Yosimura, Z.
Osaka J. Math.
**36** (1999), 747 – 765

# Quasi $KO_*$-types of $CW$-spectra $X$ with $KU_*X \cong Free \oplus Z/2^m$

Dedicated to the memory of Professor Katsuo Kawakubo

ZEN-ICHI YOSIMURA

(Received February 13, 1998)

## 1. Introduction

Let $KO$, $KU$ and $KC$ denote the real, the complex and the self-conjugate $K$-spectrum, respectively. Given $CW$-spectra $X, Y$ we say that $X$ is quasi $KO_*$-equivalent to $Y$ if $KO \wedge X$ is isomorphic to $KO \wedge Y$ as a $KO$-module spectrum, in other words, if there exists a map $h : Y \to KO \wedge X$ inducing an isomorphism $h_* : KO_*Y \to KO_*X$. Note that if $X$ is quasi $KO_*$-equivalent to $Y$, then $KU_*X$ is isomorphic to $KU_*Y$ as a ($Z/2$-graded) abelian group with involution $\psi_C^{-1}$, in this case we say that $X$ has the same $\mathcal{C}$-type as $Y$. We are interested in the determination of the quasi $KO_*$-type of any $CW$-spectrum $X$ using the information of its $KU$-homology group $KU_*X \cong KU_0X \oplus KU_1X$ with the conjugation $\psi_C^{-1}$.

Let $\eta : \Sigma^1 \to \Sigma^0$ be the stable Hopf map of order 2 and $C(\eta^l)$ denote the cofiber of the map $\eta^l : \Sigma^l \to \Sigma^0$. The sphere spectrum $S = \Sigma^0$ and the cofibers $C(\eta^l)$ $(l = 1, 2)$ are typical examples of spectra $X$ with $KU_*X$ free. In [1, Theorem 3.2] Bousfield has completely determined the quasi $KO_*$-type of a $CW$-spectrum $X$ with $KU_*X$ free.

**Bousfield's Theorem .** *Let $X$ be a $CW$-spectrum such that $KU_*X \cong KU_0X \oplus KU_1X$ is free. Then it has the same quasi $KO_*$-type as a certain wedge sum of copies of $\Sigma^i (0 \le i \le 7), \Sigma^j C(\eta) (0 \le j \le 1)$ and $\Sigma^k C(\eta^2) (0 \le k \le 3)$.* (Cf. [6, Theorem 2.4]).

Let $SZ/2^m$ denote the Moore spectrum of type $Z/2^m$. In [4] and [5] we introduced some 3-cells spectra $X_m$ and $X'_m$ constructed as the cofibers of certain maps $f : \Sigma^i \to SZ/2^m$ and $f' : \Sigma^{i-1}SZ/2^m \to \Sigma^0$ and some 4-cells spectra $XY_m, X'Y'_m$ and $Y'X_m$ obtained as the cofibers of their mixed maps. In [5, Theorems 3.3, 4.2 and 4.4] by using these small spectra we have also determined the quasi $KO_*$-type of a $CW$-spectrum $X$ such that $KU_0X \cong F \oplus Z/2^m$ with $F$ free and $KU_1X = 0$. The purpose of this note is to determine completely the quasi $KO_*$-type of a $CW$-spectrum $X$ such that $KU_*X \cong F \oplus Z/2^m$ with $F$ free and finitely generated, without the restriction that $KU_1X = 0$.

Notice that the self-conjugate $K$-spectrum $KC$ may be regarded as the fiber of the map $1-\psi_C^{-1} : KU \to KU$. For any map $f : Y \to KU \wedge X$ with $(\psi_C^{-1} \wedge 1)f = f$ we can choose a map $g : Y \to KC \wedge X$ with $(\zeta \wedge 1)g = f$ in which $\zeta : KC \to KU$ is the complexification map. In §2 we show that under a certain assumption such a map $g$ is chosen to satisfy a nice property that $g_* : KC_iY \to KC_iX (i = 0, 2)$ are nearly the canonical inclusions if $f_* : KU_*Y \to KU_*X$ is the canonical inclusion in the category $\mathcal{C}$ of abelian groups with involution $\psi_C^{-1}$. In §3 we give the most refined direct sum decomposition of $KU_*X$ in the category $\mathcal{C}$ when $KU_*X$ is free (Proposition 3.2), and then prove Bousfield's Theorem (Theorem 3.3) along the line adopted in [4, 5]. Our new proof is very simple, and it is applicable to prove our main results (Theorems 5.1, 5.2 and 5.3). In order to distinguish $CW$-spectra $X$ such that $KU_*X \cong F \oplus Z/2^m$ with $F$ free and finitely generated we divide them into ten kinds of $\mathcal{C}$-types (Proposition 4.1). In §4 we give the most refined direct sum decomposition of $KU_*X$ in the category $\mathcal{C}$ when the $\mathcal{C}$-type of $X$ is known (Proposition 4.3), and in §5 we prove our main results (Theorems 5.1, 5.2 and 5.3) by applying our method developed in [4, 5].

## 2. $K$-spectra $KO, KU$ and $KC$

Let $KO$, $KU$ and $KC$ denote the real, the complex and the self-conjugate $K$-spectrum, respectively. As relations among these $K$-spectra we have the following cofiber sequence:

(2.1)

$$\begin{array}{ll}
\text{i)} & \Sigma^1 KO \xrightarrow{\eta\wedge 1} KO \xrightarrow{\epsilon_U} KU \xrightarrow{\epsilon_O\beta_U^{-1}} \Sigma^2 KO \\
\text{ii)} & \Sigma^2 KO \xrightarrow{\eta^2\wedge 1} KO \xrightarrow{\epsilon_C} KC \xrightarrow{\tau\beta_C^{-1}} \Sigma^3 KO \\
\text{iii)} & KC \xrightarrow{\zeta} KU \xrightarrow{\beta_U^{-1}(1-\psi_C^{-1})} \Sigma^2 KU \xrightarrow{\gamma\beta_U} \Sigma^1 KC \\
\text{iv)} & \Sigma^1 KC \xrightarrow{(-\tau,\tau\beta_C^{-1})} KO\vee\Sigma^4 KO \xrightarrow{\epsilon_U\vee\beta_U^2\epsilon_U} KU \xrightarrow{\epsilon_C\epsilon_O\beta_U^{-1}} \Sigma^2 KC \\
\text{v)} & \Sigma^2 KU \xrightarrow{(-\epsilon_O\beta_U,\epsilon_O\beta_U^{-1})} KO\vee\Sigma^4 KO \xrightarrow{\epsilon_C\vee\beta_C\epsilon_C} KC \xrightarrow{\epsilon_U\tau\beta_C^{-1}} \Sigma^3 KU
\end{array}$$

where $\beta_U : \Sigma^2 KU \to KU$ and $\beta_C : \Sigma^4 KC \to KC$ are the periodicity maps satisfying $\zeta\beta_C = \beta_U^2\zeta, \beta_C\gamma = \gamma\beta_U^2$ and $\psi_C^{-1}\beta_U = -\beta_U\psi_C^{-1}$. The maps involved in (2.1) satisfy the following equalities:

$$\begin{aligned}
\zeta\epsilon_C = \epsilon_U, \tau\gamma = \epsilon_O, \epsilon_O\epsilon_U = 2, \epsilon_U\epsilon_O = 1+\psi_C^{-1}, \\
\tau\epsilon_C = \eta\wedge 1 \text{ and } \gamma\beta_U\zeta = \eta\wedge 1.
\end{aligned} \tag{2.2}$$

For any $CW$-spectrum $Y$ its $K$-homology and $K$-cohomology groups are related

**Lemma 2.2.** *For any homomorphisms $a_i : H'_i \to H_i, d_i : T'_i \to T_i, b_i : H'_i \to T_i$ and $c_i : H_i \to T'_{i+1}(i = 0,1)$ there exists a map $f : Y \to KU \wedge X$ so that $f_* : KU_iY \to KU_iX$ and $Df_* : KU_iDX \to KU_iDY(i = 0,1)$ are represented by the matrices $\begin{pmatrix} a_i & 0 \\ b_i & d_i \end{pmatrix}$ and $\begin{pmatrix} a_i^* & 0 \\ c_i & d_{i+1}^* \end{pmatrix}$, respectively.*

Proof. Choose a map $f' : Y \to KU \wedge X$ such that $f'_* : KU_iY \to KU_iX(i = 0,1)$ is represented by the matrix $\begin{pmatrix} a_i & 0 \\ b_i & d_i \end{pmatrix}$. Then $Df'_* : KU_iDX \to KU_iDY(i = 0,1)$ is represented by a certain matrix $\begin{pmatrix} a_i^* & 0 \\ x_i & d_{i+1}^* \end{pmatrix}$. Use a geometric resolution of $Y$ given in (2.5). The difference $c_i - x_i : H_i \to T'_{i+1}(i = 0,1)$ has a coextension $y_i : H_i \to KU_{i+1}DV$ satisfying $D\delta_* y_i = c_i - x_i$. Choose a map $h : \Sigma^1 V \to KU \wedge X$ such that $Dh_* : KU_iDX \to KU_{i+1}DV(i = 0,1)$ coincides with $y_i$. Setting $f = f' + h\delta : Y \to KU \wedge X$ it satisfies the desired property. □

Let $\mathcal{C}$ be the category of abelian groups with involution $\psi_C^{-1}$, modelled on $KU$-homology groups $KU_*X$. Given $CW$-spectra $X, Y$ we say that they have the same $\mathcal{C}$-type if $KU_*X$ and $KU_*Y$ are isomorphic in the category $\mathcal{C}$.

