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AFFINE FUNCTIONS ON ALEXANDROV SURFACES

YUKIHIRO MASHIKO

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0. Introduction

An *Alexandrov surface* X is by definition a 2-Hausdorff dimensional, connected, locally compact and complete length space of curvature bounded from below in the sense of Alexandrov which has no boundary points. For a point $x \in X$, Σ'_x is the set of all directions of geodesics emanating from x equipped with the angular metric $\angle$. Let Σ_x be the metric completion of Σ'_x. We call it the *space of directions at* x. This corresponds to the unit tangent sphere in Riemannian geometry. The space of directions Σ_x for each $x \in X$ is either a circle of circumference $\leq 2\pi$ or a segment of length $\leq \pi$ (see [1],[9]). Since, by definition, X has no boundary, we mean that Σ_x is a circle for every $x \in X$. A point $x \in X$ is called *singular* iff the circumference of Σ_x is less than 2π, and we denote by $Sing(X)$ the set of all singular points of X. It is a well known fact in Alexandrov geometry that the pointed Gromov-Hausdorff limit $\lim_{r\to 0}(1/rX, x)$ of the $1/r$-rescaling of the metric around x is the flat cone $(C(\Sigma_x), o^*)$ over Σ_x with vertex o^* for every $x \in X$. We call $C(\Sigma_x)$ the *tangent cone* at x, which corresponds to the tangent space in Riemannian geometry.

A real valued function $\psi : X \longrightarrow R$ on X is called *convex* iff the following inequality holds for an arbitrary geodesic $\gamma : [a, b] \longrightarrow X$ and arbitrary $\lambda \in [0, 1]$:

$$(*) \qquad \psi \circ \gamma((1-\lambda)a + \lambda b) \leq (1-\lambda) \cdot \psi \circ \gamma(a) + \lambda \cdot \psi \circ \gamma(b).$$

A convex function on X is not in general continuous, because X admits the singular set $Sing(X)$. Nevertheless, we can introduce the notion of the a-level set of ψ for each $a \in (\inf \psi, \infty)$ (see §0 of [6]). Every convex function on a complete Riemannian manifold M is always locally Lipschitz. Moreover, M is automatically noncompact if such a convex function is nonconstant. However, an Alexandrov surface X which admits a locally nonconstant convex function is not always noncompact (see Theorem A of [6]). The following results have been established by the author: Let $\psi : X \longrightarrow R$ be a convex function satisfying the condition

$$\mathrm{int}\Big(\bigcap_{a > \inf \psi} \overline{\{x \in X | \psi(x) \leq a\}} \Big) = \emptyset.$$

Then we conclude the following:

i. $\sup \psi = \infty$.

ii. Each component of the a-level set for each $a \in (\inf \psi, \infty)$ is either a simple closed curve or a line.

iii. For each $a \in (\inf \psi, \infty)$, the a-level set has at most two components. Moreover, if the a-level set for some $a \in (\inf \psi, \infty)$ has two components, then the same holds for all the b-level set with $b \in (\inf \psi, \infty)$, and in each case, the two components are both simple closed curves or both lines.

iv. X is homeomorphic to one of three spaces, R^2, $S^1 \times R$ or $(S^1 \times R)/Z_2$.

The purpose of the present paper is to determine the metric structure of X admitting a non-trivial affine function. Here, a function $\varphi : X \longrightarrow R$ is by definition *affine* iff the equality in $(*)$ always holds for arbitrary unit speed geodesic $\gamma : [a, b] \longrightarrow X$ and arbitrary $\lambda \in [0, 1]$. Letting $X_a^a := \{x \in X | \varphi(x) = a\}$ define the a-level set of φ for convenience, we specialize the above result to the case of affine functions as follows:

Theorem 1. *If an Alexandrov surface X admits a non-trivial affine function $\varphi : X \longrightarrow R$, then for every $a \in (-\infty, \infty)$ there is an isometric map*

$$I : X_a^a \times (-\infty, \infty) \longrightarrow X$$

such that $I(y, t) \in X_t^t$ for every $(y, t) \in X_a^a \times (-\infty, \infty)$. Moreover, X is isometric to either flat R^2 or flat $S^1 \times R$.

