernoulli convolutions*, Amer. J. Math. **61** (1939), 974–976.

[6] U. Frisch and G. Parisi: *On the singularity structure of fully developed turbulence*, in Proc. Int. Sch. Phys., "Enrico Fermi" Course LXXXVIII, North Holland, Amsterdam, 1985, 84–88.

[7] A.M. Garsia: *Arithmetic properties of Bernoulli convolutions*, Trans. Amer. Math. Soc. **102** (1962), 409–432.

[8] J.S. Geronimo and D.P. Hardin: *An exact formula for the measure dimensions associated with a class of piecewise linear maps*, Constr. Approx. **5** (1989), 89–98.

[9] T.C. Halsey, M.H. Jensen, L.P. Kadanoff, I. Procaccia and B.I. Shraiman: *Fractal measures and their singularities: The characterization of strange sets*, Phys. Rev. A **33** (1986), 1141–1151.

[10] H. Hentschel and I. Procaccia: *The infinite number of generalized dimensions of fractals and strange attractors*, Phys. D. **8** (1983), 435–444.

[11] S.P. Lalley: *Random series in powers of algebraic integers: Hausdorff dimension of the limit distribution*, J. London Math. Soc. **57** (1998), 629–654.

[12] K.-S. Lau: *Fractal measures and mean p-variations*, J. Funct. Anal. **108** (1992), 427–457.

[13] K.-S. Lau: *Dimension of a family of singular Bernoulli convolutions*, J. Funct. Anal. **116** (1993), 335–358.

[14] K.-S. Lau and S.-M. Ngai: *Multifractal measures and a weak separation condition*, Adv. Math. **141**, (1999), 45–96.

[15] K.-S. Lau and S.-M. Ngai: *L^q-spectrum of the Bernoulli convolution associated with the golden ratio*, Studia Math. **131** (1998), 225–251.

[16] K.-S. Lau and J. Wang: *Mean quadratic variations and Fourier asymptotics of self-similar measures*, Monatsh. Math. **115** (1993), 99–132.

[17] F. Ledrappier and A. Porzio: *A dimension formula for Bernoulli convolutions*, J. Statist. Phys. **76** (1994), 1307–1327.

[18] L. Olsen: *A multifractal formalism*, Adv. Math. **116** (1995), 82–196.

[19] R. Riedi: *An improved multifractal formalism and self-similar measures*, J. Math. Anal. Appl. **189** (1995), 462–490.

[20] R. Salem: Algebraic Numbers and Fourier Analysis, Heath Math. Monographs, Boston, 1963.

[21] E. Seneta: Non-negative Matrices; an Introduction to Theory and Applications, John Wiley, New York, 1973.

[22] R.S. Strichartz: *Self-similar measures and their Fourier transforms III*, Indiana Univ. Math. J. **42** (1993), 367–411.

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