**Proposition 2.3.** *Let $X$ and $Y$ be finite $CW$-spectra with $KU_1X$ and $KU_1Y$ free. If $X$ and $DX$ have the same $\mathcal{C}$-types as $Y$ and $DY$, respectively, then there exists a map $f : Y \to KU \wedge X$ with $(\psi_C^{-1} \wedge 1)f = f$ such that $f_* : KU_*Y \to KU_*X$ and $Df_* : KU_*DX \to KU_*DY$ are isomorphisms in the category $\mathcal{C}$.*

Proof. Identify $KU_*X$ and $KU_*DX$ with $KU_*Y$ and $KU_*DY$ in the category $\mathcal{C}$, respectively. By means of Lemma 2.2 we can choose a map $f : Y \to KU \wedge X$ such that $f_* : KU_*Y \to KU_*X$ and $Df_* : KU_*DX \to KU_*DY$ are both the identity. By virtue of Lemma 2.1 such a map $f$ satisfies the desired equality. □

For a $CW$-spectrum $X$ with $KU_*X$ free we have direct sum decompositions

$$KU_0X \cong A \oplus B \oplus C \oplus C, \quad KU_1X \cong D \oplus E \oplus F \oplus F \tag{2.7}$$

in the category $\mathcal{C}$, where $A, B, C, D, E$ and $F$ are free and $\psi_C^{-1} = 1$ on $A$ or $D$, $\psi_C^{-1} = -1$ on $B$ or $E$ and $\psi_C^{-1} = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}$ on $C \oplus C$ or $F \oplus F$. Using the cofiber sequence (2.1.iii) we can easily compute its $KC$-homology groups $KC_iX(i = 0,1,2,3)$ as follows:

$$\begin{aligned} &KC_0X \cong A \oplus C \oplus D \oplus E_2 \oplus F, \quad KC_1X \cong A_2 \oplus B \oplus C \oplus D \oplus F \\ &KC_2X \cong B \oplus C \oplus D_2 \oplus E \oplus F, \quad KC_3X \cong A \oplus B_2 \oplus C \oplus E \oplus F \end{aligned} \tag{2.8}$$

by the following universal coefficient sequences:

$$
(2.3)\quad \begin{array}{ll}
\text{i)} & 0 \to \operatorname{Ext}(KO_{3+i}Y,\ Z) \to KO^iY \to \operatorname{Hom}(KO_{4+i}Y,\ Z) \to 0 \\
\text{ii)} & 0 \to \operatorname{Ext}(KU_{5+i}Y,\ Z) \to KU^iY \to \operatorname{Hom}(KU_{6+i}Y,\ Z) \to 0 \\
\text{iii)} & 0 \to \operatorname{Ext}(KC_{6+i}Y,\ Z) \to KC^iY \to \operatorname{Hom}(KC_{7+i}Y,\ Z) \to 0.
\end{array}
$$

When $CW$-spectra $X$ and $Y$ are finite, we have a duality isomorphism

$$
(2.4)\qquad D : [Y,\ K \wedge X] \cong [DX,\ K \wedge DY]
$$

for $K = KU, KO$ or $KC$ where $DX$ and $DY$ denote the $S$-duals of $X$ and $Y$. Therefore $K^iY$ may be replaced by $K_{-i}DY$ whenever $Y$ is finite.

For any $CW$-spectrum $Y$ there exists a geometric resolution

$$
(2.5)\qquad V \xrightarrow{\psi} W \xrightarrow{\varphi} Y \xrightarrow{\delta} \Sigma^1 V
$$

so that $0 \to KU_*V \to KU_*W \to KU_*Y \to 0$ is a short exact sequence with $KU_*V$ and $KU_*W$ free. Using its geometric resolution we have the following universal coefficient sequence

$$
(2.6)\quad 0 \to \operatorname{Ext}(KU_{*-1}Y,\ KU_*X) \to [Y,\ KU \wedge X] \to \operatorname{Hom}(KU_*Y,\ KU_*X) \to 0
$$

for any $CW$-spectrum $X$.

**Lemma 2.1.** *Let $X$ and $Y$ be finite $CW$-spectra with $KU_1X$ and $KU_1Y$ free. Then a map $f : Y \to KU \wedge X$ is trivial if $f_* : KU_*Y \to KU_*X$ and $Df_* : KU_*DX \to KU_*DY$ are both trivial.*

Proof. Use a geometric resolution of $Y$ given in (2.5). Since $f_* : KU_*Y \to KU_*X$ is trivial, the composition map $f\varphi : W \to KU \wedge X$ is trivial. In other words, the composition map $(1 \wedge D\varphi)Df : DX \to KU \wedge DW$ is trivial. The $S$-dual map $D\varphi : DY \to DW$ induces a split monomorphism $D\varphi_* : KU_0DY \to KU_0DW$ under the assumption that $KU_1Y$ is free. Therefore $(D\varphi_*)^* : \operatorname{Ext}(KU_1DX,\ KU_0DY) \to \operatorname{Ext}(KU_1DX,\ KU_0DW)$ is a monomorphism. Hence the triviality of $Df_* : KU_*DX \to KU_*DY$ implies that the dual map $Df : DX \to KU \wedge DY$ is in fact trivial. $\square$

Given finite $CW$-spectra $X, Y$ we set $KU_iX \cong H_i \oplus T_i$ and $KU_iY \cong H_i' \oplus T_i'$ $(i = 0, 1)$ where $H_i, H_i'$ are free and $T_i, T_i'$ are torsion. When $H = H_i, H_i'$ and $T = T_i, T_i'$ are identified with $H^* \cong \operatorname{Hom}(H,\ Z)$ and $T^* \cong \operatorname{Ext}(T,\ Z)$, respectively, we have isomorphisms $KU_iDX \cong H_i \oplus T_{i+1}$ and $KU_iDY \cong H_i' \oplus T_{i+1}'$ $(i = 0, 1)$ where $T_2 = T_0$ and $T_2' = T_0'$.

where $G_2$ stands for the $Z/2$-module $G \otimes Z/2$.

Let $Y$ and $X$ be $CW$-spectra having direct sum decompositions $KU_0Y \cong A \oplus B \oplus C \oplus C \oplus M$ and $KU_1X \cong D \oplus E \oplus F \oplus F$ in the category $\mathcal{C}$ where $A, B, C, D, E$ and $F$ are free objects on which $\psi_C^{-1}$ behaves as stated in (2.7). Thus $KU_1X$ is assumed to be free. Note that $A \oplus C$ and $B \oplus C$ are direct summands of $KC_0Y$ and $KC_2Y$, respectively. For any map $f : Y \to KU \wedge X$ with $(\psi_C^{-1} \wedge 1)f = f$ we can choose homomorphisms

$$\alpha_0 : A \oplus C \to KC_0X, \quad \alpha_2 : B \oplus C \to KC_2X$$

such that $\zeta_*\alpha_0 = f_*\zeta_*|A \oplus C$ and $\zeta_*\alpha_2 = f_*\zeta_*|B \oplus C$.

**Lemma 2.4.** *Assume that $KU_1X$ is free. For any map $f : Y \to KU \wedge X$ with $(\psi_C^{-1} \wedge 1)f = f$ there exists a map $g : Y \to KC \wedge X$ with $(\zeta \wedge 1)g = f$ so that $g_* : KC_iY \to KC_iX (i = 0, 2)$ satisfy $g_*|A = \alpha_0|A, g_*|B = \alpha_2|B, (g_* - \alpha_0)(C) \subset E_2 \oplus F$ and $(g_* - \alpha_2)(C) \subset D_2 \oplus F$ .*

Proof. Choose a map $g' : Y \to KC \wedge X$ satisfying $(\zeta \wedge 1)g' = f$, and then set $\alpha_0' = \alpha_0 - g_*'|A \oplus C$ and $\alpha_2' = \alpha_2 - g_*'|B \oplus C$. The homomorphisms $\alpha_0' : A \oplus C \to KC_0X$ and $\alpha_2' : B \oplus C \to KC_2X$ are factorized through $D \oplus E_2 \oplus F \subset KC_0X$ and $D_2 \oplus E \oplus F \subset KC_2X$. Exchange them for the modified ones $\alpha_0''$ and $\alpha_2''$ with $\alpha_0''(C) \subset D \oplus F$ and $\alpha_2''(C) \subset E \oplus F$, respectively. Choose homomorphisms $\beta_i : KU_iY \to KU_{i+1}X (i = 0, 2)$ such that $\gamma_*\beta_0\zeta_*|A \oplus C = \alpha_0''$, $\gamma_*\beta_2\zeta_*|B \oplus C = \alpha_2''$ and $\beta_i|M = 0$. Then we get a map $h : Y \to \Sigma^{-1}KU \wedge X$ such that $h_* : KU_*Y \to KU_{*+1}X$ coincides with $\beta_0 + \beta_2 : KU_0Y \to KU_1X$. Setting $g = g' + (\gamma \wedge 1)h : Y \to KC \wedge X$, it satisfies the desired property. □

Assume that the short exact sequences

$$(2.9) \qquad 0 \to \gamma_*(KU_{i+1}X) \to KC_iX \to \zeta_*(KC_iX) \to 0 \quad (i = 0, 2)$$

are splittable, whose splitting homomorphisms are denoted by

$$\sigma_i : \zeta_*(KC_iX) \to KC_iX, \quad \rho_i : KC_iX \to \gamma_*(KU_{i+1}X).$$

Now we may take as $\alpha_0$ and $\alpha_2$ in Lemma 2.4 the restricted homomorphisms $\sigma_0 f_*\zeta_*|A \oplus C$ and $\sigma_2 f_*\zeta_*|B \oplus C$, respectively.

**Corollary 2.5.** *Assume that $KU_1X$ is free and the short exact sequences* (2.9) *are split. For any map $f : Y \to KU \wedge X$ with $(\psi_C^{-1} \wedge 1)f = f$ there exists a map $g : Y \to KC \wedge X$ with $(\zeta \wedge 1)g = f$ so that $g_* : KC_iY \to KC_iX (i = 0, 2)$ satisfy $\rho_0 g_*|A = 0, \rho_2 g_*|B = 0, \rho_0 g_*(C) \subset E_2 \oplus F$ and $\rho_2 g_*(C) \subset D_2 \oplus F$.*

Let $X$ be a finite $CW$-spectrum having a direct sum decomposition

$$KU_{-1}DX \cong D \oplus E \oplus F \oplus F \oplus N \tag{2.10}$$

in the category $\mathcal{C}$ where $D, E$ and $F$ are free objects on which $\psi_C^{-1}$ behaves as stated in (2.7). In this case we may assume that $\psi_C^{-1}$ behaves as $1 \oplus (-1)$ on the free part of $N$ itself. Note that $\gamma_* KU_{-1}DX \cong D \oplus E_2 \oplus F \oplus N_-$ and $\gamma_* KU_1 DX \cong D_2 \oplus E \oplus F \oplus N_+$ in which $N_\pm$ denotes the cokernel of $1 \pm \psi_C^{-1}$ on $N$. If $KU_1X$ is free, then it follows that

$$\operatorname{Tor} KC_0X \cong E_2 \oplus \operatorname{Tor} N_-, \quad \operatorname{Tor} KC_2X \cong D_2 \oplus \operatorname{Tor} N_+$$

because $\operatorname{Tor} KC_iX \cong \operatorname{Tor} KC_{6-i}DX$ by use of (2.3.iii) where $\operatorname{Tor} G$ stands for the torsion part of $G$.