Note that every level set of an affine function on X is totally convex, and hence such a set is either a simple closed geodesic or a straight line. In particular $Sing(X) = \emptyset$, and hence $C(\Sigma_x)$ is isometric to R^2 for all $x \in X$. Since $-\varphi$ is also affine, we conclude from (i) that the range of φ is $(-\infty, \infty)$.

The fundamental notion used here is the *directional derivative* $d\varphi(v)$ of an affine function $\varphi : X \longrightarrow R$ for $v \in \Sigma_x'$. Set

$$d\varphi(v) := (\varphi \circ \gamma_v)'_+(0), \ v \in \Sigma_x', \ x \in X,$$

where $\gamma_v : [0, l(v)] \longrightarrow X$ is a geodesic such that $\gamma_v(0) = x$ and $\dot\gamma_v(0) = v$, and $(\cdot)'_+$ is the right-hand derivative. Note that we do not take the limit in the above definition since φ is affine. We will show in Lemma 1.1 that $d\varphi : \Sigma_x' \longrightarrow R$ can be extended continuously to an affine function $d\varphi : C(\Sigma_x) \longrightarrow R$ on the whole tangent cone $C(\Sigma_x)$. Note that we use the same expression $d\varphi(v)$ for $v \in C(\Sigma_x)$. It follows from the compactness of Σ_x that the function of the directional derivative $d\varphi|_{\Sigma_x} : \Sigma_x \longrightarrow R$ attains its maximum at some (unique) direction $v_{\varphi,x} \in \Sigma_x$ (see Lemma 1.2). This allows us to introduce the *generalized gradient* $\nabla\varphi_x$ of φ at $x \in X$, that is, $\nabla\varphi_x/|\nabla\varphi_x| = v_{\varphi,x}$ realizes the maximum of $d\varphi|_{\Sigma_x}$. Here we mean by $|\cdot|$ the R^2-norm under identifing $C(\Sigma_x)$ with R^2. The following lemma on the generalized gradient plays a crucial role in our investigation:

Lemma 2. *The following statements are true:*

(1)*We have for every $x \in X$ and for every $v \in C(\Sigma_x)$*

$$d\varphi(v) = |\nabla\varphi_x||v| \cos \angle(\nabla\varphi_x, v).$$

(2)*Let a and b be arbitrary fixed numbers with $a < b$. Then for every $x \in X_a^a$ and a minimal geodesic $\sigma_x : [0, l(x)] \longrightarrow X$ from x to X_b^b, we have*

$$\dot\sigma_x(0) = v_{\varphi,x} = \nabla\varphi_x/|\nabla\varphi_x|.$$

Hence there is a unique minimal geodesic from x to X_b^b for every $x \in X_a^a$.

(3)$|\nabla\varphi_x|$ *is constant for all $x \in X$.*

To show Theorem 1, the flatness of every geodesic triangle in X is required. Therefore we prove the similarities of geodesic triangles as follows. Let a and b be as in Lemma 2(2), and let $\gamma : [0, l] \longrightarrow X$ be a geodesic from a point on X_a^a to a point on X_b^b. For every $s \in (0, l]$, let $\sigma_s : [0, l(s)] \longrightarrow X$ be the (unique) minimal geodesic from $\gamma(s)$ to X_a^a. Then it follows from Lemma 2 (1) and (3) that the angle between σ_s and γ is constant for all $s \in (0, l]$. This is true for the angle between σ_s and X_a^a. Let $\triangle(t)$ for $t \in (0, l]$ be a geodesic triangle spanned by geodesics $\{\sigma_s | 0 < s \leq t\}$. Using this and the first variation formula, we conclude the following:

Proposition 3. *With the above notation, $\triangle(t_1)$ and $\triangle(t_2)$ for all $t_1, t_2 \in (0, l]$ are similar triangles, i.e., all ratio of the lengths of corresponding edges are same.*

In §1 we prove assertions (1)-(3) of Lemma 2, and in §2 we construct the isometric map I indicated in Theorem 1.

