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LARGE DEVIATIONS AND RELATED LIL'S FOR BROWNIAN MOTIONS ON NESTED FRACTALS

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1. Introduction

In [11], Donsker and Varadhan applied their celebrated large deviation theory for general Markov processes to the symmetric stable process $X(t)$ on $\boldsymbol{R}^1$ of index α, $0 < \alpha \leq 2$, and, by making use of the scaling property of $X(t)$, they proved that the accumulation points of scale changed occupation time distributions

$$\hat{L}_t(\omega, \cdot) = \frac{1}{t}\int_0^t \chi_\cdot\left(\left(\frac{\log\log t}{t}\right)^{1/\alpha} X(s)\right) ds \tag{1.1}$$

as $t \to \infty$ in the space $\mathcal{M}$ of subprobability measures on $\boldsymbol{R}^1$ endowed with the vague topology coincide almost surely with its subspace

$$C = \{\beta \in \mathcal{M} : I(\beta) \leq 1\}, \tag{1.2}$$

where $I(\beta)$ denotes the I-function in the large deviation principle.

From this, they deduced, among other things, the "other" law of the iterated logarithm

$$\liminf_{t\to\infty}\left(\frac{\log\log t}{t}\right)^{1/\alpha} \sup_{0\leq s\leq t}|X(s)| = \ell_\alpha(>0)\ a.s. \tag{1.3}$$

and a LIL for the local time (in case $d = 1, 1 < \alpha \leq 2$),

$$\limsup_{t\to\infty}\left(\frac{t}{\log\log t}\right)^{1/\alpha}\frac{1}{t}\sup_y \ell_t(\omega, y) = d_\alpha\ a.s. \tag{1.4}$$

extending the older results for the Brownian motion (the case that $\alpha = 2$) due to Chung [4], Jain and Pruitt [21], and Kesten [22]. As compared to the ordinary law of the

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itereted logarithm, "the other" LIL (1.3) singles out those parts of the typical sample path sticking around the origin and moving much slower than the average size $t^{1/\alpha}$. Just as the Strassen law [28] covers the ordinary LIL, Donsker-Varadhan's identification of the limit points of (1.1) in the space of $\mathcal{M}$ covers the laws (1.3) and (1.4). We note that limit theorems of the type (1.3) have been demonstrated by other means for more general Lévy processes starting with the work of Taylor [32]. See Bertoin [3] and references therein in this connection. We also note that $\liminf$ counterpart of (1.4) was shown by Griffin [20] for symmetric stable processes and by Wee [33] for more general Lévy processes.

We now turn to looking at the Brownian motion $\mathbf{M} = (X_t, P_x)$ on a general unbounded nested fractal E studied by Lindstrøm [26], Kusuoka [8], Fukushima [15], Kumagai [25], Fitzsimmons,Hambly,Kumagai [13] and others. We consider a bounded nested fractal $\tilde{E} \subset \boldsymbol{R}^d$ decided by N number of α-similitudes ($\alpha > 1$). We assume that one of its boundary points is located at the origin and its diameter equals 1. Then the unbounded nested fractal E is defined by $E = \cup_{m=0}^{\infty} E^{\langle m \rangle}$ for $E^{\langle m \rangle} = \alpha^m \tilde{E}$. $\mathbf{M}$ is associated with a Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(E;\mu)$, where μ is the Hausdorff measure on E of Hausdorff dimension $d_f = \log N / \log \alpha$ such that $\mu(\tilde{E}) = 1$. This Dirichlet form involves a parameter c related to a returning probability of the approximating random walk on the pre-fractal ($0 < c < 1$). We let

$$(1.5) \qquad d_w = \frac{\log N - \log(1-c)}{\log \alpha} (> d_f), \ \gamma = \frac{1}{d_w}, \ d_s = \frac{2 d_f}{d_w} (< 2).$$

d_w and d_s are known as the walk dimension and the spectral dimension of $\mathbf{M}$ respectively.

Since $\mathbf{M}$ is μ-symmetric with a well behaved transition function satisfying conditions being formulated in [10] and in §6 in general contexts, it admits all required uniform estimates in the large deviation principle for the occupation time distributions as we shall state in §2. The principle involves the I-function which is defined on the space $\mathcal{M}$ of subprobability measures on E in terms of the Dirichlet form as

$$(1.6) \quad I_{\mathcal{E}}(\beta) = \begin{cases} \mathcal{E}(\sqrt{f}, \sqrt{f}) & \beta \prec \mu, \ \sqrt{f} \in \mathcal{F} \text{ for } f = d\beta/d\mu \\ 0 & \text{otherwise} \end{cases} \qquad \beta \in \mathcal{M}.$$

We shall also see in §2 that $\mathbf{M}$ enjoys the scaling property (a semi-stable property)

$$(1.7) \qquad X(\eta t) \text{ under } P_x \sim \eta^{\gamma}\, X(t) \text{ under } P_{\eta^{-\gamma} x},$$

holding for restricted values

$$(1.8) \qquad \eta = \left(\frac{N}{1-c}\right)^m, \text{ so that } \eta^{\gamma} = \alpha^m, \ m = 0, \pm 1, \pm 2, \cdots.$$

Accordingly, we are able to prove in this paper the counterparts of the stated results of Donsker-Varadhan in [11] by replacing the index α for the stable process with d_w for the present process $\mathbf{M}$.

More specifically, we define a sequence $\{t_m,\ m = 1, 2, \cdots\}$ of times by

$$\frac{t_m}{\log\log t_m} = \left(\frac{N}{1-c}\right)^m. \tag{1.9}$$

and set

$$\hat{L}_{t_m}(\omega, \cdot) = \frac{1}{t_m}\int_0^{t_m} I.(\alpha^{-m}\ X_s)ds.$$

In §3, we shall show that

$$\bigcap_N \overline{\bigcup_{m \geq N} \hat{L}_{t_m}(\omega, \cdot)} = C \quad P_x\text{ -a.e. } \omega,\ x \in E, \tag{1.10}$$

where C is the subspace of $\mathcal{M}$ defined as (1.2) by the present I-function (1.6).

From (1.10) and a proposition leading to it, we shall derive in §4 the identity

$$\liminf_{m \to \infty} \alpha^{-m} \sup_{0 \leq s \leq t_m} |X_s| = a_0 \quad P_x\text{ -a.e. } \omega,\ x \in E, \tag{1.11}$$

for a finite positive value

$$a_0 = \inf\{a > 0 : \kappa_a \leq 1\}, \tag{1.12}$$

where κ_a is the smallest eigenvalue of the part of the Dirichlet form $\mathcal{E}$ on the domain $G_a = \{x \in E : |x| < a\}$. As an immediate consequence of a recent work [1] by Barlow and Bass on the $0 - 1$ law for the tail σ-fields, we then get

$$\liminf_{t \to \infty} \left(\frac{\log\log t}{t}\right)^\gamma \sup_{0 \leq s \leq t} |X_s| = a_{00} \quad P_x\text{ -a.e. } \omega,\ x \in E, \tag{1.13}$$

for a positive constant a_{00} satisfying

$$\alpha^{-1} a_0 \leq a_{00} \leq a_0. \tag{1.14}$$

We do not know yet if $a_{00} = a_0$.

In a similar way, we shall derive in §4 from (1.10) that, for any $\theta > 0$,

$$\liminf_{t \to \infty} \frac{(\log\log t)^{\theta\gamma}}{t^{1+\theta\gamma}} \int_0^t |X_s|^\theta ds = A'_\theta \quad P_x\text{ -a.e. } \omega,\ x \in E, \tag{1.15}$$

where A'_θ is a constant satisfying

$$\alpha^{-\theta-d_w} A_\theta \leq A'_\theta \leq \alpha^{\theta+d_w} A_\theta \tag{1.16}$$

for the value

$$A_\theta = \inf\{\int_E |x|^\theta f^2 d\mu : f \in \mathcal{F}, \mathcal{E}(f,f) \leq 1, (f,f)_{L^2(E;\mu)} = 1\}$$

which will be shown to be strictly positive and to be a constant appearing in (1.15) if we replace t by t_m.

Since each one point set of E has a positive capacity, $\mathbf{M}$ admits a local time $\ell_t(\omega, y)$ which is actually jointly continuous in t, y and satisfies

$$\int_B \ell_t(\omega, y)\mu(dy) = \int_0^t I_B(X_s(\omega))ds \quad B \subset R^d. \tag{1.17}$$

We let

$$\hat{\ell}_{t_m}(\omega, y) = N^m \frac{1}{t_m} \ell_{t_m}(\omega, \alpha^m y)$$

and denote by $\mathcal{A}$ the space of subprobability density functions on E (with respect to μ) which are uniformly continuous. The space $\mathcal{A}$ is endowed with the topology of the uniform convergence on each compact set. It will be seen in §5 that $\hat{\ell}_{t_m}(\omega, y)$ is a member of this space almost surely and furthermore in accordance with (1.10)

$$\bigcap_N \overline{\bigcup_{m \geq N} \hat{\ell}_{t_m}(\omega, \cdot)} = \{f \in \mathcal{A} : \sqrt{f} \in \mathcal{F},\ \mathcal{E}(\sqrt{f}, \sqrt{f}) \leq 1\} \quad P_x\text{-a.e. } \omega,\ x \in E, \tag{1.18}$$

where the closure on the left hand side is taken in the space $\mathcal{A}$.

We shall derive from this the identity

$$\limsup_{m \to \infty} \left(\frac{t_m}{\log\log t_m}\right)^{d_s/2} \frac{1}{t_m} \ell_{t_m}(\omega, 0) = b_0 \quad P_x \text{ -a.e. } \omega,\ x \in E, \tag{1.19}$$

for the value

$$b_0 = \sup\{f(0) : f \in \mathcal{A}, \sqrt{f} \in \mathcal{F}, \mathcal{E}(\sqrt{f}, \sqrt{f}) \leq 1\}, \tag{1.20}$$

which will be shown to be bounded by using capacities as $1/c_{\ell,0} \leq b_0 \leq 2/c_0$. By the $0-1$ law again,

$$\limsup_{t \to \infty} \left(\frac{t}{\log\log t}\right)^{d_s/2} \frac{1}{t} \ell_t(\omega, 0) = b_{00} \quad P_x \text{ -a.e. } \omega,\ x \in E, \tag{1.21}$$

where b_{00} is a constant satisfying

$$b_0 \le b_{00} \le \frac{N}{1-c} b_0. \tag{1.22}$$

In the final section (§6), we shall present the Donsker-Varadhan large deviation principles in specific forms being used in §3 but in a more general context of symmetric Markov processes with smooth transition functions. In particluar, we need a lower estimate holding locally uniformly with respect to the space variable, which was first demonstrated in Theorem 8.1 of [10] for a general Markov process by using a Markov chain approximation. We shall give yet another proof of it by making use of a supermartingale transformation as was utilized in [9],[18],[30],[31].

2. Dirichlet forms and Brownian motions on unbounded nested fractals

For $\alpha > 1$, a mapping Ψ from $\boldsymbol{R}^d$ to $\boldsymbol{R}^d$ is said to be an *α-similitude* if $\Psi x = \alpha^{-1}Ux + \beta,\ x \in \boldsymbol{R}^d$, for some unitary map U and $\beta \in \boldsymbol{R}^d$. Given a collection $\boldsymbol{\Psi} = \{\Psi_1, \Psi_2, \cdots, \Psi_N\}$ of α-similitudes, there exists a unique compact set $\tilde{E} \subset \boldsymbol{R}^d$ such that $\tilde{E} = \bigcup_{i=1}^N \Psi_i(\tilde{E})$. The pair $(\boldsymbol{\Psi}, \tilde{E})$ is called a *self similar fractal.*

For $A \subset \boldsymbol{R}^d$ and integer $n \ge 1$, we let

$$A_{i_1\cdots i_n} = \Psi_{i_1} \circ \cdots \circ \Psi_{i_n}(A) \quad 1 \le i_1, \cdots, i_n \le N$$

$$A^{(n)} = \bigcup_{1 \le i_1, \cdots, i_n \le N} A_{i_1\cdots i_n}, \quad A^{(0)} = A.$$

We denote by F the set of all *essential fixed points* of $\boldsymbol{\Psi}$ ([26]). $\sharp F \le N$. Lindstrøm [26] calls a self similar fractal $(\boldsymbol{\Psi}, \tilde{E})$ a *nested fractal* if three axioms(axiom of connectivity,symmetry and nesting) and the open set condition are fulfilled and $\sharp F \ge 2$. We refer the readers to [26] for details but we note that the nesting axiom requires

$$\tilde{E}_{i_1\cdots i_n} \cap \tilde{E}_{j_1\cdots j_n} = F_{i_1\cdots i_n} \cap F_{j_1\cdots j_n} \quad (i_1 \cdots i_n) \ne (j_1 \cdots j_n),$$

which says that $\tilde{E}$ is finitely ramified, namely, it can be disconnected by removing certain finite number of points. Thus the family of nested fractals contains the Sierpinski gaskets on $\boldsymbol{R}^d$ and the snowfrake on $\boldsymbol{R}^2$ but excludes the Sierpinski carpets [1].

We consider a nested fractal $(\boldsymbol{\Psi}, \tilde{E})$ on $\boldsymbol{R}^d$. We assume that the origin of $\boldsymbol{R}^d$ is an essentially fixed point, $\Psi_1 x = \alpha^{-1}x,\ x \in \boldsymbol{R}^d$, and that the diameter of $\tilde{E}$ equals 1. The countable set

$$F^{(\infty)} = \bigcup_{n=0}^{\infty} F^{(n)}$$

is called a *nested pre-fractal* because $\tilde{E} = \overline{F^{(\infty)}}$.