Let $X$ and $Y$ be finite $CW$-spectra having direct sum decompositions

$$KU_{-1}DX \cong D \oplus E \oplus F \oplus F \oplus N \quad \text{and} \quad KU_{-1}DY \cong D' \oplus E' \oplus F' \oplus F' \oplus N'$$

in the category $\mathcal{C}$ as given in (2.10). When $KU_1X$ and $KU_1Y$ are free, the restricted homomorphisms $g_* : KC_iY \to KC_iX (i = 0, 2)$ to the torsion parts are given by $\tau_0(g) : E_2' \oplus \operatorname{Tor} N_-' \to E_2 \oplus \operatorname{Tor} N_-$ and $\tau_2(g) : D_2' \oplus \operatorname{Tor} N_+' \to D_2 \oplus \operatorname{Tor} N_+$ for any map $g : Y \to KC \wedge X$.

**Lemma 2.6.** *Let $f : Y \to KU \wedge X$ be a map with $(\psi_C^{-1} \wedge 1)f = f$ such that $Df_* : KU_{-1}DX \to KU_{-1}DY$ satisfies $Df_*(D \oplus E) \subset D' \oplus E'$ and $Df_*(N) \subset N'$. Assume that $KU_1X$ and $KU_1Y$ are free. For any map $g : Y \to KC \wedge X$ with $(\zeta \wedge 1)g = f$ the restricted homomorphisms $\tau_i(g)(i = 0, 2)$ are expressed as the direct sum $f_* \oplus (Df_*)^*$.*

Proof. The restricted homomorphisms $Dg_* : KC_{6-i}DX \to KC_{6-i}DY (i = 0, 2)$ to the torsion parts are induced by only $Df_*$. Hence our result is immediately shown by duality. □

Let $X$ and $Y$ be finite $CW$-spectra such that $KU_{-1}DX$ and $KU_{-1}DY$ are decomposed as previously and $KU_0Y$ is decomposed to a direct sum $A \oplus B \oplus M$ in the category $\mathcal{C}$ where $A$ and $B$ are free objects on which $\psi_C^{-1}$ behaves as stated in (2.7), and $\psi_C^{-1}$ behaves as $1 \oplus (-1)$ on the free part $H \cong H^+ \oplus H^-$ of $M$ itself. Assume that $KU_1X$ is free, and $D \oplus E_2 \oplus F \subset KC_0X$ and $D_2 \oplus E \oplus F \subset KC_2X$ are direct summands whose splitting epimorphisms are denoted by

$$\rho_0 : KC_0X \to D \oplus E_2 \oplus F \quad \text{and} \quad \rho_2 : KC_2X \to D_2 \oplus E \oplus F.$$

**Lemma 2.7.** *Let $f : Y \to KU \wedge X$ be a map with $(\psi_C^{-1} \wedge 1)f = f$ such that $Df_* : KU_{-1}DX \to KU_{-1}DY$ satisfies $Df_*(D \oplus E \oplus F \oplus F) \subset D' \oplus E' \oplus F' \oplus F'$. Assume that $KU_1X$ and $KU_1Y$ are free. Then there exists a map $g : Y \to KC \wedge X$ with $(\zeta \wedge 1)g = f$ such that $g_* : KC_iY \to KC_iX (i = 0, 2)$ satisfy $\rho_0 g_*|A = 0, \rho_2 g_*|B = 0, \rho_0 g_*(H^+) \subset E_2$ and $\rho_2 g_*(H^-) \subset D_2$.*

Proof. Take as $\alpha_0$ and $\alpha_2$ in Lemma 2.4 the restricted homomorphisms $\sigma_0' Df_* \zeta_* | D \oplus F$ and $\sigma_2' Df_* \zeta_* | E \oplus F$ , respectively, where $\sigma_0' : D' \oplus F' \to KC_{-1}DY$ and $\sigma_2' : E' \oplus F' \to KC_1DY$ are splitting monomorphisms. Then we can choose a map $Dg : DX \to KC \wedge DY$ with $(\zeta \wedge 1)Dg = Df$ such that $\rho_0' Dg_* | D \oplus F = 0$ and $\rho_2' Dg_* | E \oplus F = 0$ where $\rho_0' : KC_{-1}DY \to A \oplus H^+$ and $\rho_2' : KC_1DY \to B \oplus H^-$ are the canonical projections. Evidently $g_* : KC_iY \to KC_iX (i = 0, 2)$ satisfy $\rho_0 g_*(A \oplus H^+) \subset E_2$ and $\rho_2 g_*(B \oplus H^-) \subset D_2$ for the dual map $g$ of $Dg$. Such a map $g$ is chosen to satisfy $\rho_0 g_*|A = 0$ and $\rho_2 g_*|B = 0$ by means of Lemma 2.4. □

Let $h : V \to W$ be a map such that $h^* : [W, \Sigma^1 KU \wedge X] \to [V, \Sigma^1 KU \wedge X]$ is trivial, and $f : Y \to KU \wedge X$ be a map with $(\psi_C^{-1} \wedge 1)f = f$ where $Y$ denotes the cofiber of $h$. Assume that the composition map $(\epsilon_O \beta_U^{-1} \wedge 1) f i_Y : W \to \Sigma^2 KO \wedge X$ is trivial where $i_Y : W \to Y$ is the canonical inclusion. Then there exists a map $k : Y \to \Sigma^1 KU \wedge X$ such that $(\tau \beta_C^{-1} \wedge 1) g i_Y = (\epsilon_O \beta_U^{-1} \wedge 1) k i_Y$ for each map $g : Y \to KC \wedge X$ with $(\zeta \wedge 1)g = f$. Such a map $k$ is chosen to satisfy that the restricted homomorphism $k_* : KU_{*+1}Y \to KU_*X$ to $KU_*V$ is trivial if $KU_{*+1}Y \cong KU_{*+1}W \oplus KU_*V$. Replacing the map $g$ by $g + (\gamma \beta_U \wedge 1)k$ we can observe that

(2.11) the composition map $(\tau \beta_C^{-1} \wedge 1) g i_Y : W \to \Sigma^3 KO \wedge X$ is trivial (cf. [3, Lemma1.1]).

### 3. $CW$-spectra $X$ such that $KU_*X$ is free

In this section we deal with a $CW$-spectrum $X$ such that $KU_*X \cong KU_0X \oplus KU_1X$ is free. For such a $CW$-spectrum $X$ the $KU$-homology groups $KU_iX (i = 0, 1)$ have direct sum decompositions in the category $\mathcal{C}$ as given in (2.7) and the $KC$-homology groups $KC_iX (i = 0, 1, 2, 3)$ are computed as obtained in (2.8). Consider the induced homomorphisms

$$\varphi_i = (\epsilon_C \epsilon_O \beta_U^{-1})_* : KU_{i+2}X \to KC_iX \quad \text{and} \quad \varphi_i' = (\epsilon_U \tau \beta_C^{-1})_* : KC_iX \to KU_{i-3}X.$$

Using the equalities $\zeta_* \varphi_i = ((1 + \psi_C^{-1})\beta_U^{-1})_*, \varphi_i'(\gamma \beta_U)_* = ((1 + \psi_C^{-1})\beta_U^{-1})_*, \varphi_i' \varphi_i = 0, \varphi_i \varphi_{i+5}' = 0$ and $(\gamma \beta_U)_* \varphi_i' = \varphi_{i-2} \zeta_*$ we can easily verify that the induced homomorphisms $\varphi_0$ and $\varphi_2$ are represented by the following matrices:

(3.1)
$$\tilde{\Gamma}_0 = \begin{pmatrix} \Gamma_0 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix} : (A \oplus B) \oplus C \oplus C \to (A \oplus D \oplus E_2) \oplus C \oplus F$$
$$\tilde{\Gamma}_2 = \begin{pmatrix} \Gamma_2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} : (A \oplus B) \oplus C \oplus C \to (B \oplus D_2 \oplus E) \oplus C \oplus F$$

in which $\Gamma_0 = \begin{pmatrix} 2 & 0 \\ x & 0 \\ y & w \end{pmatrix} : A \oplus B \to A \oplus D \oplus E_2$ and $\Gamma_2 = \begin{pmatrix} 0 & 2 \\ x & z \\ 0 & w \end{pmatrix} : A \oplus B \to B \oplus D_2 \oplus E$ for some $x, y, z$ and $w$. Here the direct sum decompositions $KC_0X \cong (A \oplus C) \oplus (D \oplus E_2 \oplus F)$ and $KC_2X \cong (B \oplus C) \oplus (D_2 \oplus E \oplus F)$ might be modified suitably if necessary.