1. Proof of Lemma 2

From this point let X be an Alexandrov surface admitting an affine function $\varphi : X \longrightarrow R$. We denote by $|x, y|$ the distance between x and y for $x, y \in X$. We use the following fact through this paper:

FACT 1.0. The pointed Gromov-Hausdorff limit $\lim_{t\to 0}(1/tX, x)$ of the $(1/t)$ -rescaling of the metric around x is the flat cone $(C(\Sigma_x), o^*)$ over Σ_x with vertex o^* for every $x \in X$.

Since X admits the affine function φ, Σ_x is the circle of length 2π for all $x \in X$. Thus $C(\Sigma_x)$ is identified with R^2, and Σ_x is identified with the unit circle centered at origin of R^2. Hence we can denote an arbitrary element of $C(\Sigma_x)$ by λu for some $\lambda \in [0, \infty)$ and some $u \in \Sigma_x$.

We first discuss the directions in Σ'_x for arbitrary fixed $x \in X$. Let u, v be fixed directions in Σ'_x with $0 < \angle(u, v) < \pi$. Then we choose the direction $w_\lambda \in \Sigma'_x$ for some $\lambda \in (0, 1)$ such that (by identifying $C(\Sigma_x)$ with R^2)

$$w_\lambda = \frac{(1-\lambda)u + \lambda v}{|(1-\lambda)u + \lambda v|},$$

where $|\cdot|$ denotes the standard norm in R^2. Using this notation, the following holds:

Lemma 1.1. *We have*

$$d\varphi(w_\lambda) = [(1-\lambda)\cdot d\varphi(u) + \lambda \cdot d\varphi(v)] \cdot \frac{\sin \angle(u, w_\lambda)}{\lambda \sin \angle(u, v)}.$$

Moreover, $d\varphi : \Sigma'_x \longrightarrow R$ *has the continuous extension* $d\varphi : \Sigma_x \longrightarrow R$*, and* $d\varphi : C(\Sigma_x) \longrightarrow R$ *becomes an affine function again.*

Proof. Since the directional derivatives are defined locally, we discuss only in (sufficiently small) disk neighborhood U_x of x. The bracket part in the above equation follows from the definition of affine functions, and the other part follows from Euclidean geometry on $C(\Sigma_x)$, the sine formula and from Fact 1.0.

With the equation established, the second assertion easily follows. The third assertion follows from the property that $d\varphi(\lambda v) = \lambda d\varphi(v)$ for all $\lambda \in (0, \infty)$ and $v \in \Sigma_x$. □

For every $x \in X$, we denote by O_x the directions in Σ_x tangent to $X^{\varphi(x)}_{\varphi(x)}$. Clearly, O_x consists of exactly two elements, $O_{1,x}$ and $O_{2,x}$ such that $\angle(O_{1,x}, O_{2,x}) = \pi$ and $d\varphi(O_{1,x}) = d\varphi(O_{2,x}) = 0$. Put

$$M^x_\varphi := \{v \in \Sigma_x | d\varphi(v) = \max_{w \in \Sigma_x} d\varphi(w)\} \text{ and } m^x_\varphi := \{v \in \Sigma_x | d\varphi(v) = \min_{w \in \Sigma_x} d\varphi(w)\}.$$

Then the configuration of O_x, M^x_φ and m^x_φ is determined as follows.