Let $\pi_{xy},\ x, y \in F$, be Lindstrøm's invariant transition probability on F. It enjoys the following properties [26],[8]:

$$\pi_{xy} = \pi_{x'y'} \text{ whenever } |x-y| = |x'-y'|,$$
$$\pi_{xx} = 0\ x \in F;\ \pi_{xy} > 0\ x, y \in F, x \neq y;\ \sum_{y\in F} \pi_{xy} = 1\ x \in F.$$

π induces random walks not only on F but also on $F^{(1)}$ in a natural way. Let c be the probability that the random walk on $F^{(1)}$ starting at 0 returns to 0 before it hits other points of F. Then $0 < c < 1$ and the quadratic form

$$\begin{aligned}\tilde{\mathcal{E}}^{(n)}(u,u) = \tfrac{1}{2}(1-c)^{-n} \textstyle\sum_{1\le i_1,\cdots,i_n\le N} \sum_{x,y\in F}(u(\Psi_{i_1}\circ\cdots\circ\Psi_{i_n}x)& \\ -u(\Psi_{i_1}\circ\cdots\circ\Psi_{i_n}y))^2\pi_{xy}&\end{aligned} \tag{2.1}$$

turns out to be non-decreasing in n for any real-valued function u on $F^{(\infty)}$. If we put

$$\tilde{\mathcal{F}} = \{u : \lim_{n\to\infty} \tilde{\mathcal{E}}^{(n)}(u,u) < \infty\}, \tag{2.2}$$

then any function on $F^{(\infty)}$ belonging to the space $\tilde{\mathcal{F}}$ can be extended to a continuous function on $\tilde{E}$. Thus we ragard $\tilde{\mathcal{F}}$ as a subspace of the space $C(\tilde{E})$ of continuous functions on $\tilde{E}$. For $u, v \in \tilde{\mathcal{F}}$, we put

$$\tilde{\mathcal{E}}(u,v) = \lim_{n\to\infty} \tilde{\mathcal{E}}^{(n)}(u,v).$$

We next let

$$E^{\langle m\rangle} = \alpha^m \tilde{E},\ m = 0, \pm1, \pm2, \cdots,$$

$$E = \bigcup_{m=0}^{\infty} E^{\langle m\rangle}$$

and call the set E the *unbounded nested fractal*. Denote by μ the Hausdorff measure on E with $\mu(\tilde{E}) = 1$. A regular local Dirichlet form on $L^2(E;\mu)$ can then be defined from the above mentioned space $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ in the following manner [15].

Define a map σ_m sending a function on E to a function on E by

$$\sigma_m u(x) = u(\alpha^m x)(= u\circ\Psi_1^{-m}(x)),\ x \in \tilde{E},\ m = 0, \pm1, \pm2, \cdots. \tag{2.3}$$

Then σ_m sends the space $C(E^{\langle m\rangle})$ onto $C(\tilde{E})$ and we let

$$\mathcal{F}_{E^{\langle m\rangle}} = (\sigma_m)^{-1}\cdot\tilde{\mathcal{F}} \tag{2.4}$$

(2.5) $$\mathcal{E}_{E\langle m\rangle}(u,v) = (1-c)^m \tilde{\mathcal{E}}(\sigma_m u, \sigma_m v), \ u,v \in \mathcal{F}_{E\langle m\rangle}.$$

It is easy to see that

$$\mathcal{E}_{E\langle \ell\rangle}\left(u|_{E\langle \ell\rangle}, u|_{E\langle \ell\rangle}\right) \leq \mathcal{E}_{E\langle m\rangle}(u,u), \ \ell < m, \ u \in \mathcal{F}_{E\langle m\rangle}.$$

Accordingly we may set, denoting by $C(E)$ the space of continuous functions on E,

(2.6) $$\mathcal{F} = \left\{ u \in C(E) : \lim_{m\to\infty} \mathcal{E}_{E\langle m\rangle}(u|_{E\langle m\rangle}, u|_{E\langle m\rangle}) < \infty \right\} \cap L^2(E;\mu)$$

(2.7) $$\mathcal{E}(u,v) = \lim_{m\to\infty} \mathcal{E}_{E\langle m\rangle}\left(u|_{E\langle m\rangle}, v|_{E\langle m\rangle}\right), \ u,v \in \mathcal{F}.$$

Denoting by $(\ ,\ )$ the inner product of $L^2(E;\mu)$, we further put

$$\mathcal{E}_\beta(u,v) = \mathcal{E}(u,v) + \beta(u,v) \quad u,v \in \mathcal{F}, \ \beta > 0.$$

Proposition 2.1 ([15]). (i) *$(\mathcal{E},\mathcal{F})$ is a regular local Dirichlet form on $L^2(E;\mu)$. Each one point of E has a positive capacity with respect to this Dirichlet form.*

(ii) *$\mathcal{E}$ enjoys the scaling property*

(2.8) $$\mathcal{E}(u,v) = (1-c)\mathcal{E}(\sigma_1 u, \sigma_1 v) \quad u,v \in \mathcal{F}.$$

(iii) *The Hilbert space $(\mathcal{F},\mathcal{E}_\beta)$ admits a positive continuous reproducing kernel $g_\beta(x,y)$.*

In accordance with a general theory [16], there exists a diffusion process $\mathbf{M} = (X_t, P_x)$ on E associated with the reguler local Dirichlet form $(\mathcal{E},\mathcal{F})$ which we call a *Brownian motion* on E. Since each one point set has a positive capacity, the law P_x is uniquely decided by the Dirichlet form $\mathcal{E}$ for each $x \in E$. $\mathbf{M}$ is known to be point recurrent ([15]). Denote by p_t, G_α the transition function and the resolvent of the process $\mathbf{M}$;

$$p_t f(x) = E_x(f(X_t)), \quad G_\beta f(x) = \int_0^\infty e^{-\beta t} p_t f \, dt.$$

$G_\beta(x,\cdot)$ has $g_\beta(x,y)$ in the above proposition as its density function with respect to the measure μ.

We now collect some properties of the Brownian motion $\mathbf{M}$. We sometimes write its sample path X_t as $X(t)$.

Lemma 2.1 (semistable property of **M**). *For any Borel set B of $C([0,\infty)\to E)$,*

$$P_{\alpha x}(X(\cdot)\in B)=P_x(\alpha X(\frac{1-c}{N}\cdot)\in B),\ x\in E. \tag{2.9}$$

Proof. We first show that

$$\sigma_1(G_\beta f)=\frac{N}{1-c}G_{\frac{N}{1-c}\beta}(\sigma_1 f). \tag{2.10}$$

Note that the Hausdorff measure μ has the property

$$\int_E f\,d\mu=N\int_E \sigma_1\,f\,d\mu, \tag{2.11}$$

since this reduces to $\mu(G)=N\mu(\alpha^{-1}G),\ G\subset E$ for $f=\chi_G$. This combined with the scaling property (2.8) of $\mathcal{E}$ implies

$$\begin{aligned}\mathcal{E}_{\frac{N}{1-c}\beta}(\sigma_1(G_\beta f),\sigma_1 v) &= (1-c)^{-1}\mathcal{E}(G_\beta f,v)+\frac{N}{1-c}\beta N^{-1}(G_\beta f,v)=\\ (1-c)^{-1}\mathcal{E}_\beta(G_\beta f,v) &= (1-c)^{-1}(f,v)=N(1-c)^{-1}(\sigma_1 f,\sigma_1 v)\quad v\in\mathcal{F},\end{aligned}$$

from which follows (2.10).

(2.10) means that

$$\sigma_1(p_t f)=p_{\frac{1-c}{N}t}(\sigma_1 f) \tag{2.12}$$

or

$$E_{\alpha x}(f(X_t))=E_x\left(f\left(\alpha X_{\frac{1-c}{N}t}\right)\right).$$

We can now use (2.12) and the Markov property to get a desired identity

$$E_{\alpha x}(f_1(X_{t_1})\cdots f_n(X_{t_n}))=E_x\left(f_1(\alpha X_{\frac{1-c}{N}t_1})\cdots f_n(\alpha X_{\frac{1-c}{N}t_n})\right)$$

holding for $0\le t_1<\cdots<t_n$. □

Corollary 2.1. *The scaling property* (1.7) *holds for*

$$\eta=\left(\frac{N}{1-c}\right)^m,\quad m=0,\pm1,\pm2,\cdots. \tag{2.13}$$

The occupation time distribution L_t for $\mathbf{M}$ is defined by

$$L_t(\omega, B) = \frac{1}{t}\int_0^t \chi_B(X_s(\omega))ds, \quad B \subset E \tag{2.14}$$

For each ω, $L_t(\omega, \cdot)$ is an element of the space $\mathcal{M}$ of subprobability measures on E.

Corollary 2.2. *For any* η *of* (2.13)

$$L_t(\omega, \alpha^m \cdot) \text{ under } P_x \sim L_{\eta^{-1}t}(\omega, \cdot) \text{ under } P_{\alpha^{-m}x}. \tag{2.15}$$

Proof. For any Borel set $B \subset E$,

$$L_t(\omega, \eta^\gamma B) = \frac{1}{t}\int_0^t \chi_{\eta^\gamma B}(X(s))ds = \frac{1}{t}\int_0^t \chi_B(\eta^{-\gamma}X(s))ds,$$
$$\frac{1}{t}\int_0^t \chi_B(X(\eta^{-1}s))ds = L_{\eta^{-1}t}(\omega, B).$$

Since

$$\eta^{-\gamma}X(\cdot) \text{ under } P_x \sim X(\eta^{-1}\cdot) \text{ under } P_{\eta^{-\gamma}x},$$

by (1.7), we get (2.15) by noting $\eta^\gamma = \alpha^m$. □

The one point set $\{y\}$ for each $y \in E$ has a positive capacity by Proposition 2.1, and accordingly there exists a positive continuous additive functional $\ell_t(\omega, y)$ of $\mathbf{M}$ with Revuz measure $\delta_{\{y\}}$ the delta measure concentrated on $\{y\}$ ([16]). In the case of the Sierpinski gasket on the plane, Barlow and Perkins [2] constructed a version of $\ell_t(\omega, y)$ jointly continuous in t, y, by employing a Garsia' s type of lemma which is still valid in the present nested fractal case (see Lemma 5.4 of §5). Hence we may assume that it is jointly contiuous in t, y, in the present case as well ([25]) and we call it the local time. It is then characterized by the relation (1.17). We let

$$\ell'_t(\omega, y) = \frac{1}{t}\ell_t(\omega, y), \tag{2.16}$$

so that $\ell'_t(\omega, y)$ is the density function of the occupation time distribution L_t of (2.14).

Corollary 2.3. $\ell'_t(\omega, y)$ *enjoys the following scaling property: for any* η *of* (2.13)

$$\ell'(\omega, y) \text{ under } P_x \sim N^{-m}\ell'_{\eta^{-1}t}(\omega, \alpha^{-m}y) \text{ under } P_{\alpha^{-m}x}. \tag{2.17}$$

Proof. By virtue of (2.11),

$$L_t(\omega, \alpha^m B) = \int_B \ell'_t(\omega, \alpha^m y)\mu(\alpha^m\, dy) = N^m \int_B \ell'_t(\omega, \alpha^m y)\mu(dy).$$

Therefore, (2.17) follows from (2.15). □

In what follows, $c_{21}, c_{22}, \cdots$, will be some positive constants.

Lemma 2.2. (i) $p_t(x, \cdot)$ *is absolutely continuous with respect to* μ *and moreover* $p_t(C_\infty(E)) \subset C_\infty(E))$ *and* $p_t(\mathcal{B}_b(E)) \subset C_b(E)$ *where* $C_\infty(E)$ *denotes the space of continuous functions on* E *vanishing at infinity and the subscript* b *indicates 'bounded'.*
(ii) *For any* $\delta > 0$,

$$\sup_{x\in E} P_x(\sup_{0\le s\le t} |X_s - x| > \delta) \le c_{21}t, \ \forall t > 0.$$

(ii) follows from a stronger estimate of kumagai [25,(3.7)]. The first statement in (i) is a consequence of the μ-symmetry of p_t and the absolute continuity of the resolvent. For the second statement however, we invoke a heat kernel upper bound due to Fitzsimmons-Hambly-Kumagai [13]: $p_t(x, \cdot)$ admits a jointly continuous density $p_t(x, y)$ with respect to μ such that

$$p_t(x,y) \le c_{22}t^{-d_s/2} \exp\left(-c_{23}(d(x,y)^{\tilde{d}_w}t^{-1})^{1/(\tilde{d}_w-1)}\right), \tag{2.18}$$

where $d(x, y)$ is the intrinsic metric on E satisfiying

$$d(\alpha^m x, \alpha^m y) = \xi^m d(x, y) \tag{2.19}$$

for some constant $\xi > 1$ and $\tilde{d}_w$ denotes the walk dimension with restect to this metric. In [13], a lower bound

$$c_{24}t^{-d_s/2} \exp\left(-c_{25}(d(x,y)^{\tilde{d}_w}t^{-1})^{1/(\tilde{d}_w-1)}\right) \le p_t(x,y), \tag{2.20}$$

was also derived.