Let $D'_2$ denote the cokernel of $x : A \to D_2$. Then we have direct sum decompositions $A \cong A' \oplus G'$ and $D \cong D' \oplus G'$ so that $x : A \to D_2$ is given by $0 \oplus q : A' \oplus G' \to D'_2 \oplus G'_2$ where $q$ is the mod 2 reduction. As is easily observed, the homomorphism $\begin{pmatrix} 2 \\ x \end{pmatrix} : A \to A \oplus D$ is expressed as the matrix $\begin{pmatrix} 2 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} : A' \oplus G' \to A' \oplus G' \oplus D' \oplus G'$, although the direct sum decomposition $A \oplus D$ might be modified if necessary. Therefore its cokernel coincides with $A'_2 \oplus D' \oplus G'$, and the canonical epimorphism $\rho_0 : A' \oplus G' \oplus D' \oplus G' \to A'_2 \oplus D' \oplus G'$ is represented by the matrix $\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}$. Since the torsion subgroup of $KO_1X \oplus KO_5X$ is a $Z/2$-module, the cokernel of $\Gamma_0 : A \oplus B \to A \oplus D \oplus E_2$ coincides with $A'_2 \oplus D \oplus E'_2$ in which $E'_2$ is the cokernel of $w : B \to E_2$. Moreover the canonical epimorphism $\rho_0 : (A' \oplus G' \oplus D' \oplus G') \oplus E_2 \to (A'_2 \oplus D' \oplus G') \oplus E'_2$ is represented by the matrix

$$\Lambda_y = \begin{pmatrix} & & \Lambda & & 0 \\ 0 & 0 & 0 & \pi y_2 & \pi \end{pmatrix}$$

where $y_2 = y|G'$ and $\pi : E_2 \to E'_2$ is the canonical projection.
We obtain a similar result for $\Gamma_2 : A \oplus B \to B \oplus D_2 \oplus E$.

**Lemma 3.1.** *The cokernels of* $(\epsilon_C \epsilon_O \beta_U^{-1})_* : KU_{i+2}X \to KC_iX (i = 0, 2)$ *coincide with* $A'_2 \oplus D \oplus E'_2 \oplus F$ *and* $B'_2 \oplus E \oplus D'_2 \oplus F$*, respectively, and the canonical* *epimorphisms*

$$\rho_0 : (A' \oplus G' \oplus D' \oplus G' \oplus E_2) \oplus C \oplus F \to (A'_2 \oplus D' \oplus G' \oplus E'_2) \oplus F$$
$$\rho_2 : (B' \oplus G'' \oplus E' \oplus G'' \oplus D_2) \oplus C \oplus F \to (B'_2 \oplus E' \oplus G'' \oplus D'_2) \oplus F$$

*are represented by the matrices* $\tilde{\Lambda}_y = \begin{pmatrix} \Lambda_y & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ *and* $\tilde{\Lambda}_z = \begin{pmatrix} \Lambda_z & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, *respectively, where* $A \cong A' \oplus G', D \cong D' \oplus G', B \cong B' \oplus G''$ *and* $E \cong E' \oplus G''$ *for some* $G', G''$.

There exist two monomorphisms

$$\theta_0 : A'_2 \oplus D \oplus E'_2 \oplus F \to KO_1X \oplus KO_5X$$
$$\theta_2 : B'_2 \oplus E \oplus D'_2 \oplus F \to KO_3X \oplus KO_7X$$

so that $(-\tau, \tau\beta_C)_* : KC_iX \to KO_{i+1}X \oplus KO_{i+5}X (i = 0, 2)$ are factorized as $\theta_i\rho_i$. Consider the restricted homomorphism $\theta_0 : A'_2 \oplus E'_2 \to KO_1X \oplus KO_5X$. Then we can choose a basis $\{a_i, b_j\}$ of $A'_2$ such that $\theta_0(a_i) = (x_i, 0)$ for $1 \le i \le m+p$ and $\theta_0(b_j) = (0, y_j)$ for $1 \le j \le n+q$, although the direct sum decomposition $A'_2 \oplus E'_2$ might be modified if necessary. We next choose a basis $\{c_i, d_j\}$ of $\theta_0^{-1}(KO_1X \oplus \{0\} \cup \{0\} \oplus KO_5X) \cap E'_2$ such that $\theta_0(c_i) = (z_i, 0)$ for $1 \le i \le r$ and $\theta_0(d_j) = (0, w_j)$ for $1 \le j \le s$, and moreover extend it to a basis $\{c_i, d_j, e_k, f_l\}$ of $\theta_0^{-1}(KO_1X \oplus L_y \cup L_x \oplus KO_5X) \cap E'_2$ where $L_x \cong Z/2\{x_1, \dots, x_{m+p}\}$ and $L_y \cong Z/2\{y_1, \dots, y_{n+q}\}$. Here we may take as $\theta_0(e_k) = (x_{m+k}, v_k)$ for $1 \le k \le p$ and $\theta_0(f_l) = (u_l, y_{n+l})$ for $1 \le l \le q$ by relabelling $\{x_i, y_j\}$. As is easily observed, the set $\{c_i, d_j, e_k, f_l\}$ forms a basis of the whole $E'_2$. However the elements given in the forms of $\{f_l\}$ can be removed by setting $a_{m+p+l} = b_{n+l} + f_l, x_{m+p+l} = u_l, e_{p+l} = f_l$ and $v_{p+l} = y_{n+l}$. Thus there exist a basis $\{a_i, b_j\}$ of $A'_2$ and a basis $\{c_i, d_j, e_k\}$ of $E'_2$ such that

$$\begin{aligned} &\theta_0(a_i) = (x_i, 0) \quad \text{for} \quad 1 \le i \le m+p, \quad \theta_0(b_j) = (0, y_j) \quad \text{for} \quad 1 \le j \le n \\ &\theta_0(c_i) = (z_i, 0) \quad \text{for} \quad 1 \le i \le r, \quad \theta_0(d_j) = (0, w_j) \quad \text{for} \quad 1 \le j \le s \\ &\theta_0(e_k) = (x_{m+k}, v_k) \quad \text{for} \quad 1 \le k \le p. \end{aligned} \tag{3.2}$$

Similarly we can choose bases of $B'_2$ and $D'_2$ using the restricted homomorphism $\theta_2 : B'_2 \oplus D'_2 \to KO_3X \oplus KO_7X$.

**Proposition 3.2.** *Let* $X$ *be a* $CW$*-spectrum with* $KU_*X$ *free. Then there are* *direct sum decompositions*

$$A \cong A^0 \oplus A^4 \oplus G^0 \oplus G', \quad E \cong E^3 \oplus E^7 \oplus G^0 \oplus G''$$
$$B \cong B^2 \oplus B^6 \oplus G^2 \oplus G'', \quad D \cong D^1 \oplus D^5 \oplus G^2 \oplus G'$$
$$\mathrm{Tor}\, KO_1X \cong A_2^0 \oplus E_2^7 \oplus G_2^0, \quad \mathrm{Tor}\, KO_5X \cong A_2^4 \oplus E_2^3 \oplus G_2^0$$
$$\mathrm{Tor}\, KO_3X \cong B_2^2 \oplus D_2^1 \oplus G_2^2, \quad \mathrm{Tor}\, KO_7X \cong B_2^6 \oplus D_2^5 \oplus G_2^2$$

*so that* $\theta_0|A_2^0 \oplus A_2^4 \oplus E_2^3 \oplus E_2^7$ *and* $\theta_2|B_2^2 \oplus B_2^6 \oplus D_2^1 \oplus D_2^5$ *behave identically, and* $\theta_0|G_2^0 \oplus G_2^0$ *and* $\theta_2|G_2^2 \oplus G_2^2$ *behave as the automorphism represented by the matrix* $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Given $CW$-spectra $X, Y$ we say that they have the same quasi $KO_*$-type if $KO \wedge Y$ is isomorphic to $KO \wedge X$ as a $KO$-module spectrum. The following result shown in [4, Proposition 1.1] is very useful in proving our main theorems.

(3.3) $CW$-spectra $X$ and $Y$ have the same quasi $KO_*$-type if and only if there exists a map $h : Y \to KO \wedge X$ inducing an isomorphism $((\epsilon_U \wedge 1)h)_* : KU_*Y \to KU_*X$.

Applying Corollary 2.5, Lemma 3.1 and Proposition 3.2 we can show

**Theorem 3.3.** *Let $X$ be a $CW$-spectrum with $KU_*X$ free. Then there exist free abelian groups $A^i (0 \le i \le 7), C^j (0 \le j \le 1)$ and $G^k (0 \le k \le 3)$ so that $X$ has the same quasi $KO_*$-type as the wedge sum $Y = (\bigvee_i \Sigma^i SA^i) \vee (\bigvee_j \Sigma^j C(\eta) \wedge SC^j) \vee (\bigvee_k \Sigma^k C(\eta^2) \wedge SG^k)$ where $SH$ denotes the Moore spectrum of type $H$ and $C(\eta^l)$ denotes the cofiber of the map $\eta^l : \Sigma^l \to \Sigma^0 (l = 1, 2)$.* (Cf. [1, Theorem 3.2] and [6, Theorem 2.4]).

Proof. Using the free abelian groups chosen in Proposition 3.2 we set $A^{1+i} = D^{1+i}, A^{2+i} = B^{2+i}, A^{3+i} = E^{3+i} (i = 0, 4), C^0 = C, C^1 = F, G^1 = G'$ and $G^3 = G''$. For each component $Y_H$ of the wedge sum $Y$ there exists a unique map $f_H : Y_H \to KU \wedge X$ such that $f_{H*} : KU_*Y_H \to KU_*X$ is the z inclusion, where $H$ is taken to be $A^i (0 \le i \le 7), C^j (0 \le j \le 1)$ or $G^k (0 \le k \le 3)$. Choose a map $g_H : Y_H \to KC \wedge X$ with $(\zeta \wedge 1)g_H = f_H$ as given in Corollary 2.5. Applying our method developed in [4, 5] we can easily find a map $h_H : Y_H \to KO \wedge X$ with $(\epsilon_U \wedge 1)h_H = f_H$, by means of Lemma 3.1 and Proposition 3.2. For example, in case of $H = G^1$ we get a map $h_1 : \Sigma^1 SG^1 \to KO \wedge X$ satisfying $h_1(1 \wedge j_Q) = (\tau\beta_C^{-1} \wedge 1)g_H$ because $(\tau\beta_C^{-1} \wedge 1)g_H(1 \wedge i_Q)(\eta \wedge 1) = 0$ where $i_Q : \Sigma^0 \to C(\eta^2)$ and $j_Q : C(\eta^2) \to \Sigma^3$ denote the bottom cell inclusion and collapsing. Here the map $g_H$ might be modified slightly by means of (2.11), but still it satisfies the property as given in Corollary 2.5. The map $h_1$ is factorized as $(\eta \wedge 1)h_1'$ for some $h_1'$ because it has at most order 4. Since the composition map $(\epsilon_O\beta_U^{-1} \wedge 1)f_H$ becomes trivial, there exists a map $h_H$ with $(\epsilon_U \wedge 1)h_H = f_H$ as desired. Our result is now established by virtue of (3.3). □