Lemma 1.2. *For every* $v \in M^x_\varphi$ *and* $u \in m^x_\varphi$*, we have*

$$\angle(O_x, v) = \angle(O_x, u) = \frac{\pi}{2}.$$

Hence each of the sets M_φ^x and m_φ^x consists of only one element.

Proof. Suppose that $\angle(O_x, v) \neq \pi/2$ for some $v \in M_\varphi^x$. Since $\angle(O_{1,x}, O_{2,x}) = \pi$, we can choose a direction $w \in \Sigma_x$ such that $d\varphi(w) > 0$ and $\angle(O_x, w) = \pi/2$. Here we assume that $\angle(O_{1,x}, v) > \pi/2$. Then, under identifying $C(\Sigma_x)$ with R^2, we have $w = [(1-\lambda)O_{1,x} + \lambda v]/|(1-\lambda)O_{1,x} + \lambda v|$ for some $\lambda \in (0,1)$. Therefore, from the equation in Lemma 1.1, we have

$$d\varphi(w) = [(1-\lambda)d\varphi(O_{1,x}) + \lambda \cdot d\varphi(v)] \cdot \frac{\sin \angle(O_{1,x}, w)}{\lambda \sin \angle(O_{1,x}, v)}$$

$$= \frac{1}{\sin \angle(O_{1,x}, v)} \cdot d\varphi(v) > d\varphi(v).$$

This contradicts the choice of $v \in M_\varphi^x$.

Since $-\varphi$ is also affine, $\angle(O_x, u) = \pi/2$ follows for every $u \in m_\varphi^x$. □

Proof of Lemma 2 (1). We see from Lemma 1.1 that $d\varphi$ is an affine function on $C(\Sigma_x)$ isometric to R^2. We can easily see a fact that every affine function on R^2 satisfies the equation in Lemma 2 (1). □

Proof of Lemma 2 (2). Suppose that $\dot{\sigma}_x(0) \neq \nabla\varphi_x$ for some minimal geodesic $\sigma_x : [0, l(x)] \longrightarrow X$ from x to X_b^b. Then we construct a broken geodesic segment

$$\xi = \bigcup_i \gamma_i : [0, l(\xi)] \longrightarrow X$$

such that $(\varphi \circ \xi)'_+(s) > d\varphi(\dot{\sigma}_x(0))$ for every $s \in [0, l(\xi))$ and $\xi(0) = x$, $\xi(l(\xi)) \in X_b^b$. The construction of ξ is achieved by inductive steps as follows. First of all, we note that $d\varphi(\dot{\gamma}(s))$ is constant in s on each geodesic $\gamma : [0, l(\gamma)] \longrightarrow X$. By the continuity of $d\varphi : \Sigma_x \longrightarrow R$, we can find a direction $v_1 \in \Sigma'_x$ such that $d\varphi(v_1) > d\varphi(\dot{\sigma}_x(0))$. Let $\gamma_1 : [0, l_1] \longrightarrow X$ be a maximal geodesic tangent to v_1. If $\gamma_i(l_i)$ does not reach X_b^b for the i-th maximal geodesic $\gamma_i : [0, l_i] \longrightarrow X$ tangent to $v_i \in \Sigma_{\gamma_i(0)}$, then using the continuity of $d\varphi : \Sigma_x \longrightarrow R$ and Lemma 2(1), we can find a direction $v_{i+1} \in \Sigma_{\gamma_i(l_i)}$ such that $d\varphi(v_{i+1}) > d\varphi(\dot{\sigma}_x(0))$, and we denote the maximal geodesic tangent to v_{i+1} by $\gamma_{i+1} : [0, l_{i+1}] \longrightarrow X$. Then, put a broken geodesic segment $\xi := \bigcup_i \gamma_i : [0, \sum_i l_i] \longrightarrow X$, and $x_1 := \xi(\sum_i l_i)$, $l(\xi) := \sum_i l_i$.