We prepare one more lemma for later use. For an open set $G \subset E$, we set

$$\mathcal{F}_G^0 = \{u \in \mathcal{F} : u(x) = 0 \ \forall x \in E - G\}. \tag{2.21}$$

The restriction of the form $\mathcal{E}$ to the space $\mathcal{F}_G^0$ is called *the part* of $\mathcal{E}$ on the set G and is denoted by $(\mathcal{E}, \mathcal{F}_G^0)$. It is a regular local Dirichlet form on $L^2(G; \mu)$ and is associated with the part of $\mathbf{M}$ on the set G, namely the process $\mathbf{M}$ being killed upon the leaving time τ_G from G. Let $\tilde{E}_0$ and $E_0^{\langle m\rangle}$ be the interior of $\tilde{E}$ and $E^{\langle m\rangle}$ respectively; $\tilde{E}_0 = \tilde{E} - F$, $E_0^{\langle m\rangle} = E^{\langle m\rangle} - \alpha^m F$. The space $\mathcal{F}_G^0$ for $G = \tilde{E}_0$ and for $G = E_0^{\langle m\rangle}$ will be designated by $\tilde{\mathcal{F}}^0$ and $\mathcal{F}_{\langle m\rangle}^0$ respectively for simplicity. They are related by

$$\mathcal{F}_{\langle m\rangle}^0 = (\sigma_m)^{-1} \cdot \tilde{\mathcal{F}}^0$$

just as (2.4).

$\kappa \geq 0$ is called an *eigenvalue* of $(\mathcal{E}, \mathcal{F}_G^0)$ if there exists a function (called an *eigenfunction*) $f \in \mathcal{F}_G^0$ such that

$$\mathcal{E}(f,g) = \kappa(f,g)_{L^2(G;\mu)}, \quad \forall g \in \mathcal{F}_G^0.$$

Lemma 2.3. (i) *Suppose G is a bounded connected open set. $(\mathcal{E}, \mathcal{F}_G^0)$ then admits a positive smallest eigenvalue with an associated eigenfunction being strictly positive on G. Further $(\mathcal{E}_\lambda, \mathcal{F}_G^0)$ has a reproducing kernel $g_\lambda^{0,G}(x,y)$ which is continuous and strictly positive on $G \times G$.*
(ii) *κ is an eigenvalue for $(\mathcal{E}, \tilde{\mathcal{F}}^0)$ if and only if so is $\left(\frac{1-c}{N}\right)^m \kappa$ for $(\mathcal{E}, \mathcal{F}_{\langle m\rangle}^0)$, $m = 0, \pm 1, \pm 2, \cdots$.*

Proof. (i) From Kusuoka's estimate [8,(4.14)]

$$\sup_{x,y\in\tilde{E}} |u(x) - u(y)| \leq c_{26}\sqrt{\tilde{\mathcal{E}}(u,u)}, \; u \in \tilde{\mathcal{F}} \tag{2.22}$$

and (2.5),(2,7), we can derive a bound

$$\sup_{x\in G} |u(x)| \leq c_{27}\sqrt{\mathcal{E}_\lambda(u,u)}, \; u \in \mathcal{F}_G^0, \; \lambda \geq 0, \tag{2.23}$$

together with a Poincaré inequality

$$(u,u)_{L^2(G;\mu)} \leq c_{28}\mathcal{E}(u,u), \; u \in \mathcal{F}_G^0. \tag{2.24}$$

(2.24) implies that the smallest eigenvalue of $(\mathcal{E}, \mathcal{F}_G^0)$ is not less than $c_{28}^{-1} > 0$. Observe that the Dirichlet form $(\mathcal{E}, \mathcal{F}_G^0)$ on $L^2(G,\mu)$ is irreducible because of the inclusion $\mathcal{F}_G^0 \subset C(G)$ ([16, Lemma 4.6.2]). Hence the smallest eigenvalue admits a corresponding eigenfunction which is strictly positive on G ([6,Theorem 1.4.3]).

(2.23) implies the existence of the reproducing kernel $g_\lambda^{0,G}(x,y)$ for $(\mathcal{E}_\lambda, \mathcal{F}_G^0)$. Its joint continuity and strict positivity can be shown in exactly the same way as the proof of [15,Theorem 2.3] except that the positivity of

$$g_\lambda^{0,G}(x,y)/g_\lambda^{0,G}(y,y) = E_x(e^{-\lambda\sigma_{\{y\}}}; \sigma_{\{y\}} < \tau_G)$$

for $x, y \in G$ can now be derived from [16,Theorem 4.6.6 (i)].
(ii) This scaling property of eigenvalues can be shown in the same way as [15,Corollary 3.2]. □

We are now in a position to formulate the large deviation principle in the present context. The occupation time distribution (2.14) is, for each $t > 0$, an $\mathcal{M}$-valued random variable. We denote its distribution with respect to P_x by $Q_{t,x}$:

$$Q_{t,x}(A) = P_x(L_t(\omega, \cdot) \in A), \quad A \subset \mathcal{M}.$$

$\mathcal{M}$ is endowed with the vague topology. We also consider the space $\mathcal{M}_1$ of all probability measures on E endowed with the weak topology. The I-function $I_{\mathcal{E}}$ is defined by (1.6).

Theorem 2.1. (i) *For any closed subset K of $\mathcal{M}$,*

$$\limsup_{t\to\infty} \frac{1}{t} \sup_{x\in E} \log Q_{t,x}(K) \leq - \inf_{\beta\in K} I_{\mathcal{E}}(\beta). \tag{2.25}$$

(ii) *Let β be a probability measure on E with $\beta(G) = 1$ for a bounded connected open set $G \subset E$. Let O be a neighbourhood of β in $\mathcal{M}_1$ and G' be a bounded connected open set with $G' \supset \overline{G}$. Then*

$$\liminf_{t\to\infty} \frac{1}{t} \log \inf_{x\in G} P_x\left(L_t(\omega, \cdot) \in O, t < \tau_{G'}\right) \geq -I_{\mathcal{E}}(\beta). \tag{2.26}$$

Proof. In view of the heat kernal bounds (2.18) and (2.20), conditions $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$ of §6 are fulfilled and hence Theorem 2.1 follows from Theorem 6.1, Theorem 6.2 together with Proposition 6.2. Here we give a more direct proof using the preceding lemmas.
(i) follows from Lemma 2.2 (i) and Theorem 6.1.
(ii). It suffices to check hypotheses $H_1, \sim, H_5$ in Theorem 8.1 of Donsker-Varadhan [10] for the part $\mathbf{M}^0_{G'}$ of the process $\mathbf{M}$ on the set G'. Since the resolvent of $\mathbf{M}^0_{G'}$ possesses the reproducing kernel $g^{0,G'}_\lambda(x, y)$ appearing in Lemma 2.3 as its density with respect to μ, both H_4 and H_5 are fulfilled. The absolute continuity of its transtion function $p^{0,G'}_t$(hypothsis H_1) also holds because of the symmetry. Since $p^{0,G'}_t$ sends $L^2(G'; \mu)$ into $\mathcal{F}^0_{G'}(\subset C_\infty(G'))$, it makes the space $C_\infty(G')$ invariant and accordingly give rise to a strongly continuous semigroup on this space. Hence H_2 is satisfied by $B_{00} = C_\infty(G')$. H_3 is clear since $C_\infty(G') \subset B_0$. □

Finally we quote a powerful $0 - 1$ law from [1,Theorem 8.4 and Remark 8.5].

Theorem 2.2. (Barlow and Bass [1]) *Suppose Γ is a tail event: $\Gamma \in \bigcap_t \sigma\{X_u, u \geq t\}$. Then either $P_x(\Gamma)$ is 0 for all x or else it is 1 for all x.*

3. Limit points of scale changed occupation time distributions

From now on, we work with the Brownian motion $\mathbf{M} = (X_t, P_x)$ on the unbounded nested fractal E associated with the regular local Dirichlet form $(\mathcal{E}, \mathcal{F})$ of Proposition 2.1. Let $L_t(\omega, B),\ B \in \mathcal{B}(E)$, be the occupation time distribution defined by (2.14) and put

$$\hat{L}_{t_m}(\omega, B) = L_{t_m}(\omega, \alpha^m B), \quad B \in \mathcal{B}(E), \tag{3.1}$$

where t_m is the solution of equation (1.9).

We define by (1.6) the functional $I_{\mathcal{E}}$ on the space $\mathcal{M}$ of subprobability measures on E endowed with the vague topology and we set

$$\begin{aligned} C &= \{\beta \in \mathcal{M} : I_{\mathcal{E}}(\beta) \le 1\} \\ (&= \{d\beta = f \cdot d\mu \in \mathcal{M} : \sqrt{f} \in \mathcal{F},\ \mathcal{E}(\sqrt{f}, \sqrt{f}) \le 1\}\ ). \end{aligned}$$

Proposition 3.1. *For P_x-a.e. ω*

$$\bigcap_N \overline{\bigcup_{m \ge N} \{\hat{L}_{t_m}(\omega, \cdot)\}} \subset C \tag{3.2}$$

for each $x \in E$.

Proof. We can follow the arguments in the proof of Theorem 2.8 of [11]. Take any open set $N \subset \mathcal{M}$ such that $C \subset N$ and let $\theta = \inf_{\beta \in N^c} I_{\mathcal{E}}(\beta)$. Since $I_{\mathcal{E}}$ is lower semicontinuous function on $\mathcal{M}$ on account of the equation (6.5) in §6 and N^c is compact, we have that $\theta > 1$. Take θ' such that $\theta > \theta' > 1$. Then, by virtue of Theorem 2.1, we have for sufficiently large m

$$Q_{\log\log t_m,\ \alpha^{-m}x}(N^c) \le \exp(-(\log\log t_m)\theta') = \frac{1}{(\log t_m)^{\theta'}}. \tag{3.3}$$

On the other hand, we see from (1.9)

$$\alpha^m = \left(\frac{t_m}{\log\log t_m}\right)^{\gamma} \tag{3.4}$$

which, together with Corollary 2.2, implies

$$P_x\left(\hat{L}_{t_m}(\omega, \cdot) \in N^c\right) = Q_{\log\log t_m,\ \alpha^{-m}x}(N^c). \tag{3.5}$$

Since $\log t_m \geq m \log \left(\frac{N}{1-c}\right)$, we get from (3.3) and (3.5)

$$\sum_m P_x\left(\hat{L}_{t_m}(\omega,\cdot) \in N^c\right) \leq \sum_m \frac{1}{\left(m \log \frac{N}{1-c}\right)^{\theta'}} < \infty.$$

Borel-Cantelli's lemma implies that $\hat{L}_{t_m}(\omega,\cdot) \in N$ from some m on for P_x-a.e. ω. □

Lemma 3.1. (i) *There exist positive c_m with $\lim_{m\to\infty} c_m = 1$ such that*

$$t_m = c_m \left(\frac{N}{1-c}\right)^m \log m.$$

(ii) *For any $k > 1$, there is an increasing sequence $\{m_n\}_{n=1}^{\infty}$ of positive integers such that*

$$\log\log t_{m_n} \sim k \log n \quad \text{as } n \to \infty \tag{3.6}$$

$$t_{m_n}/t_{m_{n+1}} \leq \exp(-c_{31} n^{k-1}) \quad \text{for some } c_{31} > 0. \tag{3.7}$$

Proof. In what follows, $\eta_{1m}, \cdots, \eta_{4m}$ denote some numbers tending to 0 as $m \to \infty$.

(i) Since

$$\log t_m = m \log \frac{N}{1-c} + \log^{(3)} t_m = m(1+\eta_{1m}) \log \frac{N}{1-c},$$

we have

$$\log^{(2)} t_m = (1+\eta_{2m}) \log m, \ \log^{(3)} t_m = \log^{(2)} m + \eta_{3m}, \tag{3.8}$$

and hence

$$\log t_m = m \log \frac{N}{1-c} + \log^{(2)} m + \eta_{3m}. \tag{3.9}$$

(ii) Let $m_n = [n^k]$ the integer part of n^k. Then $\log m_n \sim k \log n$, and (3.6) follows from (3.8). Since $(n+1)^k - n^k \geq kn^{k-1}$, we have from (3.9)

$$\log t_{m_n} - \log t_{m_{n+1}} \leq -kn^{k-1} \log \frac{N}{1-c} + \eta_{4m}.$$

□

Proposition 3.2. *Consider a probability measure β satisfying $I_{\mathcal{E}}(\beta) < 1$ and $\beta(G) = 1$ for a bounded open set G containing the origin. Let O be a neighbourhood of β in $\mathcal{M}_1$ and G' be a bounded open set with $G' \supset \overline{G}$. We put*

$$J_{t_m} = \{\omega : \hat{L}_{t_m}(\omega, \cdot) \in O,\ \tau_{\alpha^m \cdot G'} > t_m\}.$$

Then

$$\omega \in J_{t_m} \quad \text{for infinitely often } m \tag{3.10}$$

for P_x-a.e. ω for each $x \in E$.

Proof. We can proceed along the same line as in the proof of Theorem 2.15 of [11]. We take a neighbourhood O_1 of β in the space $\mathcal{M}_1$ such that $\beta \in O_1 \subset \overline{O_1} \subset O$. By the assumption $\theta = I_{\mathcal{E}}(\beta) < 1$. Choose $k > 1$ such that

$$\theta' = (2k-1) \cdot \frac{1}{2}(1+\theta) < 1. \tag{3.11}$$

Take a sequence $m_1 < m_2 < \cdots < m_n < \cdots$ satisfying conditions of Lemma 3.1 (ii) for this k.