## 4. $KU_*X$ containing only one 2-torsion cyclic group $Z/2^m$

In this section we deal with a $CW$-spectrum $X$ such that $KU_0X \cong H \oplus Z/2^m$ and $KU_1X \cong K$ with $H, K$ free and finitely generated. In this case we may assume that $\psi_C^{-1} = 1$ or $1 + 2^{m-1}$ on $Z/2^m$ itself because $X$ is replaced by $\Sigma^2 X$ if $\psi_C^{-1} = -1$ or $-1 + 2^{m-1}$ on $Z/2^m$. Given such a $CW$-spectrum $X$ we admit direct sum decompositions

$$KU_0X \cong A \oplus B \oplus C \oplus C \oplus M, \quad KU_1X \cong D \oplus E \oplus F \oplus F \tag{4.1}$$

in the category $\mathcal{C}$ , where $A, B, C, D, E$ and $F$ are free objects on which $\psi_C^{-1}$ behaves as stated in (2.7) and $M$ is one of the objects given in the following forms:

(4.2)

| | (I) | (II) | (III) | (IV) | (V) |
|---|---|---|---|---|---|
| $M$ | $Z/2^m$ | $Z \oplus Z/2^m$ | $Z \oplus Z/2^m$ | $Z \oplus Z \oplus Z/2^m$ | $Z/2^m (m \geq 3)$ |
| $\psi_C^{-1}$ | $1$ | $\begin{pmatrix} 1 & 0 \\ 2^{m-1} & 1 \end{pmatrix}$ | $\begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 2^{m-1} & -1 & 1 \end{pmatrix}$ | $1 + 2^{m-1}$. |

According to Bousfield[1, Theorem 11.1] any $CW$-spectrum $X$ has the same $K_*$-local type as a certain finite $CW$-spectrum $Y$ if $KU_*X$ is finitely generated. So we may assume that a $CW$-spectrum $X$ satisfying (4.1) is finite in our discussion. In order to distinguish such a $CW$-spectrum $X$ we define its $\mathcal{C}$-type to be the pair (J, J') when the components $M$ in $KU_0X$ and $KU_{-1}DX$ are given in forms of (J) and (J'), respectively.

**Proposition 4.1.** *Let $X$ be a $CW$-spectrum satisfying* (4.1). *Then it has one of the following ten $\mathcal{C}$-types:* (I, I), (II, II), (III, III), (V, V), (I, II), (I, III), (I, IV), (II, I), (III, I) *and* (IV, I).

Proof. It is sufficient to show that $X$ never has the $\mathcal{C}$-types (II, III), (II, IV), (III, IV) and (IV, IV). Assume that the $\mathcal{C}$-type of $X$ is (II, III). Thus we have the following direct sum decompositions $KU_0X \cong A \oplus B \oplus C \oplus C \oplus (Z^A \oplus Z/2^m), KU_1X \cong D \oplus E \oplus F \oplus F \oplus Z^E, KU_{-1}DX \cong D \oplus E \oplus F \oplus F \oplus (Z^E \oplus Z/2^m), KU_0DX \cong A \oplus B \oplus C \oplus C \oplus Z^A$ in which $Z^A \cong Z^E \cong Z, \psi_C^{-1} = \begin{pmatrix} 1 & 0 \\ 2^{m-1} & 1 \end{pmatrix}$ on $Z^A \oplus Z/2^m$ and $\psi_C^{-1} = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}$ on $Z^E \oplus Z/2^m$. By the aid of (2.1.iii) and (2.3.iii) we can easily calculate $KC_0X \cong A \oplus C \oplus D \oplus E_2 \oplus F \oplus Z^A \oplus Z/2^{m+1}$ and $KC_2X \cong B \oplus C \oplus D_2 \oplus E \oplus F \oplus Z^E$. Consider the induced homomorphisms $\varphi_2 = (\epsilon_C \epsilon_O \beta_U^{-1})_* : KU_4X \to KC_2X$ and $\varphi_2' = (\epsilon_U \tau \beta_C^{-1})_* : KC_2X \to KU_{-1}X$. Using the equalities $\zeta_* \varphi_2 = ((1 + \psi_C^{-1})\beta_U^{-1})_*$ and $\varphi_2'(\gamma\beta_U)_* = ((1 + \psi_C^{-1})\beta_U^{-1})_*$ we can observe that $\pi_Z \varphi_2' \varphi_2 | Z^A$ is non-trivial where $\pi_Z$ denotes the projection onto $Z^E$. This is a contradiction to $\varphi_2'\varphi_2 = 0$. The other cases are similarly shown. □

Let $X$ be a $CW$-spectrum whose $\mathcal{C}$-type is one of the following seven types : (I, I), (II, I), (III, I), (IV, I), (II, II), (III, III) and (V, V). Thus we admit direct sum decompositions given in the following forms:

$$\begin{aligned} KU_0X &\cong A \oplus B \oplus C \oplus C \oplus M, \quad KU_1X \cong D \oplus E \oplus F \oplus F \oplus K \\ KU_{-1}DX &\cong D \oplus E \oplus F \oplus F \oplus N, \quad KU_0DX \cong A \oplus B \oplus C \oplus C \oplus H \end{aligned} \tag{4.3}$$

in the category $\mathcal{C}$. Here $A, B, C, D, E$ and $F$ are free objects on which $\psi_C^{-1}$ behaves as stated in (2.7), and $M \cong H \oplus Z/2^m$ and $N \cong K \oplus Z/2^m$ are the objects in the category $\mathcal{C}$ tabled below:

$$\begin{array}{ccccc} & \text{(I, I)} & \text{(II, I)} & \text{(III, I)} & \text{(IV, I)} \\ M & Z/2^m & Z \oplus Z/2^m & Z \oplus Z/2^m & Z \oplus Z \oplus Z/2^m \\ \psi_C^{-1} & 1 & \begin{pmatrix} 1 & 0 \\ 2^{m-1} & 1 \end{pmatrix} & \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 2^{m-1} & -1 & 1 \end{pmatrix} \\ N & Z/2^m & Z/2^m & Z/2^m & Z/2^m \\ \psi_C^{-1} & 1 & 1 & 1 & 1 \\ & \text{(II, II)} & \text{(III, III)} & \text{(V, V)} & \\ M & Z \oplus Z/2^m & Z \oplus Z/2^m & Z/2^m & \\ \psi_C^{-1} & \begin{pmatrix} 1 & 0 \\ 2^{m-1} & 1 \end{pmatrix} & \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} & 1 + 2^{m-1} & \\ N & Z \oplus Z/2^m & Z \oplus Z/2^m & Z/2^m & \\ \psi_C^{-1} & \begin{pmatrix} 1 & 0 \\ 2^{m-1} & 1 \end{pmatrix} & \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} & 1 + 2^{m-1}. & \end{array} \tag{4.4}$$

By the aid of (2.1.iii) and (2.3.iii) we can calculate the $KC$-homology groups $KC_iX (i = 0, 1, 2, 3)$ as follows:

(4.5)
$$\begin{aligned} KC_0X &\cong A \oplus C \oplus D \oplus E_2 \oplus F \oplus M^0, \quad KC_1X \cong A_2 \oplus B \oplus C \oplus D \oplus F \oplus M^1 \\ KC_2X &\cong B \oplus C \oplus D_2 \oplus E \oplus F \oplus M^2, \quad KC_3X \cong A \oplus B_2 \oplus C \oplus E \oplus F \oplus M^3 \end{aligned}$$

in which $M^i (i = 0, 1, 2, 3)$ are the abelian groups tabled below:

$$\begin{array}{ccccc} & \text{(I, I)} & \text{(II, I)} & \text{(III, I)} & \text{(IV, I)} \\ M^0 & Z/2^m & Z \oplus Z/2^m & Z/2^m & Z \oplus Z/2^m \\ M^1 & Z/2 & (*)_m & Z & Z \oplus Z/2 \\ M^2 & Z/2 & Z/2 & Z \oplus Z/2 & Z \oplus Z/2 \\ M^3 & Z/2^m & Z \oplus Z/2^{m-1} & Z/2^{m+1} & Z \oplus Z/2^m \end{array} \tag{4.6}$$

| | (II, II) | (III, III) | (V, V) |
|---|---|---|---|
| $M^0$ | $Z \oplus Z \oplus Z/2^{m-1}$ | $Z/2^{m+1}$ | $Z/2^{m-1}$ |
| $M^1$ | $Z \oplus (*)_m$ | $Z$ | $Z/2$ |
| $M^2$ | $(*)_m$ | $Z \oplus Z$ | $Z/2$ |
| $M^3$ | $Z \oplus Z/2^{m-1}$ | $Z \oplus Z/2^{m+1}$ | $Z/2^{m-1}$ |

where $(*)_1 \cong Z/4$ and $(*)_m \cong Z/2 \oplus Z/2$ if $m \geq 2$.