It may happen that the endpoint x_1 of ξ does not reach to X_b^b. We then join x to x_1 by a minimal geodesic $\alpha : [0, |x, x_1|] \longrightarrow X$. By the minimizing property of α, we see that $d\varphi(\dot{\alpha}(|x, x_1|)) \geq (\varphi \circ \xi)'_+(s)$ for all $s \in [0, l(\xi)]$. Since $d\varphi(\dot{\alpha}(|x, x_1|)) > d\varphi(\dot{\sigma}_x(0))$, using the continuity of $d\varphi : \Sigma_x \longrightarrow R$, we can find a direction $w_1 \in \Sigma'_x$ with $d\varphi(w_1) > d\varphi(\dot{\sigma}_x(0))$, and hence we proceed with inductive steps to construct ξ.

From the above reason, we may assume that $x_1 \in X_b^b$. Clearly, we have

$$\int_0^{l(\xi)} (\varphi \circ \xi)'_+(s)ds > \int_0^{l(\xi)} d\varphi(\dot{\sigma}_x(0))ds.$$

Moreover, we conclude that $l(x) > l(\xi)$ since $\varphi \circ \xi$ is almost everywhere differentiable. This contradicts the minimizing property of σ_x. $\square$

Proof. Proof of Lemma 2 (3) We prove that $|\nabla\varphi_{x_1}| = |\nabla\varphi_{x_2}|$ for every $x_1, x_2 \in X$. The first step of the proof is to show that $|\nabla\varphi_x|$ is constant for all $x \in X_a^a$ and for arbitrary fixed $a \in (-\infty, \infty)$. Choose $x_1, x_2 \in X_a^a$ and let $\tau : [0, |x_1, x_2|] \longrightarrow X$ be a minimal geodesic from x_1 to x_2. Necessarily, $\tau \subset X_a^a$. Set $\sigma_s : [0, l(s)] \longrightarrow X$ for the minimal geodesic from $\tau(s)$ to X_b^b. Then it follows from (1) and (2) of Lemma 2 and the first variation formula that the function $g = g(s) := l(s)$ is differentiable in $s \in (0, |x_1, x_2|)$, and $\frac{dg}{ds} = 0$ for all $s \in (0, |x_1, x_2|)$. This therefore implies that $|\nabla\varphi_{x_1}| = (b-a)/l(0) = (b-a)/l(|x_1, x_2|) = |\nabla\varphi_{x_2}|$.

The second step of the proof is to show that $|\nabla\varphi_{x_1}| = |\nabla\varphi_{x_2}|$ when $x_1 \in X_a^a$ and $x_2 \in X_b^b$ for distinct numbers $a, b \in (-\infty, \infty)$. Here we assume $a < b$. Set $\sigma_{x_1} : [0, l(x_1)] \longrightarrow X$ for the minimal geodesic from x_1 to X_b^b and $z := \sigma_{x_1}(l(x_1))$. Then it follows from (1) and (2) of Lemma 2 that $|\nabla\varphi_{x_1}| = |\nabla\varphi_z|$. From the first step of the proof, we see that $|\nabla\varphi_z| = |\nabla\varphi_{x_2}|$, and hence $|\nabla\varphi_{x_1}| = |\nabla\varphi_{x_2}|$. $\square$

2. Proof of Theorem 1

In this section, we construct a isometric map I in Theorem 1. Lemma 2 (2) guarantees that for an arbitrary fixed $a \in (-\infty, \infty)$ there exist the gradient flow $\phi_x : (-\infty, \infty) \longrightarrow X$ passing through $x \in X_a^a$ such that $\phi_x(t) \in X_t^t$ for every $t \in (-\infty, \infty)$. Then the required bijective map $I : X_a^a \times (-\infty, \infty) \longrightarrow X$ is obtained by $I(x, t) := \phi_x(t)$ for $(x, t) \in X_a^a \times (-\infty, \infty)$. We will verify that the map $I : X_a^a \times (-\infty, \infty) \longrightarrow X$ satisfies the following:

$$|I(x_1, t_1), I(x_2, t_2)|^2 = |x_1, x_2|^2 + |t_1 - t_2|^2$$

for every $(x_1, t_1), (x_2, t_2) \in X_a^a \times (-\infty, \infty)$.