Put

$$\tilde{L}_{t_{m_{n-1}}, t_{m_n}}(\omega, B) = \frac{1}{t_{m_n} - t_{m_{n-1}}} \int_{t_{m_{n-1}}}^{t_{m_n}} I_{\alpha^{m_n} B}(X_s)\, ds.$$

Then the total variation

$$\|\tilde{L}_{t_{m_{n-1}}, t_{m_n}}(\omega, \cdot) - \hat{L}_{t_{m_n}}(\omega, \cdot)\|$$

is not greater than $2t_{m_{n-1}}/t_{m_n}$, which tends to zero as $n \to \infty$ by Lemma 3.1. Consequently, for the proof of Proposition 3.2, it suffices to prove that

$$\omega \in \tilde{J}_n \text{ infinitely often } P_x\text{-a.e. } \omega,\ x \in E \tag{3.12}$$

for the event

$$\tilde{J}_n = \{\omega : \tilde{L}_{t_{m_{n-1}}, t_{m_n}}(\omega, \cdot) \in O_1,\ \tau_{\alpha^{m_n} \cdot G} > t_{m_{n-1}},\ \tau_{\alpha^{m_n} \cdot G'} > t_{m_n}\}. \tag{3.13}$$

We let

$$\mathcal{F}_{n-1} = \sigma\{X_s; 0 \le s \le t_{m_{n-1}}\},\ B_{n-1} = \{\omega : \tau_{\alpha^{m_n} \cdot G} > t_{m_{n-1}}\}$$

and we further put $\tilde{t}_n = t_{m_n} - t_{m_{n-1}}$ and

$$q_n(x) = P_x\left(\tau_{\alpha^{m_n}\cdot G'} > \tilde{t}_n,\ \frac{1}{\tilde{t}_n}\int_0^{\tilde{t}_n} I_{\alpha^{m_n}\cdot}(X_s)\, ds \in O_1\right).$$

Then, by virtue of the Markov property

$$P_x(\tilde{J}_n|\mathcal{F}_{n-1}) \geq I_{B_{n-1}}(\omega)\cdot \inf_{x\in\alpha^{m_n}\cdot G'} q_n(x) \quad P_x\text{-a.e. } \omega. \tag{3.14}$$

On the other hand, if we let

$$\eta_n = \left(\frac{N}{1-c}\right)^{m_n},\ s_n = \frac{\tilde{t}_n}{\eta_n}, \tag{3.15}$$

we find from (1.7) that

$$\begin{aligned} q_n(\alpha^{m_n}x) &= P_{\alpha^{m_n}x}\left(\alpha^{-m_n}X_s \in G'\ \forall s\in[0,\tilde{t}_n],\ \frac{1}{\tilde{t}_n}\int_0^{\tilde{t}_n} I.(\alpha^{-m_n}X_s)ds \in O_1\right) \\ &= P_x\left(X_{\eta_n^{-1}s} \in G'\ \forall s\in[0,\tilde{t}_n],\ \frac{1}{\tilde{t}_n}\int_0^{\tilde{t}_n} I.(X_{\eta_n^{-1}s})ds \in O_1\right) \\ &= P_x\left(X_s\in G'\ \forall s\in[0,s_n],\ L_{s_n}(\omega,\cdot)\in G_1\right). \end{aligned}$$

and consequently

$$\inf_{x\in\alpha^{m_n}\cdot G} q_n(x) = \inf_{x\in G} P_x\left(\tau_{G'} > s_n,\ L_{s_n}(\omega,\cdot)\in O_1\right). \tag{3.16}$$

On account of (1.9),(3.15) and Lemma 3.1,

$$s_n = \frac{t_{m_n} - t_{m_{n-1}}}{t_{m_n}}\cdot \log\log t_{m_n} \sim k\ \log\ n,\quad n\to\infty.$$

Hence, in virtue of Theorem 2.1 (ii), we see that, for sufficiently large n, the right hand side of (3.16) is not less than

$$\begin{aligned} \exp(-s_n\cdot\frac{1}{2}(1+\theta))) &\geq \exp\{-(2k-1)\frac{1}{2}(1+\theta)\log\ n\} \\ =\exp(-\theta'\ \log\ n) &= \frac{1}{n^{\theta'}}. \end{aligned}$$

In view of (3.14) and (3.16), there exists N such that

$$\sum_{n=N}^{\infty} P_x(\tilde{J}_n|\mathcal{F}_{n-1}) \geq \sum_{n=N}^{\infty}\frac{I_{B_{n-1}}}{n^{\theta'}} \quad P_x\text{-a.e. } \omega. \tag{3.17}$$

Suppose that

$$\sum_{n=1}^{\infty} P_x(B_n^c) < \infty. \tag{3.18}$$

Then, by Borel-Cantelli's lemma, the right hand side of (3.17) diverges for P_x-a.e. ω, and accordingly we can get to the desired conclusion (3.12) owing to the Borel-Cantelli type lemma [11,Lemma 2.14].

It only remains to show (3.18). Using the scaling property (1.7) again,

$$\begin{aligned} P_x(B_n^c) &= P_x\left(\alpha^{-m_{n+1}} X_s \notin G \text{ for some } s \in (0, t_{m_n}]\right) \\ &= P_{\alpha^{-m_{n+1}}x}\left(X_{\eta_{n+1}^{-1}s} \notin G \text{ for some } \ s \in (0, t_{m_n}]\right) \\ &= P_{\alpha^{-m_{n+1}}x}\left(\tau_G \le \eta_{n+1}^{-1} t_{m_n}\right). \end{aligned}$$

Let ℓ be an integer such that $E^{\langle \ell \rangle} \subset G$. By Lemma 2.2, we have, for sufficiently large n such that $\alpha^{-m_{n+1}}x \in E^{\langle \ell \rangle}$,

$$P_x(B_n^c) \le c_{21}\eta_{n+1}^{-1} t_{m_n} = c_{21}\frac{t_{m_n}}{t_{m_{n+1}}} \log\log t_{m_{n+1}}.$$

The sum of the right hand side is finite by Lemma 3.1. □

Lemma 3.2. *Let $\mathcal{D}$ be the set of probability measures β on E with compact support and $I_{\mathcal{E}}(\beta) < 1$. Then*

$$C \subset \overline{\mathcal{D}}, \tag{3.19}$$

where the closure is taken in the space $\mathcal{M}$.

Proof. We first show that

$$\forall \beta \in C,\ \exists \gamma_n \in \mathcal{M}_1,\ I_{\mathcal{E}}(\gamma_n) < 1,\ \lim_{n\to\infty} \gamma_n = \beta \text{ vaguely.} \tag{3.20}$$

Take $\lambda_n \uparrow 1$ and put $\beta_n = \lambda_n \beta$. Then

$$I_{\mathcal{E}}(\beta_n) = \lambda_n I_{\mathcal{E}}(\beta) \le \lambda_n,\ a_n = \beta_n(E) \le \lambda_n < 1.$$

We next take $\nu = \phi^2 \cdot \mu \in \mathcal{M}_1$ with non-negative bounded ϕ and $I_{\mathcal{E}}(\nu) < \infty$ and we set

$$\nu_m = \frac{1}{N^m}\phi(\alpha^{-m}\cdot)^2\mu.$$

Then, by the scaling property of μ,

$$\nu_m(E) = \frac{1}{N^m}\int_E \phi(\alpha^{-m}x)^2\mu(dx) = \int_E \phi^2 d\mu = 1$$

and for any compact $K \subset E$

$$\nu_m(K) \leq N^{-m}\|\phi\|_\infty^2 \mu(K) \to 0, \ m \to \infty.$$

Namely

$$\nu_m \in \mathcal{M}_1 \quad \lim_{m\to\infty} \nu_m = 0 \text{ vaguely.}$$

Furthermore, by (2.8),

$$\begin{aligned} I_{\mathcal{E}}(\nu_m) &= \frac{1}{N^m}\mathcal{E}(\phi(\alpha^{-m}\cdot), \phi(\alpha^{-m}\cdot)) \\ &= \left(\frac{1-c}{N}\right)^m \mathcal{E}(\phi,\phi) = \left(\frac{1-c}{N}\right)^m I_{\mathcal{E}}(\nu). \end{aligned}$$

We let

$$\gamma_{n,m} = \beta_n + (1-a_n)\nu_m (\in \mathcal{M}_1)$$

Then

$$I_{\mathcal{E}}(\gamma_{n,m}) \leq I_{\mathcal{E}}(\beta_n) + (1-a_n)I_{\mathcal{E}}(\nu_m) \leq \lambda_n + \left(\frac{1-c}{N}\right)^m I_{\mathcal{E}}(\nu),$$

which can be made to be less than 1 for a large m, say m_n. Now $\gamma_n = \gamma_{n,m_n}$ has the required property (3.20).

Next take any $\nu = f^2 \cdot \mu \in \mathcal{M}_1$ with $\mathcal{E}(f,f) = \lambda < 1$. Since the Dirichlet form $(\mathcal{E},\mathcal{F})$ is regular, there exist $f_n \in \mathcal{F}$, $n = 1, 2, \cdots$, which are non-negative, with compact support and satisfying, for $\lambda < \lambda' < 1$,

$$\lim_{n\to\infty} \mathcal{E}_1(f - f_n, f - f_n) = 0 \quad \mathcal{E}(f_n, f_n) < \lambda'.$$

Let $\nu_n = (f_n/b_n)^2 \cdot \mu$ where b_n is the $L^2(E;\mu)$-norm of f_n. Then, $\nu_n \in \mathcal{M}_1$, $\lim_{n\to\infty} \nu_n = \nu$ weakly and

$$\limsup_{n\to\infty} I_{\mathcal{E}}(\nu_n) = \limsup_{n\to\infty} b_n^{-2}\mathcal{E}(f_n, f_n) \leq \lambda' < 1,$$

which means that $\nu_n \in \mathcal{D}$ for large n. □

Theorem 3.1. *The identity (1.10) is valid.*

In fact, the inclusion $\subset$ is proven by Proposition 3.1. The left hand side of (1.10) includes the space $\mathcal{D}$ according to Proposition 3.2, and hence the space C by virtue of Lemma 3.2, proving the other inclusion $\supset$. Theorem 3.1 implies the following ([11]):

Theorem 3.2. *If Φ is a functional on $\mathcal{M}$ which is lower semicontinuous in the vague topology, then, for P_x-a.e. ω,*

$$\limsup_{m\to\infty} \Phi(\hat{L}_{t_m}(\omega,\cdot)) \geq \sup_{\beta\in C} \Phi(\beta), \tag{3.21}$$

and if Φ is a funtional on $\mathcal{M}$ which is upper semicontinuous in the vague topology, then, for P_x-a.e. ω,

$$\limsup_{m\to\infty} \Phi(\hat{L}_{t_m}(\omega,\cdot)) \leq \sup_{\beta\in C} \Phi(\beta). \tag{3.22}$$

4. Other laws of the iterated logarithm

In order to derive the identity (1.11) from Theorem 3.2 and Proposition 3.2, we consider the functionals Φ_a on $\mathcal{M}$ defined by

$$\Phi_a(\beta) = \beta(\overline{G}_a) \quad \Phi'_a(\beta) = \beta(G_a), \quad a > 0, \tag{4.1}$$

where

$$G_a = \{x \in E : |x| < a\}, \quad a > 0.$$

G_a is bounded open and connected because of the relation

$$|x|^{d_c} \asymp d(0,x)$$

for the shortest path distance (intrisic metric) d and a chemical exponent d_c ([13,Remark 3.7]). Since the measure μ is σ-finite, there exists a countable dense subset D of $[0,\infty)$ such that

$$\mu(\overline{G}_a \setminus G_a) = 0 \quad \forall a \in D.$$

All elements of C is absolutely continuous with respect to the measure μ and accordingly

$$\sup_{\beta\in C} \Phi_a(\beta) = \sup_{\beta\in C} \Phi'_a(\beta) \quad a \in D. \tag{4.2}$$

Furthermore Φ_a (resp.Φ'_a) is upper (resp. lower) semicontinuous and we have from Theorem 3.2 that

$$\limsup_{m\to\infty} \hat{L}_{t_m}(\omega, G_a) = r_a \quad P_x\text{-a.e.} \quad a \in D. \tag{4.3}$$

Here we denote by r_a the common value in (4.2). Obviously $0 \le r_a \le 1$.

Lemma 4.1. *For $a \in D$, $r_a = 1$ if and only if there exists a function $u \in \mathcal{F}$ vanishing on $E \setminus G_a$ such that*

$$(u,u)_{L^2(E;\mu)} = 1 \quad \mathcal{E}(u,u) \le 1.$$

Proof. We see from (1.6) that

$$r_a = \sup\{\beta(G_a) : \beta = v^2 d\mu,\ v \in \mathcal{F}, (v,v)_{L^2(E;\mu)} \le 1,\ \mathcal{E}(v,v) \le 1\}.$$

Suppose $r_a = 1$. We can find $v_n \in \mathcal{F}$ such that

$$(v_n, v_n) \le 1,\ \lim_{n\to\infty} \int_{G_a} v_n^2\, d\mu = 1,\ \mathcal{E}(v_n, v_n) \le 1.$$

Then there exists a subsequence $\{v_k\}$ of $\{v_n\}$ such that $v_k^2\mu$ vaguely converges to $\nu \in \mathcal{M}$ and $\nu(G_a) \ge \limsup_{k\to\infty} \int_{G_a} v_k^2 d\mu = 1$. Moreover, by the lower semicontinuity of $I_{\mathcal{E}}$, $I_{\mathcal{E}}(\nu) \le \liminf_{k\to\infty} \mathcal{E}(v_k, v_k) \le 1$ and thus ν can be written as $\nu = u^2\mu$, $u \in \mathcal{F}$. Since u is continuous, it vanishes on $E \setminus G_a$ identically. The 'if' part is clear. □

Let κ_a be the smallest eigenvalue of the Dirichlet form $(\mathcal{E}, \mathcal{F}^0_{G_a})$ the part of $\mathcal{E}$ on the open set G_a. Since

$$\kappa_a = \inf\{\mathcal{E}(u,u) : u \in \mathcal{F},\ u = 0 \text{ on } E \setminus G_a, (u,u)_{L^2(E;\mu)} = 1\},$$

we get from the above lemma that

Corollary 4.1. $r_a = 1$ *if and only if* $\kappa_a \le 1$.