Similarly to (3.1) we can observe that the induced homomorphisms $\varphi_i = (\epsilon_C \epsilon_O \beta_U^{-1})_* : KU_{i+2}X \to KC_iX (i = 0, 2)$ are represented by the following matrices

$$(4.7) \quad \begin{aligned} &\begin{pmatrix} \tilde{\Gamma}_0 & 0 \\ \beta_0 & \gamma_0 \end{pmatrix} : (A \oplus B \oplus C \oplus C) \oplus M \to (A \oplus D \oplus E_2 \oplus C \oplus F) \oplus M^0 \\ &\begin{pmatrix} \tilde{\Gamma}_2 & 0 \\ \beta_2 & \gamma_2 \end{pmatrix} : (A \oplus B \oplus C \oplus C) \oplus M \to (B \oplus D_2 \oplus E \oplus C \oplus F) \oplus M^2. \end{aligned}$$

Here $\tilde{\Gamma}_i (i = 0, 2)$ are the same matrices given in (3.1), $\beta_0 = 0$ unless the $\mathcal{C}$-type of $X$ is (III, III), $\beta_2 = 0$ unless the $\mathcal{C}$-type of $X$ is (II, II), and $\gamma_i (i = 0, 2)$ are expressed as the matrices tabled below:

(4.8)

| | $\gamma_0 : M \to M^0$ | $\gamma_2 : M \to M^2$ | $M$ | $M^0$ | $M^2$ |
|---|---|---|---|---|---|
| (I, I) | $2$ | $0$ | $Z/2^m$ | $Z/2^m$ | $Z/2$ |
| (II, I) | $\begin{pmatrix} 1 & 0 \\ 2^{m-1} & 2 \end{pmatrix}$ | $(1\ 0)$ | $Z \oplus Z/2^m$ | $Z \oplus Z/2^m$ | $Z/2$ |
| (III, I) | $(-1\ 2)$ | $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ | $Z \oplus Z/2^m$ | $Z/2^m$ | $Z \oplus Z/2$ |
| (IV, I) | $\begin{pmatrix} 1 & 0 & 0 \\ 2^{m-1} & -1 & 2 \end{pmatrix}$ | $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ | $Z \oplus Z \oplus Z/2^m$ | $Z \oplus Z/2^m$ | $Z \oplus Z/2$ |
| (II, II) | $\begin{pmatrix} 1 & 0 \\ -2^{m-1} & 0 \\ 0 & 1 \end{pmatrix}$ | $\begin{cases} (\pm 1\ 2) \\ \begin{pmatrix} 1 & 0 \\ \epsilon & 1 \end{pmatrix} \end{cases}$ | $Z \oplus Z/2^m$ | $Z \oplus Z \oplus Z/2^{m-1}$ | $\begin{cases} Z/4 \\ Z/2 \oplus Z/2 \end{cases}$ |
| (III, III) | $(-1 + 2^m\epsilon\ 2)$ | $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ | $Z \oplus Z/2^m$ | $Z/2^{m+1}$ | $Z \oplus Z$ |
| (V, V) | $1$ | $1$ | $Z/2^m$ | $Z/2^{m-1}$ | $Z/2$ |

where $\epsilon = 0$ or $1$.

Let us denote by $L^i (i = 0, 2)$ the cokernel of $\gamma_i : M \to M^i$. Note that $L^0 = \{0\}$ unless the $\mathcal{C}$-type of $X$ is (I, I), (II, I) or (II, II), and $L^2 = \{0\}$ unless the $\mathcal{C}$-type of $X$ is (I, I), (III, I) or (III, III). In the non-zero cases the canonical epimorphisms $\rho'_i : M^i \to L^i (i = 0, 2)$ are represented by the rows tabled below:

(4.9)

| | $\rho'_0 : M^0 \to L^0$ | | $\rho'_2 : M^2 \to L^2$ |
|---|---|---|---|
| (I, I) | $1 : Z/2^m \to Z/2$ | (I, I) | $1 : Z/2 \to Z/2$ |
| (II, I) | $(2^{m-1}\ 1) : Z \oplus Z/2^m \to Z/2$ | (III, I) | $(0\ 1) : Z \oplus Z/2 \to Z/2$ |
| (II, II) | $(2^{m-1}\ 1\ 0) : Z \oplus Z \oplus Z/2^{m-1} \to Z$ | (III, III) | $(0\ 1) : Z \oplus Z \to Z$ |

**Lemma 4.2.** *The cokernels of* $(\epsilon_C \epsilon_O \beta_U^{-1})_* : KU_{i+2}X \to KC_iX (i = 0, 2)$ *coincide with* $A_2' \oplus D \oplus E_2' \oplus F \oplus L^0$ *and* $B_2' \oplus E \oplus D_2' \oplus F \oplus L^2$*, respectively, and the canonical epimorphisms*

$$\tilde{\rho}_0 : (A' \oplus G' \oplus D' \oplus G' \oplus E_2 \oplus C \oplus F) \oplus M^0 \to (A_2' \oplus D' \oplus G' \oplus E_2' \oplus F) \oplus L^0$$
$$\tilde{\rho}_2 : (B' \oplus G'' \oplus E' \oplus G'' \oplus D_2 \oplus C \oplus F) \oplus M^2 \to (B_2' \oplus E' \oplus G'' \oplus D_2' \oplus F) \oplus L^2$$

*are expressed as the direct sums* $\rho_0 \oplus \rho_0'$ *and* $\rho_2 \oplus \rho_2'$*, respectively. Here* $\rho_i (i = 0, 2)$ *are the same epimorphisms as given in Lemma* 3.1.

There exist two monomorphisms

$$\theta_0 : A_2' \oplus D \oplus E_2' \oplus F \oplus L^0 \to KO_1X \oplus KO_5X$$
$$\theta_2 : B_2' \oplus E \oplus D_2' \oplus F \oplus L^2 \to KO_3X \oplus KO_7X$$

so that $(-\tau, \tau\beta_C)_* : KC_iX \to KO_{i+1}X \oplus KO_{i+5}X (i = 0, 2)$ are factorized as $\theta_i \tilde{\rho}_i$. Consider the restricted homomorphism $\theta_0 : A_2' \oplus E_2' \oplus L^0 \to KO_1X \oplus KO_5X$ when the $\mathcal{C}$-type of $X$ is (I, I) or (II, I). For the generator $g$ of $L^0 \cong Z/2$ we set $\theta_0(g) = (x_0, y_0)$ in $KO_1X \oplus KO_5X$. Then the pair $(x_0, y_0)$ is divided into the three types:

$$\text{i)} \quad x_0 \neq 0, \quad y_0 = 0 \qquad \text{ii)} \quad x_0 = 0, \quad y_0 \neq 0 \qquad \text{iii)} \quad x_0 \neq 0, \quad y_0 \neq 0.$$

Here we may assume that the set $\theta_0^{-1}(KO_1X \oplus \{y_0\}) \cap E_2'$ is empty in case of ii) and the set $\theta_0^{-1}(KO_1X \oplus \{y_0\} \cup \{x_0\} \oplus KO_5X) \cap E_2'$ is empty in case of iii), although the generator $g$ might be changed by using a suitable transformation on $E_2' \oplus L^0$. As in (3.2) we can choose a basis $\{a_i, b_j\}$ of $A_2'$ and a basis $\{c_i, d_j, e_k\}$ of $E_2'$ such that

$$\begin{aligned} &\theta_0(a_i) = (x_i, 0) \quad \text{for} \quad 1 \leq i \leq m+p, \quad \theta_0(b_j) = (0, y_j) \quad \text{for} \quad 1 \leq j \leq n \\ &\theta_0(c_i) = (z_i, 0) \quad \text{for} \quad 1 \leq i \leq r, \quad \theta_0(d_j) = (0, w_j) \quad \text{for} \quad 1 \leq j \leq s \\ &\theta_0(e_k) = (x_k, v_k) \quad \text{for} \quad \epsilon \leq k \leq p. \end{aligned} \tag{4.10}$$

Here $\epsilon = 0$ or $1$ in case of i), $\epsilon = 1$ in cases of ii) and iii), and $x_0 = x_{p+1}$ in case of iii). Similarly we can choose bases of $L^2 \cong Z/2, B_2'$ and $D_2'$ using the restricted homomorphism $\theta_2 : B_2' \oplus D_2' \oplus L^2 \to KO_3X \oplus KO_7X$ when the $\mathcal{C}$-type of $X$ is (I, I) or (III, I).

**Proposition 4.3.** i) *Let* $X$ *be a CW-spectrum whose* $\mathcal{C}$*-type is* (I, I) *or* (II, I). *Then it admits one of four kinds of direct sum decompositions given in the following forms:*

$$\begin{array}{lrl}
(A1) & A \cong A^0 \oplus A^4 \oplus G^0 \oplus G', & E \cong E^3 \oplus E^7 \oplus G^0 \oplus G'' \\
& \operatorname{Tor} KO_1X \cong A_2^0 \oplus E_2^7 \oplus G_2^0 \oplus L^0, & \operatorname{Tor} KO_5X \cong A_2^4 \oplus E_2^3 \oplus G_2^0 \\
(A2) & A \cong A^0 \oplus A^4 \oplus G^0 \oplus G', & E \cong E^3 \oplus E^7 \oplus G^0 \oplus G'' \\
& \operatorname{Tor} KO_1X \cong A_2^0 \oplus E_2^7 \oplus G_2^0, & \operatorname{Tor} KO_5X \cong A_2^4 \oplus E_2^3 \oplus G_2^0 \oplus L^0 \\
(A3) & A \cong A^0 \oplus A^4 \oplus Z^A \oplus G^0 \oplus G', & E \cong E^3 \oplus E^7 \oplus G^0 \oplus G'' \\
& \operatorname{Tor} KO_1X \cong A_2^0 \oplus E_2^7 \oplus G_2^0 \oplus L^0, & \operatorname{Tor} KO_5X \cong A_2^4 \oplus E_2^3 \oplus G_2^0 \oplus L^0 \\
(A4) & A \cong A^0 \oplus A^4 \oplus G^0 \oplus G', & E \cong E^3 \oplus E^7 \oplus Z^E \oplus G^0 \oplus G'' \\
& \operatorname{Tor} KO_1X \cong A_2^0 \oplus E_2^7 \oplus G_2^0 \oplus L^0, & \operatorname{Tor} KO_5X \cong A_2^4 \oplus E_2^3 \oplus G_2^0 \oplus Z/2.
\end{array}$$

*Here* $\theta_0|A_2^0\oplus A_2^4\oplus E_2^3\oplus E_2^7$ *and* $\theta_0|G_2^0\oplus G_2^0$ *behave as in Proposition 3.2,* $\theta_0|L^0$ *behaves identically, and* $\theta_0|Z_2^A \oplus L^0$ *and* $\theta_0|Z_2^E \oplus L^0$ *behave as the automorphisms represented by the matrices* $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ *and* $\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}$, *respectively, in which* $Z^A \cong Z^E \cong Z$ *and* $Z_2^A \cong Z_2^E \cong L^0 \cong Z/2$.

ii) *Let* $X$ *be a* $CW$*-spectrum whose* $\mathcal{C}$*-type is* (I, I) *or* (III, I). *Then it admits one of four kinds of direct sum decompositions* (B1), (B2), (B3) *and* (B4) *given in similar forms to* (A1), (A2), (A3) *and* (A4).

iii) *Let* $X$ *be a* $CW$*-spectrum whose* $\mathcal{C}$*-type is* (IV, I), (II, II), (III, III) *or* (V, V). *Then it admits only one kind of direct sum decompositions as given in Proposition 3.2.*

For a $CW$-spectrum $X$ of $\mathcal{C}$-type (I, I) we define its $\theta$-type to be the pair (Ai, Bj) if it admits direct sum decompositios given in (Ai) and (Bj). Similarly we define its $\theta$-type (Ai) or (Bj) for a $CW$-spectrum $X$ of $\mathcal{C}$-type (II, I) or (III, I).