It follows from Lemma 2 and the first variation formula that this flow ϕ_x satisfies the following:

(2.1) ϕ_x is perpendicular to X_t^t for every $t \in (-\infty, \infty)$.

(2.2) $|\phi_{x_1}(t), \phi_{x_2}(t)|$ is constant for all $t \in (-\infty, \infty)$.

We first normalize φ so that $|\nabla\varphi_x| = 1$ for all $x \in X$. From (2.2), we may assume without loss of generality that the geodesic $\gamma : [0,l] \longrightarrow X$ in Proposition 3 is a minimal geodesic from $I(x_2,t_2) \in X_b^b$ to $I(x_1,t_1) \in X_a^a$. Put $\theta := \angle(\gamma, X_b^b) \in [0,\pi/2]$. Then it suffices to prove the distance-preserving property of I in the case that $\theta \neq 0, \pi/2$. With the same notation as in Proposition 3, if we denote by $\overline{\triangle}$ the $1/t$-rescaling limit triangle of $\triangle(t)$ for $t \to 0+$, fixing the vertex $\gamma(0)$ of $\triangle(t)$, it follows from Proposition 3 that $\overline{\triangle}$ and $\triangle(l)$ are similar triangles. Moreover, it follows from Fact 1.0 that $\overline{\triangle}$ is a flat right triangle with an inner angle θ. Together with this and the similarity of $\overline{\triangle}$ and $\triangle(l)$, we observe that

$$|I(x_1,t_1), I(x_2,t_2)|^2 = \frac{1}{\cos^2\theta}|x_1,x_2|^2$$

$$= |x_1,x_2|^2 + \tan^2\theta|x_1,x_2|^2.$$

Using again the similarity of $\overline{\triangle}$ and $\triangle(l)$, we have $\tan\theta = |t_1 - t_2|/|x_1,x_2|$. Hence the proof is complete.

References

[1] Yu.D.Burago, M.Gromov and G.Perelman: *A. D. Alexandrov's spaces with curvatures bounded below* (english version), Russian Math. Surveys, **47**:2 (1992), 1-58

[2] J.Cheeger and D.Gromoll: *On the structure of complete manifolds of nonnegative curvature*, Ann. of Math. **96** (1972), 413-443

[3] J.Cheeger and D.Gromoll: *The splitting theorem for manifolds of nonnegative Ricci curvature*, J. Diff. Geom. **61** (1971), 119-128

[4] R.E.Greene and K.Shiohama: *Convex functions on complete noncompact manifolds: Topological structure*, Invent. Math. **63** (1981), 129-157

[5] R.E.Greene and K.Shiohama: *The isometry groups of manifolds admitting nonconstant convex functions*, J. Math. Soc. Japan, **39** (no.1) (1987), 1-16

[6] Y.Mashiko: *Convex functions on Alexandrov surfaces*, to appear in Trans. AMS

[7] Y.Otsu and T.Shioya: *The Riemannian structure of Alexandrov spaces*, J. Diff. Geom. **39** (1994), 629-658

[8] V.A.Sharafutdinov: *The Pogorelov-Klingenberg theorem for manifolds homeomorphic to R^n*, Transl. from Sib. Mat. Zhur. **18** (4) (1977), 649-657

[9] K.Shiohama: An Introduction to the Geometry of Alexandrov Spaces, Seoul National univ. Lecture notes series 8, 1992

Faculty of Science and Engineering
Saga University
1 Honjyoumachi.
Saga 840-8502,Japan
e-mail: mashiko@ms.saga-u.ac.jp