On account of Lemma 2.3, we see that κ_a is strictly smaller(resp. greater) than 1 if $a > 0$ is large(resp. small). So the value a_0 defined by (1.12) is finite, strictly positive and

$$a_0 = \inf\{a \in D : \kappa_a \le 1\}. \tag{4.4}$$

Lemma 4.2. *The identity* (1.11) *is valid.*

Proof. κ_a can be seen to be strictly decreasing in a because a corresponding eigenfunction is striclty positive on G_a by virtue of Lemma 2.3. Take any $a' > a > a_0$. Then $\kappa_a < 1$ and accordingly there exists a function $u \in \mathcal{F}$ vanishing on $E \setminus G_a$ such that

$$(u, u)_{L^2(E;\mu)} = 1, \quad \mathcal{E}(u, u) < 1.$$

Hence the measure $\beta = u^2 \cdot \mu$ satisfies conditions in Proposition 3.2 for $G = G_a$ and in particular

$$\tau_{\alpha^m \cdot G_{a'}} > t_m \text{ infnitely often, } P_x\text{−a.e. } \omega, \ x \in E,$$

in other words

$$\liminf_{m\to\infty} \alpha^{-m} \sup_{0\le s\le t_m} |X_s| \le a' \quad P_x\text{−a.e. } \omega, \ x \in E.$$

On the other hand, for any $a < a_0$, $\kappa_a > 1$ and hence $r_a < 1 - \delta$ for some $\delta > 0$ by Corollary 4.1. By virtue of (4.3), there exists a positive integer N_0 such that

$$\frac{1}{t_m}\text{meas}\{s \in [0, t_m] : X_s \in \alpha^m \cdot G_a\} < 1 - \delta, \ \forall m > N_0, \ P_x\text{−a.e.},$$

which means

$$\alpha^{-m} \sup_{0\le s\le t_m} |X_s| > a, \ \forall m > N_0, \ P_x\text{−a.e.}$$

Therefore

$$\liminf_{m\to\infty} \alpha^{-m} \sup_{0\le s\le t_m} |X_s| \ge a. \qquad \square$$

Theorem 4.1. *The identity* (1.13) *holds for a positive constant* a_{00} *satisfying* (1.14).

Proof. The random variable appearing in the left hand side of (1.13) is measurable with respect to the tail σ-field $\cap_t \sigma\{X_u; u \ge t\}$. Owing to Theorem 2.2, it is a constant a_{00} P_x-a.e. for any $x \in E$. If $t \in (t_{m-1}, t_m]$ and t is large, then

$$\frac{1}{\alpha^m} \sup_{0\le s\le t_{m-1}} |X_s| \le \left(\frac{\log\log t}{t}\right)^{\gamma} \sup_{0\le s\le t} |X_s| \le \frac{1}{\alpha^{m-1}} \sup_{0\le s\le t_m} |X_s|,$$

which, combined with the preceding lemma, leads us to the bound (1.14) of the constant a_{00}. □

Let $V(x)$ be a continuous function on E with $V(x) \to \infty$ as $x \to \infty$. As in [11], one can use Theorem 3.2 to get asymptotics of additive functionals of type

$$A_t = \int_0^t V(X_s)ds.$$

We prepare a lemma. Denoting by $\|\cdot\|_2$ the norm in $L^2(E;\mu)$, we set

$$\mathcal{A} = \{f \in \mathcal{F} : \mathcal{E}(f,f) \leq 1, \ \|f\|_2 = 1\} \text{ and}$$
$$\mathcal{A}_0 = \{f \in \mathcal{F} \cap C_0(E) : \mathcal{E}(f,f) < 1, \ \|f\|_2 = 1\}.$$

Lemma 4.3.

$$\inf_{\phi \in \mathcal{A}} \int_E V\phi^2 d\mu = \inf_{\phi \in \mathcal{A}_0} \int_E V\phi^2 d\mu.$$

Proof. Let ψ be a function in $\mathcal{A}_0$. Then, for any $\phi \in \mathcal{A}$ and $0 < \epsilon < 1$,

$$I_{\mathcal{E}}\left(\left((1-\epsilon)\phi^2 + \epsilon\psi^2\right)\mu\right) \leq (1-\epsilon)I_{\mathcal{E}}(\phi^2\mu) + \epsilon I_{\mathcal{E}}(\psi^2\mu) < 1$$

and

$$\lim_{\epsilon \to 0} \int_E V\left((1-\epsilon)\phi^2 + \epsilon\psi^2\right) d\mu = \int_E V\phi^2 d\mu.$$

Hence, we obtain

$$\inf_{\phi \in \mathcal{A}} \int_E V\phi^2 d\mu = \inf_{\substack{\phi \in \mathcal{A} \\ \mathcal{E}(\phi,\phi)<1}} \int_E V\phi^2 d\mu.$$

Since $(\mathcal{E}, \mathcal{F})$ is recurrent, there exists a sequence $\{f_n\}_{n=1}^{\infty} \subset \mathcal{F} \cap C_0(E)$ such that $0 \leq f_n \leq 1$, $\mathcal{E}(f_n, f_n) \to 0$ and $f_n \to 1$ μ-a.e. as $n \to \infty$ ([16,Theorem 1.6.5]). By selecting a subsequence if necessary, we may assume that $\lim_{n\to\infty} n^2\mathcal{E}(f_n, f_n) = 0$. For any $\varphi \in \mathcal{A}$ with $\mathcal{E}(\varphi,\varphi) < 1$, let $\psi^{(n)} = ((-n) \vee \varphi) \wedge n$ and $\psi_n = \varphi^{(n)} f_n/m_n$ ($m_n = \|\varphi^{(n)} f_n\|_2$). Then, by Theorem 1.4.2 (ii) in [16]

$$\limsup_{n\to\infty} \mathcal{E}(\psi_n,\psi_n)^{1/2} = \limsup_{n\to\infty} \frac{1}{m_n}\mathcal{E}(\varphi^{(n)} f_n, \varphi^{(n)} f_n)^{1/2}$$
$$\leq \limsup_{n\to\infty} \frac{1}{m_n}\left(n\mathcal{E}(f_n, f_n)^{1/2} + \mathcal{E}(\varphi,\varphi)^{1/2}\right) < 1.$$

Noting that $\lim_{n\to\infty} \int_E V\psi_n^2 d\mu = \int_E V\varphi^2 d\mu$, we obtain the lemma. □

Using Proposition 3.2 and Lemma 4.3, we obtain, by the same argument as in Example 2 of [11],

$$\liminf_{m\to\infty} \frac{1}{t_m} \int_0^{t_m} V\left(X_s\left(\frac{\log\log t_m}{t_m}\right)^{\gamma}\right) ds = \inf_{f\in\mathcal{A}} \int_E V f^2 d\mu. \tag{4.5}$$

Let us consider a special case when $V(x) = |x|^\theta$.

Lemma 4.4. *For $\theta > 0$,*

$$\inf_{f\in\mathcal{A}} \int_E |x|^\theta f^2 d\mu > 0.$$

Proof. Suppose that $\inf_{f\in\mathcal{A}} \int_E |x|^\theta f^2 d\mu = 0$. Then there exists a sequence $\{f_n\} \subset \mathcal{A}$ such that $\lim_{n\to\infty} \int_E |x|^\theta f_n^2 d\mu \to 0$ and $f_n \to 0$ μ-a.e. as $n \to \infty$. Let $K = \{|x| \le R\} \cap E$ $(R > 0)$. We then see from Lemma 4.5 below that f_n^2 converges to 0 in $L^1(K;\mu)$ and thus $\lim_{n\to\infty} \int_{K^c} f_n^2 d\mu = 1$. Therefore,

$$0 = \lim_{n\to\infty} \int_E |x|^\theta f_n^2 d\mu \ge R^\theta \lim_{n\to\infty} \int_{K^c} f_n^2 d\mu = R^\theta,$$

which is contradictory. □

Lemma 4.5. *For any compact set $K \subset E$, $\{f^2\}_{f\in\mathcal{A}}$ is uniformly integrable on K.*

Proof. Since $p_t(x,y) \le c_{22} t^{-d_s/2}$ by (2.18),

$$\|p_t\|_{\infty,2} \le e^{M(t)}, \ M(t) = (1/2)\log c_{22} - (d_s/4)\log t,$$

by Lemma 2.1.2 in [6], and for any $f \in \mathcal{F}$ and $\epsilon > 0$

$$\frac{1}{2}\int_E f^2 \log_+ f^2 d\mu \le \epsilon\mathcal{E}(f,f) + (M(\epsilon/4)+2)\|f\|_2^2 + \|f\|_2^2 \log\|f\|_2$$

by virtue of Theorem 2.2.4 in [6], where $\|p_t\|_{\infty,2}$ denotes the norm of the operator p_t from $L^2(E;\mu)$ to $L^\infty(E;\mu)$. In particular,

$$\sup_{f\in\mathcal{A}} \int_E f^2 \log_+ f^2 d\mu \le 2\epsilon + 2M(\epsilon/4) + 4 < \infty.$$

Hence, Theorem 22 in [7] leads us to this lemma. □

We now obtain the next theorem.

Theorem 4.2. *For any $\theta > 0$, it holds that*

$$\liminf_{t\to\infty} \frac{(\log\log)^{\theta\gamma}}{t^{1+\theta\gamma}} \int_0^t |X_s|^\theta ds = A'_\theta.$$

for a constant A'_θ satisfying (1.16).

Proof. Put

$$A_\theta = \inf_{f\in\mathcal{A}} \int_E |x|^\theta f^2 d\mu.$$

We then see from (4.5) that

$$\liminf_{m\to\infty} L_m = A_\theta \quad P_x\text{-a.e. } \forall x \in E,$$

where

$$L_m = \frac{(\log\log t_m)^{\gamma\theta}}{t_m{}^{1+\gamma\theta}} \int_0^{t_m} |X_s|^\theta ds.$$

If $t \in (t_{m-1}, t_m]$ and t is large, then

$$\alpha^{-\theta}\frac{t_{m-1}}{t_m} L_m \le \frac{(\log\log t)^{\gamma\theta}}{t^{1+\gamma\theta}} \int_0^t |X_s|^\theta ds \le \alpha^\theta \frac{t_m}{t_{m-1}} L_m$$

and, by lemma 3.1 (i),

$$\alpha^{-\theta-d_w} A_\theta \le \liminf_{t\to\infty} \frac{(\log\log t)^{\gamma\theta}}{t^{1+\gamma\theta}} \int_0^t |X_s|^\theta ds \le \alpha^{\theta+d_w} A_\theta.$$

which leads us to the bound (1.16) of the constant A'_θ on accounts of Theorem 2.2 and Lemma 4.4. □

5. Law of the iterated logarithm for local times

In this section, we aim at establishing (1.19) by showing ,just as in [11], the equi-uniform continuity of $\hat{\ell}_{t_m}(\omega, \cdot)$ as m tends to infinity. We need to prepare a series of lemmas to this end. $\tilde{d}_w$ (resp. $\tilde{d}_f$) will denote the walk (resp. Hausdorff) dimension with respect to the intrinsic metric d ([13]). we also let $\tilde{\gamma} = 1/\tilde{d}_w$.

Lemma 5.1. *Let $\phi(x)$ be a function on E with a compact support, and $2h$ a diameter of the support. We further suppose $\int_E \phi d\mu = 0$ and $\int_E |\phi| d\mu = e < \infty$. Fix*

$C > 0$ and $\theta \in (0, \tilde{d}_w(1 - d_s/2) \wedge \tilde{d}_w/(\tilde{d}_w - 1))$. Then, for every positive integer n, we have

$$\sup_{\substack{u\in[0,C]\\ x\in E}} E_x[(\int_0^u \phi(X_s)ds)^{2n}] \leq \frac{(2n)!B^n h^{\theta n} C^{\tau n}}{\Gamma(1+n\tau)},$$

where $\tau = 2 - \tilde{\gamma}\theta - d_s$, and B is a constant depending only on e and θ.