## 5. Main results

We now recall several small spectra constructed in [4] and [5]. Let $SZ/2^m$ be the Moore spectrum of type $Z/2^m$ with the bottom cell inclusion $i$ and the top cell projection $j$. Denote by $M_m, N_m, Q_m, R_m, M'_m, N'_m, Q'_m, R'_m, V_m$ and $W_m$ the cofibers of the following maps:

$$\begin{array}{lll}
i\eta : \Sigma^1 \to SZ/2^m, & i\eta^2 : \Sigma^2 \to SZ/2^m, & \tilde{\eta}\eta : \Sigma^3 \to SZ/2^m, \\
\tilde{\eta}\eta^2 : \Sigma^4 \to SZ/2^m, & \eta j : SZ/2^m \to \Sigma^0, & \eta^2 j : \Sigma^1 SZ/2^m \to \Sigma^0, \\
\eta\bar{\eta} : \Sigma^2 SZ/2^m \to \Sigma^0, & \eta^2\bar{\eta} : \Sigma^3 SZ/2^m \to \Sigma^0, & \\
i\bar{\eta} : \Sigma^1 SZ/2 \to SZ/2^m, & i\bar{\eta} + \tilde{\eta}j : \Sigma^1 SZ/2 \to SZ/2^m, &
\end{array}$$

respectively. Here $\tilde{\eta} : \Sigma^2 \to SZ/2^m$ and $\bar{\eta} : \Sigma^1 SZ/2^m \to \Sigma^0$ are a coextension and an extension of $\eta : \Sigma^1 \to \Sigma^0$. Given two cofibers $X_m, Y_m$ of any maps $f : \Sigma^i \to SZ/2^m$, $g : \Sigma^j \to SZ/2^m (i \leq j)$ we denote by $XY_m$ the cofiber of the map $f \vee g : \Sigma^i \vee \Sigma^j \to SZ/2^m$. Dually we denote by $X'Y'_m$ the cofiber of the map $(f,\ g)$ :

$\Sigma^j SZ/2^m \to \Sigma^{j-i} \vee \Sigma^0$ for two cofibers $X'_m, Y'_m$ of any maps $f : \Sigma^i SZ/2^m \to \Sigma^0$, $g : \Sigma^j SZ/2^m \to \Sigma^0 (i \le j)$. Moreover we denote by $M'M_m, N'M_m, N'N_m, Q'Q_m$, $R'Q_m$ and $R'R_m$ the cofibers of the following maps:

$$\begin{array}{lll} \eta k_M : M_m \to \Sigma^0, & \eta^2 k_M : \Sigma^1 M_m \to \Sigma^0, & \eta^2 k_N : \Sigma^1 N_m \to \Sigma^0, \\ \eta \bar{k}_Q : \Sigma^2 Q_m \to \Sigma^0, & \eta^2 \bar{k}_Q : \Sigma^3 Q_m \to \Sigma^0, & \eta^2 \bar{k}_R : \Sigma^3 R_m \to \Sigma^0, \end{array}$$

respectively. Here the map $k_M : M_m \to \Sigma^1$, $k_N : N_m \to \Sigma^1$, $\bar{k}_Q : \Sigma^1 Q_m \to \Sigma^0$ and $\bar{k}_R : \Sigma^1 R_m \to \Sigma^0$ satisfy $k_M i_M = k_N i_N = j$ and $\bar{k}_Q i_Q = \bar{k}_R i_R = \bar{\eta}$ in which $i_X : SZ/2^m \to X_m$ denotes the canonical inclusion.

The small spectra $SZ/2^m, V_m, N_m, R_m, \Sigma^2 N'_m, R'_m, NR_m, N'R'_m, \Sigma^2 N'N_m$ and $R'R_m$ have the $\mathcal{C}$-type (I, I). As is easily observed (cf. [5, Lemma 3.2]), their $\theta$-types are tabled as follows:

$$(5.1) \qquad \begin{array}{cccccc} SZ/2^m & V_m & N_m & R_m & \Sigma^2 N'_m & R'_m \\ (A1, B1) & (A2, B1) & (A4, B1) & (A1, B4) & (A2, B3) & (A3, B2) \\ NR_m & N'R'_m & \Sigma^2 N'N_m & R'R_m & & \\ (A4, B4) & (A3, B3) & (A4, B3) & (A3, B4) & & \end{array}$$

The small spectra $Q_m, NQ_m$ and $R'Q_m$ have the $\mathcal{C}$-type (II, I), and $M_m, MR_m$ and $\Sigma^2 N'M_m$ have the $\mathcal{C}$-type (III, I). Their $\theta$-types are tabled as follows:

$$(5.2) \qquad \begin{array}{cccccc} Q_m & NQ_m & R'Q_m & M_m & MR_m & \Sigma^2 N'M_m \\ (A1) & (A4) & (A3) & (B1) & (B4) & (B3) \end{array}$$

The small spectra $MQ_m, \Sigma^1 Q'Q_m, \Sigma^3 M'M_m$ and $W_m$ have the $\mathcal{C}$-types (IV, I), (II, II), (III, III) and (V, V), respectively.

Applying Lemma 4.2 and Proposition 4.3 we can show the following three main results.

**Theorem 5.1.** *Let $X$ be a $CW$-spectrum whose $\mathcal{C}$-type is* (I, I). *Then there exist free abelian groups $A^i (0 \le i \le 7), C^j (0 \le j \le 1), G^k (0 \le k \le 3)$ and a certain small spectrum $Y$ so that $X$ has the same quasi $KO_*$-type as the wedge sum $(\vee_i \Sigma^i SA^i) \vee (\vee_j \Sigma^j C(\eta) \wedge SC^j) \vee (\vee_k \Sigma^k C(\eta^2) \wedge SG^k) \vee Y$. Here $Y$ is taken to be one of the following small spectra:*

$$\Sigma^l SZ/2^m, \Sigma^l V_m, \Sigma^l N_m, \Sigma^l R_m, \Sigma^{2+l} N'_m, \Sigma^l R'_m, NR_m, N'R'_m, \Sigma^2 N'N_m, R'R_m$$

*for $l = 0, 4$.* (Cf. [5, Theorem 4.2]).

Proof. Let $Y_{ij}$ denote the small spectrum of $\theta$-type (Ai, Bj) as listed in (5.1). When the $\theta$-type of $X$ is (Ai, Bj), there exists a map $f : Y_{ij} \to KU \wedge X$ such that $f_* : KU_*Y_{ij} \to KU_*X$ is the canonical inclusion in the category $\mathcal{C}$. By virtue of Proposition 2.3 such a map $f$ is chosen to satisfy $(\psi_C^{-1} \wedge 1)f = f$. Then we get a map $g : Y_{ij} \to KC \wedge X$ with $(\zeta \wedge 1)g = f$ such that $g_* : KC_iY_{ij} \to KC_iX (i = 0, 2)$ are the canonical inclusions because of Corollary 2.5 and Lemma 2.6. It is sufficient to find a map $h : Y_{ij} \to KO \wedge X$ with $(\epsilon_U \wedge 1)h = f$ for each $\theta$-type (Ai, Bj) by applying our method developed in [4, 5], the remaining cases being quite similarly shown to Theorem 3.3. For example, in case of $\theta$-type (A3, B4) we get a map $k_1 : \Sigma^1 R_m \to KO \wedge X$ such that $k_1 j_{R'R,R} = (\tau\beta_C^{-1} \wedge 1)g$ for the bottom cell collapsing $j_{R'R,R} : R'R_m \to \Sigma^4 R_m$. As is easily checked, the composition map $(\eta \wedge 1)k_1 i_R : \Sigma^2 SZ/2^m \to KO \wedge X$ is trivial where $i_R : SZ/2^m \to R_m$ is the canonical inclusion. Therefore there exists a map $h_1 : \Sigma^6 \to KO \wedge X$ such that $h_1 j_{R'R} = (\tau\beta_C^{-1} \wedge 1)g$ for the top cell projection $j_{R'R} : R'R_m \to \Sigma^9$. Here the map $g$ might be modified slightly by means of (2.11), but still it satisfies the property as given in Lemma 2.6. Since the composition map $h_1\eta$ is trivial, we can find a map $h : R'R_m \to KO \wedge X$ with $(\epsilon_U \wedge 1)h = f$ as desired. The other cases are similarly established. $\square$

**Theorem 5.2.** *Let $X$ be a $CW$-spectrum whose $\mathcal{C}$-type is* (II, I) *or* (III, I). *Then there exist free abelian groups $A^i (0 \leq i \leq 7), C^j (0 \leq j \leq 1), G^k (0 \leq k \leq 3)$ and a certain small spectrum $Y$ so that $X$ has the same quasi $KO_*$-type as the wedge sum $(\vee_i \Sigma^i SA^i) \vee (\vee_j \Sigma^j C(\eta) \wedge SC^j) \vee (\vee_k \Sigma^k C(\eta^2) \wedge SG^k) \vee Y$. Here $Y$ is taken to be one of the following small spectra:*

i) $Q_m$, $\Sigma^4 Q_m$, $NQ_m$, $R'Q_m$ *in case of $\mathcal{C}$-type* (II, I) ;
ii) $M_m$, $\Sigma^4 M_m$, $MR_m$, $\Sigma^2 N'M_m$ *in case of $\mathcal{C}$-type* (III, I) .

(Cf. [5, Theorem 4.4]).

Proof. Set $Y_1 = Q_m, Y_2 = \Sigma^4 Q_m, Y_3 = R'Q_m$ and $Y_4 = NQ_m$ if the $\mathcal{C}$-type of $X$ is (II, I), and $Y_1 = M_m, Y_2 = \Sigma^4 M_m, Y_3 = \Sigma^2 N'M_m$ and $Y_4 = MR_m$ if the $\mathcal{C}$-type of $X$ is (III, I). When the $\theta$-type of $X$ is (Ak) or (Bk), there exists a map $f : Y_k \to KU \wedge X$ such that $f_* : KU_*Y_k \to KU_*X$ is the canonical inclusion in the category $\mathcal{C}$. Since such a map $f$ is chosen to satisfy $(\psi_C^{-1} \wedge 1)f = f$, we get a map $g : Y_k \to KC \wedge X$ with $(\zeta \wedge 1)g = f$ such that $g_* : KC_iY_k \to KC_iX (i = 0, 2)$ are nearly the canonical inclusions because of Lemmas 2.6 and 2.7. It is sufficient to find a map $h : Y_k \to KO \wedge X$ with $(\epsilon_U \wedge 1)h = f$ for each $\theta$-type (Ak) or (Bk) by applying our method developed in [4, 5]. For example, in case of $\theta$-type (A4) we get a map $k_1 : \Sigma^1 Q_m \to KO \wedge X$ such that $k_1 j_{R'Q,Q} = (\tau\beta_C^{-1} \wedge 1)g$ for the bottom cell collapsing $j_{R'Q,Q} : R'Q_m \to \Sigma^4 Q_m$. Since the composition map $(\eta \wedge 1)k_1 i_Q : \Sigma^2 SZ/2^m \to KO \wedge X$ is trivial for the canonical inclusion $i_Q : SZ/2^m \to Q_m$, there exists a map $h_1 : \Sigma^5 \to KO \wedge X$ such that $h_1 j_{R'Q} = (\tau\beta_C^{-1} \wedge 1)$ where $j_{R'Q} : R'Q_m \to \Sigma^8$

is the top cell projection. Here the map $g$ might be modified slightly by means of (2.11), but still it satisfies the property that $\rho_0 g_*(H^+) \subset E_2$ and $\rho_2 g_*(H^-) \subset D_2$ given in Lemma 2.7. The map $h_1$ is factorized as $h'_1\eta$ for some $h'_1$ because it has at most order 4. Recall that $R'Q_m$ is the cofiber of the map $\tilde{h}_R\eta : \Sigma^7 \to R'_m$ where the map $\tilde{h}_R$ satisfies $j'_R\tilde{h}_R = \tilde{\eta}$ for the bottom cell collapsing $j'_R : R'_m \to \Sigma^4 SZ/2^m$. Evidently the composition map $h_1\eta j_{R'Q}$ becomes trivial. Consequently we can find a map $h : R'Q_m \to KO \wedge X$ with $(\epsilon_U \wedge 1)h = f$ as desired. The other cases are similarly established. $\square$

**Theorem 5.3.** *Let $X$ be a CW-spectrum whose $\mathcal{C}$-type is* (IV, I), (II, II), (III, III) *or* (V, V). *Then there exist free abelian groups $A^i (0 \leq i \leq 7), C^j (0 \leq j \leq 1), G^k (0 \leq k \leq 3)$ and only a certain small spectrum $Y$ so that $X$ has the same quasi $KO_*$-type as the wedge sum $(\underset{i}{\vee}\Sigma^i SA^i) \vee (\underset{j}{\vee}\Sigma^j C(\eta) \wedge SC^j) \vee (\underset{k}{\vee}\Sigma^k C(\eta^2) \wedge SG^k) \vee Y$. Here $Y$ is taken to be the following small spectrum corresponding to the $\mathcal{C}$-type of $X$:*

$$\begin{array}{rcllll} \mathcal{C}\text{-type} & = & (\mathrm{IV}, \mathrm{I}) & (\mathrm{II}, \mathrm{II}) & (\mathrm{III}, \mathrm{III}) & (\mathrm{V}, \mathrm{V}) \\ Y & = & MQ_m & \Sigma^1 Q'Q_m & \Sigma^3 M'M_m & W_m \end{array}$$

(Cf. [5, Theorem 3.3]).

Proof. When the $\mathcal{C}$-type of $X$ is (II, II), there exists a map $f : \Sigma^1 Q'Q_m \to KU \wedge X$ such that $f_* : KU_{*-1}Q'Q_m \to KU_*X$ is the canonical inclusion in the category $\mathcal{C}$. Since such a map $f$ is chosen to satisfy $(\psi_C^{-1} \wedge 1)f = f$, we get a map $g : \Sigma^1 Q'Q_m \to KC \wedge X$ with $(\zeta \wedge 1)g = f$ such that $g_* : KC_{i-1}Q'Q_m \to KC_iX (i = 0, 2)$ are nearly the canonical inclusions because of Lemmas 2.6 and 2.7. More precisely, $g_* : KC_{-1}Q'Q_m \to KC_0X$ is represented by the matrix $\begin{pmatrix} x & y & 0 \\ 1 & 0 & 0 \\ 2w & 1 & 0 \\ w & 0 & 1 \end{pmatrix} : Z \oplus Z \oplus Z/2^{m-1} \to E_2 \oplus Z \oplus Z \oplus Z/2^{m-1}$ for some $x, y, w$ and $g_* : KC_1Q'Q_m \to KC_2X$ is given by the identity on $(*)_m \cong Z/4$ or $Z/2 \oplus Z/2$ in essence. Evidently we get a map $k_1 : \Sigma^1 Q_m \to KO \wedge X$ such that $k_1 j_{Q'Q,Q} = (\tau\beta_C^{-1} \wedge 1)g$ for the bottom cell collapsing $j_{Q'Q,Q} : Q'Q_m \to \Sigma^3 Q_m$. Here the map $g$ might be modified slightly by means of (2.11), but still it satisfies the property mentioned above. Note that the composition map $k_1 i_Q i : \Sigma^1 \to KO \wedge X$ is factorized as $k_1 i_Q i = k'_1\eta$ for some $k'_1$. This implies that $k_1 i_Q = k'_1\tilde{\eta} + lj : \Sigma^1 SZ/2^m \to KO \wedge X$ for some $l$. On the other hand, it is easily checked that the composition map $(\eta \wedge 1)k_1 i_Q$ is expressed as $k'_1\eta\tilde{\eta}$. Hence there exists a map $h_1 : \Sigma^5 \to KO \wedge X$ such that $h_1 j_{Q'Q} = (\tau\beta_C^{-1} \wedge 1)g$ for the top cell projection $j_{Q'Q} : Q'Q_m \to \Sigma^7$. Here the map $g$ might be modified again, but it still satisfies the property mentioned previously. Since the composition map $h_1\eta^2$ is trivial, we get a map $\lambda : \Sigma^8 \to KC \wedge X$ with $(\tau\beta_C^{-1} \wedge 1)\lambda = h_1$. Such a map $\lambda$ is chosen to be expressed as $(\alpha,\ 0,\ t,\ 0)$ in $KC_8X \cong (A \oplus C \oplus D \oplus E_2 \oplus$

$F) \oplus Z \oplus Z \oplus Z/2^{m-1}$ because of (4.5). Note that the element $\lambda = (\alpha,\ 0,\ t,\ 0)$ is carried to $\zeta_*\lambda = (\beta,\ 0,\ -t)$ in $KU_8X \cong (A \oplus B \oplus C \oplus C) \oplus Z \oplus Z/2^m$ via $\zeta_* : KC_8X \to KU_8X$. Replacing the map $g$ by $g - \lambda j_{Q'Q}$ we can observe that $(\tau\beta_C^{-1} \wedge 1)g = 0$ and $((\zeta \wedge 1)g)_* : KU_{-1}Q'Q_m \to KU_0X$ is represented by the matrix $\begin{pmatrix} -\beta & 0 \\ 1 & 0 \\ t & 1 \end{pmatrix} : Z \oplus Z/2^m \to (A \oplus B \oplus C \oplus C) \oplus Z \oplus Z/2^m$. Consequently we can find a map $h : \Sigma^1 Q'Q_m \to KO \wedge X$ with $(\epsilon_U \wedge 1)h = f$ although the map $f : \Sigma^1 Q'Q_m \to KU \wedge X$ might be replaced suitably. Our result is now established by virtue of (3.3).

The case of $\mathcal{C}$-type (III, III) is established by a parallel discussion to the above one. On the other hand, the case of $\mathcal{C}$-type (IV, I) is similarly shown to Theorem 5.2. The remaining case of $\mathcal{C}$-type (V, V) is easy. $\square$

---

## References


[1] A. K. Bousfield: *A classification of K-local spectra*, J. Pure and Applied algebra, **66** (1990), 121–163.

[2] M. Mimura, S. Oka and M. Yasuo: *K-theory and the homotopy types of some function spaces*, Contemporary Math., **146** (1993), 309–338.

[3] Z. Yosimura: *The quasi KO-homology types of the real projective spaces*, Proc. Int. Conf. at Kinosaki, Springer-Verlag, **1418** (1990), 156–174.

[4] Z. Yosimura: *Quasi K-homology equivalences, I*, Osaka J. Math., **27** (1990), 465–498.

[5] Z. Yosimura: *Quasi K-homology equivalences, II*, Osaka J. Math., **27** (1990), 499–528.

[6] Z. Yosimura: *Quasi $KO_*$-equivalences; Wood and Anderson spectra*, Math. J. Okayama Univ., **32** (1990), 165–185.



Department of Intelligence and Computer Science,
Nagoya Institute of Technology,
Gokiso, Showa, Nagoya 466-0061, Japan
e-mail : yosimura@math.kyy.nitech.ac.jp