Proof. We first note that $0 < \tilde{d}_w(1 - d_s/2) \wedge \tilde{d}_w/(\tilde{d}_w - 1)$ since $d_s < 2$, $\tilde{d}_w > 1$ ([15],[13]). Let $\Xi(x) = c_{22}\exp(-c_{23}|x|^{\tilde{d}_w/(\tilde{d}_w-1)})$. At (2.18), we have already noted that $t^{-d_s/2}\Xi(d(x,y)/t^{\tilde{\gamma}})$ bounds the transition density function from above. Thus, we have

$$\begin{aligned}
&E_x\left[\left(\int_0^u \phi(X_s)ds\right)^{2n}\right]\\
&\leq (2n)! \int_{0\leq s_1\leq\cdots\leq s_{2n}\leq u}\int_{E^{2n}} ds_1\cdots ds_{2n}\mu(dx_1)\cdots\mu(dx_{2n})\\
&\prod_{i=1}^{2n}\frac{\phi(x_i)}{(s_i - s_{i-1})^{d_s/2}}\prod_{j=0}^{n-1}\left\{\Xi\left(\frac{d(x_{2j+1},x_{2j})}{(s_{2j+1}-s_{2j})^{\tilde{\gamma}}}\right)\Xi\left(\frac{d(x_{2j+2},x_{2j+1})}{(s_{2j+2}-s_{2j+1})^{\tilde{\gamma}}}\right) - \Xi^2(0)\right\},
\end{aligned}$$

where $s_0 = 0$ and $x_0 = x$, and we used the assumption $\int_E \phi d\mu = 0$. We note that $|\Xi(x)| \leq c_{22}$, and $|\Xi(x) - \Xi(0)| \leq (c_{22} \vee c_{22}c_{23})|x|^\theta$. Hence, we have

$$\begin{aligned}
&\left|\Xi\left(\frac{d(x_{2j+1},x_{2j})}{(s_{2j+1}-s_{2j})^{\tilde{\gamma}}}\right)\Xi\left(\frac{d(x_{2j+2},x_{2j+1})}{(s_{2j+2}-s_{2j+1})^{\tilde{\gamma}}}\right) - \Xi^2(0)\right|\\
&\leq c_{22}K\left\{\left|\frac{d(x_{2j+1},x_{2j})}{(s_{2j+1}-s_{2j})^{\tilde{\gamma}}}\right|^\theta + \left|\frac{d(x_{2j+2},x_{2j+1})}{(s_{2j+2}-s_{2j+1})^{\tilde{\gamma}}}\right|^\theta\right\},
\end{aligned}$$

where $K = c_{22} \vee c_{22}c_{23}$. Using this and hypotheses, we obtain that

$$\begin{aligned}
&E_x\left[\left(\int_0^u \phi(X_s)ds\right)^{2n}\right] \leq (2n)!(e^2 c_{22}(2h)^\theta K)^n \int_{0\leq S_1\leq\cdots\leq S_{2n}\leq C} ds_1\cdots ds_{2n}\\
&\prod_{k=0}^{n-1}\frac{1}{|s_{2k+1}-s_{2k}|^{d_s/2}}\frac{1}{|s_{2k+2}-s_{2k+1}|^{d_s/2}}\left\{\frac{1}{|s_{2k+1}-s_{2k}|^{\theta\tilde{\gamma}}} + \frac{1}{|s_{2k+2}-s_{2k+1}|^{\theta\tilde{\gamma}}}\right\}.
\end{aligned}$$

From this, through the same computation as [11], we arrive at the desired estimate.

□

Lemma 5.2. *Let ϕ, θ and C be as in Lemma 5.1. Then there exists a positive constant $c_{51} = c_{51}(e, C, \theta)$ such that*

$$\sup_{\substack{u\in[0,C]\\x\in E}} E_x\left[\exp\left\{c_{51}\left|\frac{1}{h^{\theta/2}}\int_0^u \phi(X_s)ds\right|^\rho\right\}\right] < \infty,$$

where $\rho = 2/(2-\tau)$.

Proof. For $k > 0$,

$$\begin{aligned} E_x\left[\exp\left\{k\left|\int_0^u \phi(X_s)ds\right|^\rho\right\}\right] &= \sum_{n=0}^\infty \frac{k^n}{n!}E_x\left[\left|\int_0^u \phi(X_s)ds\right|^{n\rho}\right] \\ &= \sum_{n=0}^\infty \frac{k^n}{n!}\left\{E_x\left[\left|\int_0^u \phi(X_s)ds\right|^{n\rho\cdot 2/\rho}\right]\right\}^{\rho/2}. \end{aligned}$$

We here used Hölder's inequality since $2/\rho > 1$, which comes from the fact $d_s \geq 1$ (see [23]). Consequently we get

$$E_x\left[\exp\left\{k\left|\int_0^u \phi(X_s)ds\right|^\rho\right\}\right] \leq \sum_{n=0}^\infty \frac{k^n}{n!}\left\{\frac{(2n)!B^n h^{\theta n}C^{\tau n}}{\Gamma(1+n\tau)}\right\}^{\rho/2}. \tag{5.1}$$

Let $k = k_0 h^{-\rho\theta/2}$. Then, for sufficiently large n,

$$k^n((2n)!)^{\rho/2}B^{\rho n/2}h^{\rho\theta n/2}C^{\tau\rho n/2}/(n!\Gamma(1+n\tau)^{\rho/2})$$

behaves like $n^{-1/2}(k_0 B^{\rho/2}2^\rho C^{\rho-1}\tau^{1-\rho})^n$. Hence, for sufficiently small k_0, the series on the right of (5.1) converges. □

In the following, we deal with $\ell'_t(\omega,x) = \ell_t(\omega,x)/t$ instead of the local time $\ell_t(\omega,x)$ (see (2.16)).

Lemma 5.3. *Let θ and ρ be as in Lemma 5.1 and 5.2, and fix $C > 1$. Then there exists $c_{52} = c_{52}(\theta, C)$ such that, for every $u \in [1,C]$, $x \in E$ and P_x almost all ω,*

$$\iint_{d(y_1,y_2)\leq 1}\left[\exp\left\{c_{52}\left|\frac{\ell'_u(\omega,y_1)-\ell'_u(\omega,y_2)}{d(y_1,y_2)^{\theta/2}}\right|^\rho\right\}-1\right]\mu(dy_1)\mu(dy_2)$$
$$= F(u,\omega) < \infty.$$

Furthermore, $\sup_{u\in[1,C],\,x\in E} E_x[F(u,\omega)] < \infty$

Proof. We call the set, $\alpha^m \tilde{E}_{i_1,\cdots,i_{m+n}}$ $(m+n \geq 0)$, n-complex. For $x \in E$, let $\Delta_n(x)$ be the union of all n-complexes including x. Let $\hat{F}^{(n)}$ be the union of $\alpha^m F^{(n+m)}$ over the integer m satisfying $n+m \geq 0$. Take $p \in \hat{F}^{(n)}$ arbitrary, and fix $y_1,\ y_2 \in \Delta_n(p)$. We then have, by strong Markov property,

$$\begin{aligned} &E_x\left[\exp\left\{k\left|\frac{\ell'_u(\omega,y_1)-\ell'_u(\omega,y_2)}{d(y_1,y_2)^{\theta/2}}\right|^\rho\right\}-1\right] \\ &= E_x\left[1_{\{\tau_p<u\}}(\omega)E_{X_{\tau_p}(\omega)}\left[\exp\left\{\frac{k}{u^\rho}\right.\right.\right. \\ &\quad \times\left.\left.\left.\left|\frac{\ell_{u-\tau_p(\omega)}(\omega',y_1)-\ell_{u-\tau_p(\omega)}(\omega',y_2)}{d(y_1,y_2)^{\theta/2}}\right|^\rho\right\}-1\right]\right], \end{aligned}$$

where $\tau_p(\omega)=\inf\{t\geq 0:\ X_t(\omega)\in\Delta_n(p)\}$. If $\tau_p(\omega)<u$, then, for almost all ω',

$$\ell_{u-\tau_p(\omega)}(\omega',y_i)=\lim_{m\to\infty}\frac{1}{\mu(\Delta_m(y_i))}\int_0^{u-\tau_p(\omega)}1_{\Delta_m(y_i)}(X_s(\omega'))ds, \qquad i=1,\,2.$$

Let $\phi_m(x)=1_{\Delta_m(y_1)}(x)/\mu(\Delta_m(y_1))-1_{\Delta_m(y_2)}(x)/\mu(\Delta_m(y_2))$. Using Fatou's lemma and the assumption, $u\geq 1$, we have

$$\begin{aligned} &E_{X_{\tau_p}(\omega)}\left[\exp\left\{\frac{k}{u^\rho}\left|\frac{\ell_{u-\tau_p(\omega)}(\omega',y_1)-\ell_{u-\tau_p(\omega)}(\omega',y_2)}{d(y_1,y_2)^{\theta/2}}\right|^\rho\right\}-1\right] \\ &\leq \liminf_{m\to\infty} E_{X_{\tau_p}(\omega)}\left[\exp\left\{k\left|\frac{\int_0^{u-\tau_p(\omega)}\phi_m(X_s(\omega'))ds}{d(y_1,y_2)^{\theta/2}}\right|^\rho\right\}-1\right]. \end{aligned} \tag{5.2}$$

Since the diameter of supp(ϕ_m) approximates to $d(y_1,y_2)$ as $m\to\infty$, by virtue of Lemma 5.2, there exists k such that the right hand side of (5.2) is bounded by a deterministic constant K_1 independent of u and p. We hence conclude that

$$E_x\left[\exp\left\{k\left|\frac{\ell'_u(\omega,y_1)-\ell'_u(\omega,y_2)}{d(y_1,y_2)^{\theta/2}}\right|^\rho\right\}-1\right]\leq K_1P_x(\tau_p<u).$$

Let $\delta_0=\min\{d(x,y):\ x,y\in F,\ x\neq y\}$, and take an integer n_0 such that $\delta_0\xi^{n_0-1}\leq 1<\delta_0\xi^{n_0}$. If $d(y_1,y_2)\leq 1$, because of (2.19), we can find $p\in\hat{F}^{(-n_0)}$ such that $\Delta_{-n_0}(p)\ni y_1,\ y_2$. Therefore $\{(y_1,\ y_2)\in E\times E:\ d(y_1,y_2)\leq 1\}\subset\cup_{p\in\hat{F}^{(-n_0)}}\Delta_{-n_0}(p)\times\Delta_{-n_0}(p)$, and we have

$$\iint_{d(y_1,y_2)\leq 1}E_x\left[\exp\left\{k\left|\frac{\ell'_u(\omega,y_1)-\ell'_u(\omega,y_2)}{d(y_1,y_2)^{\theta/2}}\right|^\rho\right\}-1\right]\mu(dy_1)\mu(dy_2)$$

$$\leq \sum_{p\in\hat{F}^{(-n_0)}} \iint_{\Delta_{-n_0}(p)\times\Delta_{-n_0}(p)} E_x\left[\exp\left\{k\left|\frac{\ell'_u(\omega,y_1)-\ell'_u(\omega,y_2)}{d(y_1,y_2)^{\theta/2}}\right|^{\rho}\right\}-1\right]$$
$$\mu(dy_1)\mu(dy_2)$$
$$\leq K_2 \sum_{p\in\hat{F}^{(-n_0)}} P_x(\tau_p < C).$$

Finally, we verify that $\sum_p P_x(\tau_p < C) < \infty$. We then complete the proof by using Fubini's theorem.

For $p \in \hat{F}^{(n)}$, let $f_p^t(x) = P_x(\tau_p \leq t)$, and $g_p^t(x) = E_x[\int_0^{t+1} 1_{\widetilde{\Delta}_n(p)}(X_s)ds]$, where $\widetilde{\Delta}_n(p)$ is union of all n-complexes intersecting with $\Delta_n(p)$. We then have

$$\begin{aligned} g_p^t(x) &\geq E_x[\int_{\tau_p}^{\tau_p+1} 1_{\widetilde{\Delta}_n(p)}(X_s)ds;\ \tau_p \leq t] \\ &\geq E_x[\int_{\tau_p}^{\tau_p+1} 1_{\Delta_n(X_{\tau_p})}(X_s)ds;\ \tau_p \leq t] \\ &= E_x[E_{X_{\tau_p}}[\int_0^1 1_{\Delta_n(X_0)}(X_s)ds];\ \tau_p \leq t]. \end{aligned}$$

Clearly, $E_{X_{\tau_p}}[\int_0^1 1_{\Delta_n(X_0)}(X_s)ds]$ is bounded from below by some constant $c = c(n) > 0$. Hence, $g_p^t(x) \geq cf_p^t(x)$. Furthermore, since $\cup_p\widetilde{\Delta}_n(p)$ covers E finite times, we can take a constant $c' > 0$ such that

$$\begin{aligned} \sum_{p\in\hat{F}^{(n)}} g_p^t(x) &= E_x[\sum_{p\in\hat{F}^{(n)}} \int_{\widetilde{\Delta}_n(p)} \ell_{t+1}(y)\mu(dy)] \\ &\leq c'E_x[\int_E \ell_{t+1}(y)\mu(dy)] \\ &= c'(t+1). \end{aligned}$$

Combining these results, we get the estimate $\sum_p P_x(\tau_p \leq t) \leq c'(t+1)/c$. □

To prove Lemma 5.5, we introduce a version of Garsia's lemma, which is exactly the same as the one presented in [2,Lemma 6.1] except that we use a closed ball $B(x,r) = \{y \in E : d(x,y) \leq r\}$ with respect to the intrinsic metric d. The proof is also the same since, in our case, it still holds that there exists constants $c_1(E)$, $c_2(E)$ such that,

$$c_1(E)r^{\tilde{d}_f} \leq \mu(B(x,y)) \leq c_2(E)r^{\tilde{d}_f}.$$

Lemma 5.4. *Let p be an increasing function on $[0,\infty)$ with $p(0) = 0$, and ψ be a non-negative symmetric convex function, with $\lim_{u\to\infty}\psi(u) = \infty$. Let H be a compact* *subset in E containing a closed ball B, and let $f : H \to \mathbf{R}$ be a continuous function. Suppose that*

$$\Theta = \iint_{H\times H} \psi\left(\frac{|f(x)-f(y)|}{p(d(x,y))}\right)\mu(dx)\mu(dy) < \infty.$$

Then there exits a constant c_{53} depending only on E such that

$$|f(x)-f(y)| \le 8\int_0^{d(x,y)} \psi^{-1}\left(\frac{c_{53}\Theta}{u^{2\tilde{d}_f}}\right)p(du),$$

for every $(x,y) \in B \times B$.

Lemma 5.5. *Let θ and ρ be as in Lemma 5.1 and 5.2. Then there exist positive constants c_{54} and c_{55} independent of $x \in E$ such that, for all $a > 0$ and $1 > \delta > 0$,*

$$P_x\left\{\sup_{d(y_1,y_2)\le\delta} |\ell'_t(\omega,y_1) - \ell'_t(\omega,y_2)| \ge a\right\} \le c_{54}t^{d_s}\delta^{-2\tilde{d}_f}\exp[-c_{55}ta^\rho\delta^{-\rho\theta/2}].$$

Proof. Let $n(t)$ be the largest integer not greater than $\log t/\log\frac{N}{1-c}$, and $\zeta(t) = t/(\frac{N}{1-c})^{n(t)}$. Then, by Corollary 2.3, the distribution of the random variable $\ell'_t(\omega,y)$ under P_x is the same as that of $N^{-n(t)}\ell'_{\zeta(t)}(\omega,\alpha^{-n(t)}y)$ under $P_{x/\alpha^{n(t)}}$. We thus have

$$\begin{aligned} &P_x\left\{\sup_{d(y_1,y_2)\le\delta} |\ell'_t(\omega,y_1) - \ell'_t(\omega,y_2)| \ge a\right\} \\ &= P_{x/\alpha^{n(t)}}\left\{\sup_{d(y_1,y_2)\le\delta/\xi^{n(t)}} \left|\ell'_{\zeta(t)}(\omega,y_1) - \ell'_{\zeta(t)}(\omega,y_2)\right| \ge N^{n(t)}a\right\}. \end{aligned}$$

In the following, we will estimate the right hand side.

We first note that $1 \le \zeta(t) < \frac{N}{1-c}$. By setting $\frac{N}{1-c}$ for C in Lemma 5.3, the hypotheses of Lemma 5.4 is satisfied with $\psi(x) = \exp(c_{52}|x|^\rho) - 1$, $p(y) = y^{\theta/2}$ and with $f(x) = \ell'_{\zeta(t)}(\omega,x)$. Using Lemma 5.4, for every y_1, $y_2 \in E$ such as $d(y_1,y_2) \le \delta/\xi^{n(t)}$, we have

$$\begin{aligned} &\left[\ell'_{\zeta(t)}(\omega,y_1) - \ell'_{\zeta(t)}(\omega,y_2)\right] \le 8\int_0^{\delta/\xi^{n(t)}} \left\{\frac{1}{c_{52}}\log\left(\frac{c_{53}\Theta}{u^{2\tilde{d}_f}}+1\right)\right\}^{1/\rho}\frac{\theta}{2}u^{\theta/2-1}du \\ &\le \frac{4\theta}{c_{52}^{1/\rho}}\int_0^{\delta/\zeta^{n(t)}} \left\{\log\left(\frac{c_{53}F(\zeta(t),\omega)}{u^{2\tilde{d}_f}}+1\right)\right\}^{1/\rho} u^{\theta/2-1}du \\ &= \frac{4\theta D}{c_{52}^{1/\rho}}\int_b^\infty \frac{[\log(v+1)]^{1/\rho}}{v^q}dv, \end{aligned}$$

where $D = (c_{53}F(\zeta(t),\omega))^{\theta/4\tilde{d}_f}/2\tilde{d}_f$, $b = c_{53}F(\zeta(t),\omega)(\xi^{n(t)}/\delta)^{2\tilde{d}_f}$ and $q = 1 + \theta/4\tilde{d}_f$. Since $\int_b^\infty [\log(v+1)]^{1/\rho}/v^q dv \le (\log(b+1) + 1/(q-1))^{1/\rho}/((q-1)b^{q-1})$, we conclude that

$$\sup_{d(y_1,y_2)\le\delta/\xi^{n(t)}} \left|\ell'_{\zeta(t)}(\omega,y_1) - \ell'_{\zeta(t)}(\omega,y_2)\right| \le 8\left(\frac{\delta}{\xi^{n(t)}}\right)^{\theta/2}\left(\frac{\log(b+1) + 4\tilde{d}_f/\theta}{c_{52}}\right)^{1/\rho}.$$

From this, it follows that

$$\begin{aligned} & P_{x/\alpha^{n(t)}}\left\{\sup_{d(y_1,y_2)\le\delta/\xi^{n(t)}} \left|\ell'_{\zeta(t)}(\omega,y_1) - \ell'_{\zeta(t)}(\omega,y_2)\right| \ge N^{n(t)}a\right\} \\ \le\ & P_{x/\alpha^{n(t)}}\left\{\log(b+1) \ge c_{52}N^{n(t)\rho}(\frac{a}{8})^\rho(\frac{\xi^{n(t)}}{\delta})^{\rho\theta/2} - \frac{4\tilde{d}_f}{\theta}\right\}. \end{aligned}$$

Note that $N^{n(t)\rho}\xi^{n(t)\rho\theta/2} = (N/(1-c))^{n(t)}$ since

$$\tilde{d}_w = \frac{\log N - \log(1-c)}{\log\xi}, \quad \tilde{d}_f = \frac{\log N}{\log\xi}.$$

Then, we get the following estimate:

$$\begin{aligned} & P_x\left\{\sup_{d(y_1,y_2)\le\delta} |\ell'_t(\omega,y_1) - \ell'_t(\omega,y_2)| \ge a\right\} \\ \le\ & C_1\left(\frac{\xi^{n(t)}}{\delta}\right)^{2\tilde{d}_f} \exp\left\{-C_2a^\rho\left(\frac{N}{1-c}\right)^{n(t)}\delta^{-\rho\theta/2}\right\} E_{x/\alpha^{n(t)}}[F(\zeta(t),\omega)]. \end{aligned}$$

Since $\sup_{u\in[1,N/(1-c)],\,x\in E} E_x[F(u,\omega)] < \infty$, and $(N/(1-c))^{n(t)} \le t < (N/(1-c))^{n(t)+1}$, we have the lemma. □

We have defined an increasing sequence $t_m \to \infty$ as a unique solution of (1.9). Let

$$\hat{\ell}_{t_m}(\omega,y) = N^m\ell'_{t_m}(\omega,\alpha^m y).$$

Theorem 5.1. *For each $a > 0$, there exists $\delta > 0$ such that*

$$P_x\left\{\limsup_{m\to\infty}\sup_{d(y_1,y_2)\le\delta} |\hat{\ell}_{t_m}(\omega,y_1) - \hat{\ell}_{t_m}(\omega,y_2)| \ge a\right\} = 0,$$

for every $x \in E$.

Proof. On account of the Borel-Cantelli lemma, it suffices to prove

$$\sum_m P_x \left\{ \sup_{d(y_1,y_2)\le\delta} \left|\hat{\ell}_{t_m}(\omega, y_1) - \hat{\ell}_{t_m}(\omega, y_2)\right| > a \right\} < \infty. \tag{5.3}$$

We can see from Corollary 2.3 that $\hat{\ell}_{t_m}(\omega, y)$ under P_x has the same distribution as $\ell'_{\log\log t_m}(\omega, y)$ under P_{x/α^m}. Using Lemma 5.5, we get the following estimate:

$$\begin{aligned} & P_x \left\{ \sup_{d(y_1,y_2)\le\delta} \left|\hat{\ell}_{t_m}(\omega, y_1) - \hat{\ell}_{t_m}(\omega, y_2)\right| > a \right\} \\ \le\ & P_{x/\alpha^m} \left\{ \sup_{d(y_1,y_2)\le\delta} \left|\ell'_{\log\log t_m}(\omega, y_1) - \ell'_{\log\log t_m}(\omega, y_2)\right| \ge a \right\} \\ \le\ & c_{54}(\log\log t_m)^{d_s}\delta^{-2\tilde{d}_f}(\log t_m)^{-c_{55}a^\rho/\delta^{\rho\theta/2}}. \end{aligned}$$

Because of Lemma 3.1, $\log t_m = O(m)$ and $\log\log t_m = O(\log m)$. Hence, for any $a > 0$, we can find $\delta > 0$ which makes the series (5.3) converge. □

Let $\mathcal{A}$ be the totality of non-negative uniformly continuous functions f on E with $\int_E f d\mu \le 1$. The space $\mathcal{A}$ is equipped with the topology of uniform convergence on compact subsets of E. For $f \in \mathcal{A}$, we denote $I_\mathcal{E}(f d\mu)$ by $I_\mathcal{E}(f)$.

Theorem 5.2. *For P_x- a.e.,*

$$\bigcap_N \overline{\bigcup_{m\ge N} \{\hat{\ell}_{t_m}(\omega, \cdot)\}} = \{ f \in \mathcal{A} : I_\mathcal{E}(f) \le 1\}.$$

Proof. Let $\mathcal{G}$ be a continuous one to one map from $\mathcal{A}$ into $\mathcal{M}$ defined by $\mathcal{G}(f)(B) = \int_B f d\mu$ for any Borel set B. We note that $\hat{\ell}_{t_m} \in \mathcal{A}$, and $\mathcal{G}(\hat{\ell}_{t_m}(\omega, \cdot)) = \hat{L}_{t_m}(\omega, \cdot)$. Because of the continuity of $\mathcal{G}$, we see that

$$\mathcal{G}\left(\cap_N \overline{\cup_{m\ge N}\{\hat{\ell}_{t_m}(\omega, \cdot)\}}\right) \subset \cap_N \overline{\cup_{m\ge N}\{\hat{L}_{t_m}(\omega, \cdot)\}}.$$

We hence obtain $\cap_N \overline{\cup_{m\ge N}\{\hat{\ell}_{t_m}(\omega, \cdot)\}} \subset \{f \in \mathcal{A} : I_\mathcal{E}(f) \le 1\}$ from Theorem 3.1. Let $\{a_n\}$ be a sequence such that $a_n \searrow 0$. On account of Theorem 5.1, for each a_n, we can find $\delta_n > 0$ and a set Ω_n with $P_x(\Omega_n) = 1$ such that, for $\omega \in \Omega_n$,

$$\limsup_{m\to\infty} \sup_{d(y_1,y_2)\le\delta_n} \left|\hat{\ell}_{t_m}(\omega, y_1) - \hat{\ell}_{t_m}(\omega, y_2)\right| < a_n.$$

Therefore, for $\omega \in \cap_n \Omega$, $\overline{\cup_{m\geq N}\{\hat{\ell}_{t_m}(\omega,\cdot)\}}$ is uniformly bounded and equicontinuous. By Ascoli–Arzella's theorem, $\overline{\cup_{m\geq N}\{\hat{\ell}_{t_m}(\omega,\cdot)\}}$ is compact, and hence $\mathcal{G}\left(\overline{\cup_{m\geq N}\{\hat{\ell}_{t_m}(\omega,\cdot)\}}\right) \supset \overline{\cup_{m\geq N}\{\hat{L}_{t_m}(\omega,\cdot)\}}$. Using Theorem 3.1, we have

$$\mathcal{G}\left(\cap_N \overline{\cup_{m\geq N}\{\hat{\ell}_{t_m}(\omega,\cdot)\}}\right) \supset \{\beta \in \mathcal{M} : I_{\mathcal{E}}(\beta) \leq 1\}.$$

□

The following corollary is obvious from Theorem 5.2.

Corollary 5.1. *For a lower semicontinuous functional Φ on $\mathcal{A}$,*

$$\limsup_{m\to\infty} \Phi(\hat{\ell}_{t_m}(\omega,\cdot)) \geq \sup_{f\in\{f\in\mathcal{A}:\, I_{\mathcal{E}}(f)\leq 1\}} \Phi(f), \qquad P_x\text{- a.e.},$$

and for an upper semicontinuous functional Φ on $\mathcal{A}$,

$$\limsup_{m\to\infty} \Phi(\hat{\ell}_{t_m}(\omega,\cdot)) \leq \sup_{f\in\{f\in\mathcal{A}:\, I_{\mathcal{E}}(f)\leq 1\}} \Phi(f), \qquad P_x\text{- a.e.}.$$

Now we show the validity of (1.19) by using Corollary 5.1. We first prepare the following lemma.

Lemma 5.6. *Let b_0 be defined by* (1.20). *Denote by c_0 (resp. $c_{\ell,0}$) the 1-capacity of the one point set $\{0\}$ with respect to the Dirichlet form $(\mathcal{E},\mathcal{F})$ (resp. its part on the set $E_0^{\langle\ell\rangle}$, see §2). Then we have for any positive integer ℓ*

$$\frac{1}{c_{\ell,0}} \leq b_0 \leq \frac{2}{c_0}.$$

Proof. We first note the implication:

$$\inf\{\mathcal{E}_1(f,f) : f\in\mathcal{F},\, f(0)\geq\sqrt{a}\} \geq 2 \Rightarrow b_0 \leq a.$$

Indeed, if $b_0 > a$, there exists $f\in\mathcal{A}$ such that $\mathcal{E}_1(\sqrt{f},\sqrt{f}) \leq 2$ and $\sqrt{f(0)} > \sqrt{a}$, which means that $\inf\{\mathcal{E}_1(f,f) : f\in\mathcal{F},\, f(0)\geq\sqrt{a}\} < 2$. Obviously, $\mathcal{E}_1(f,f)\geq 2$ for any $f\in\mathcal{F}$ such that $f(0)\geq\sqrt{2/c_0}$, and hence we get that $b_0\leq 2/c_0$.

For the lower bound, consider the 1-equilibrium potential $e_{\ell,0}$ of the one point set $\{0\}$ relative to the part of the Dirichlet form $\mathcal{E}$ on the set $E_0^{\langle\ell\rangle}$. $e_{\ell,0}$ is continuous, vanishing outside $E_0^{\langle\ell\rangle}$ and consequently uniformly continuous on E. Further $\mathcal{E}_1(e_{\ell,0}/\sqrt{c_{\ell,0}}, e_{\ell,0}/\sqrt{c_{\ell,0}}) = 1$. Therefore $b_0 \geq e_{\ell,0}(0)^2/c_{\ell,0} = 1/c_{\ell,0}$. □

Theorem 5.3. *The identity* (1.19) *is valid.* (1.21) *holds for a positive constant* b_{00} *satifying* (1.22).

Proof. (1.19) is obtained by applying Corollary 5.1 to the functional $\Phi(f) = f(0)$ for $f \in \mathcal{A}$. To prove the second assertion, we put

$$w_t(\omega) = \left(\frac{t}{\log\log t}\right)^{d_s/2} \frac{1}{t}\ell_t(\omega, 0).$$

If $t \in (t_{m-1}, t_m]$ and t is large, then

$$\frac{t_{m-1}}{t_m} w_{t_{m-1}}(\omega) \leq w_t(\omega) \leq \frac{t_m}{t_{m-1}} w_{t_m}(\omega).$$

It then suffices to use (1.19), Lemma 3.1 and the $0-1$ law (Theorem 2.2). □

6. Uniform bounds in the large deviation principle

Let X be a locally compact separable metric space and m a positive Radon measure on X with full support. Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet space on $L^2(X; m)$ and $\mathbf{M} = (\Omega, X_t, \mathcal{F}, \mathcal{F}_t, P_x, \zeta)$ a corresponding m-symmetric Hunt process. Here $\mathcal{F}_t$ is the minimum completed admissible filtration and ζ is the lifetime. Let us denote by $\{p_t\}_{t\geq 0}$ the semi-group associated to $\mathbf{M}$, i.e., $p_t f(x) = E_x(f(X_t))$. Throughout this section, we make following assumptions for $\mathbf{M}$:

$\mathbf{A}_1$ (Irreducibility) If a Borel set A is p_t-invariant, i.e., $p_t(\chi_A f)(x) = \chi_A p_t f(x)$ m-a.e. x for any $f \in L^2(X; m) \cap \mathcal{B}(X)$ and $t > 0$, then A satisfies either $m(A) = 0$ or $m(X \setminus A) = 0$. Here $\mathcal{B}(X)$ is the space of Borel functions on X.

$\mathbf{A}_2$ (Feller property) $p_t(C_\infty(X)) \subset C_\infty(X)$, where $C_\infty(X)$ is the space of continuous functions vanishing at infinity.

$\mathbf{A}_3$ (Strong Feller property) $p_t(\mathcal{B}_b(X)) \subset C_b(X)$, where $\mathcal{B}_b(X)$ and $C_b(X)$ are spaces of bounded Borel functions and bounded continuous functions, respectively .

REMARK 6.1. (i) The symmetry of $\mathbf{M}$ and $\mathbf{A}_3$ imply the absolute continuity of transition function, $p_t(x, dy) = p_t(x, y)m(dy)$, for each $t > 0$ and $x \in X$.
(ii) By the right continuity of sample paths of $\mathbf{M}$, $p_t f(x)$, $f \in C_\infty(X)$, converges to $f(x)$ for each $x \in X$. Hence, $\mathbf{A}_2$ implies that the semigroup p_t is strongly continuous on $C_\infty(X)$ ([27]).
(iii) Due to the heat kernel bounds (2.18) and (2.20), the Brownian motion on the nested fractal satisfies $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$. By the same reason, the diffusion processes on Sierpinski carpets recently constructed by Barlow and Bass [1] also satisfy them.

Lemma 6.1. *For any Borel set $B \in \mathcal{B}(X)$ with* $\mathrm{Cap}(B) > 0$,

$$P_x(\sigma_B < \zeta) > 0 \quad \textit{for any } x \in X,$$

where $\sigma_B = \inf\{t > 0 : X_t \in B\}$.

Proof. By $\mathbf{A}_1$ and Theorem 4.6.6 in [16],

$$P_x(\sigma_B < \zeta) > 0 \qquad \text{for q.e. } x \in X.$$

Since a set of zero capacity is polar by Remark 6.1(i) and [16,Theorem 4.1.2],

$$\begin{aligned} P_x(\sigma_B < \zeta) &\geq P_x(\sigma_B(\theta_\epsilon) < \zeta) \\ &= \int_X p_\epsilon(x,y) P_y(\sigma_B < \zeta) m(dy) > 0 \end{aligned}$$

for any $x \in X$. □

Let us denote the resolvent $\{R_\alpha\}_{\alpha>0}$ of $\mathbf{M}$,

$$R_\alpha f(x) = E_x\left(\int_0^\zeta e^{-\alpha t} f(X_t)dt\right) \quad \text{for } f \in C_\infty(X),$$

and define the generator A by

$$Au = \alpha u - f \quad \text{for } u = R_\alpha f,\ f \in C_\infty(X).$$

Set

$$\mathcal{D}^+(A) = \left\{R_\alpha f:\ \alpha > 0,\ f \in C_\infty^+(X) \cap L^2(X;m), \text{ and } f \not\equiv 0\right\}.$$

Here $C_\infty^+(X)$ denotes the set of non-negative continuous functions in $C_\infty(X)$. Note that any function in $\mathcal{D}^+(A)$ is strictly positive. Indeed, Let $\phi = R_\alpha f \in \mathcal{D}^+(A)$. Since the set, $O = \{x \in X : f(x) > 0\}$, is a non-empty open set, $P_x(\sigma_O < \zeta) > 0$ for any $x \in X$ by Lemma 6.1. Hence, $R_\alpha f(x) > 0$ for any $x \in X$.

For $\phi = R_\alpha g \in \mathcal{D}^+(A)$, let $M_t^{[\phi]}$ denote the martingale additive functional

$$M_t^{[\phi]} = \phi(X_t) - \phi(X_0) - \int_0^t A\phi(X_s)ds.$$

and $N^{\phi,\epsilon}$ $(\epsilon \geq 0)$ the multiplicative functional defined by

$$N_t^{\phi,\epsilon} = \frac{\phi(X_t)+\epsilon}{\phi(X_0)+\epsilon}\exp\left(-\int_0^t \frac{A\phi}{\phi+\epsilon}(X_s)ds\right). \tag{6.1}$$

Let τ_n be the first leaving time from the set $F_n = \{x \in X : \phi(x) \geq 1/n\}$. Since, by Itō formula

$$N^{\phi,\epsilon}_{t\wedge\tau_n} - 1 = \frac{1}{\phi(X_0)+\epsilon}\int_0^{t\wedge\tau_n} \exp\left(-\int_0^s \frac{A\phi}{\phi+\epsilon}(X_u)du\right) dM^{[\phi]}_s,$$

we see

$$E_x(N^{\phi,\epsilon}_t; t<\zeta) \leq \liminf_{n\to\infty} E_x(N^{\phi,\epsilon}_{t\wedge\tau_n}) = 1. \tag{6.2}$$

Let $\mathbf{M}^\phi = (\Omega, X_t, P^\phi_x, \zeta)$ the transformed process of $\mathbf{M}$ by $N^{\phi,0}$. We then see from Lemma 6.3.1 and Theorem 6.3.2 in [16] that $\mathbf{M}^\phi$ is $\phi^2 m$-symmetric and conservative, $P^\phi_x(\zeta=\infty)=1$ for any $x \in X$. Moreover, we can show

Proposition 6.1. ([18],[30]) $\mathbf{M}^\phi$ *is ergodic in the sense that if* $\Lambda \in \mathcal{F}$ *is* θ_t*-invariant,* $(\theta_t)^{-1}(\Lambda) = \Lambda$*, then* $P^\phi_{\phi^2 m}(\Lambda) = 0$ *or* $P^\phi_{\phi^2 m}(\Omega\setminus\Lambda)=0$.

Let $\mathcal{M}$ be the set of positive measures μ on X with $\mu(X)\leq 1$. We equip $\mathcal{M}$ with the vague topology. Define the function $I_\mathcal{E}$ on $\mathcal{M}$ by

$$I_\mathcal{E}(\mu) = \begin{cases} \mathcal{E}(\sqrt{f},\sqrt{f}) & \text{if } \mu = f\cdot m,\ \sqrt{f}\in\mathcal{F} \\ \infty & \text{otherwise.}\end{cases}$$

For $\omega\in\Omega$ and $0<t<\zeta(\omega)$, we define the occupation distribution $L_t(\omega)$ by

$$L_t(\omega)(A) = \frac{1}{t}\int_0^t \chi_A(X_s(\omega))ds, \quad A\in\mathcal{B}(X).$$

Theorem 6.1. *Assume* $\mathbf{A}_2$ *and the absolute continuity of transition function. For any closed set* K *of* $\mathcal{M}$

$$\limsup_{t\to\infty}\frac{1}{t}\log\sup_{x\in X} P_x(L_t\in K, t<\zeta) \leq -\inf_{\mu\in K} I_\mathcal{E}(\mu).$$

REMARK 6.2. This theorem holds only by assuming the Feller property $\mathbf{A}_2$ and the absolute continuity of the transition function $p_t(x,\cdot)$with respect to m for each $t>0$ and $x\in X$.

Proof. Let $u\in\mathcal{D}^+(A)$. By virtue of (6.2), for any $\epsilon>0$

$$E_x\left(\exp\left(-\int_0^t \frac{Au}{u+\epsilon}(X_s)ds\right); t<\zeta\right) \leq \frac{u(x)+\epsilon}{\epsilon},$$

and thus, for any Borel set C of $\mathcal{M}$

$$\limsup_{t\to\infty}\frac{1}{t}\log\sup_{x\in X}P_x(L_t\in C, t<\zeta)\le\inf_{\substack{u\in\mathcal{D}^+(A)\\ \epsilon>0}}\sup_{\mu\in C}\int_X\frac{Au}{u+\epsilon}d\mu. \tag{6.3}$$

Let K be a closed set of $\mathcal{M}$ and set

$$\ell=\sup_{\mu\in K}\inf_{\substack{u\in\mathcal{D}^+(A)\\ \epsilon>0}}\int_X\frac{Au}{u+\epsilon}d\mu.$$

Then, for any $\delta>0$ and $\mu\in K$, there exist $u_\mu\in\mathcal{D}^+(A)$ and $\epsilon_\mu>0$ such that

$$\int_X\frac{Au_\mu}{u_\mu+\epsilon_\mu}d\mu\le\ell+\delta.$$

Since the function $\dfrac{Au_\mu}{u_\mu+\epsilon_\mu}$ belongs to $C_\infty(X)$, there exists a neighbourhood $N(\mu)$ of μ such that

$$\int_X\frac{Au_\mu}{u_\mu+\epsilon_\mu}d\nu\le\ell+2\delta\quad\text{for any }\nu\in N(\mu).$$

Since $\cup_{\mu\in K}N(\mu)$ is an open covering of K, there exist $\mu_1,\dots,\mu_k$ in K such that $K\subset\cup_{j=1}^k N(\mu_j)$. Put $K_j=N(\mu_j)$. We then have for $1\le j\le k$

$$\sup_{\mu\in K_j}\int_X\frac{Au_{\mu_j}}{u_{\mu_j}+\epsilon_{\mu_j}}d\mu\le\ell+2\delta,$$

and thus

$$\max_{1\le j\le k}\inf_{\substack{u\in\mathcal{D}^+(A)\\ \epsilon>0}}\sup_{\mu\in K_j}\int_X\frac{Au}{u+\epsilon}d\mu\le\ell+2\delta.$$

Therefore, by (6.3)

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{t}\log\sup_{x\in X}P_x(L_t\in K,t<\zeta) &\le \max_{1\le j\le k}\limsup_{t\to\infty}\frac{1}{t}\log\sup_{x\in X}P_x(L_t\in K_j,t<\zeta)\\ &\le \max_{1\le j\le k}\inf_{\substack{u\in\mathcal{D}^+(A)\\ \epsilon>0}}\sup_{\mu\in K_j}\int_X\frac{Au}{u+\epsilon}d\mu\\ &\le \ell+2\delta. \end{aligned}\tag{6.4}$$

Since

$$-\inf_{\substack{u\in\mathcal{D}^+(A)\\ \epsilon>0}}\int_X\frac{Au}{u+\epsilon}d\mu=I_{\mathcal{E}}(\mu) \tag{6.5}$$

by Proposition 4.1 in [30], the proof is completed. □

Lemma 6.2. *If a set $E \subset X$ is of positive m-measure, then $R_\alpha I_E(x) > 0$ for any $x \in X$.*

Proof. There exists a positive constant ϵ such that $m(\{x \in X : R_\alpha I_E(x) > \epsilon\}) > 0$. Let $\sigma = \inf\{t > 0 : R_\alpha I_E(X_t) > \epsilon\}$. Then, for any $x \in X$

$$\begin{aligned} R_\alpha I_E(x) &\geq E_x \left(\int_\sigma^\zeta e^{-\alpha t} I_E(X_t) dt \right) = E_x \left(e^{-\alpha\sigma} R_\alpha I_E(X_\sigma) \right) \\ &\geq \epsilon E_x(e^{-\alpha\sigma}) > 0 \end{aligned}$$

according to Lemma 6.1. □

$\mathcal{M}_1$ denotes the set of probability measures on X equipped the weak topology.

Theorem 6.2. *Let F be a compact set of X and O an open set of $\mathcal{M}_1$. Then*

$$\liminf_{t\to\infty} \frac{1}{t} \log \inf_{x\in F} P_x(L_t \in O, t < \zeta) \geq - \inf_{\mu\in O} I_{\mathcal{E}}(\mu). \tag{6.6}$$

Proof. We have only to check hypotheses $\mathbf{H}_1 \sim \mathbf{H}_5$ in [10]. $\mathbf{H}_1$ follows from Remark 6.1(i), and $\mathbf{H}_2$ is fulfilled by taking $C_\infty(X)$ as $\mathbf{B}_{00}$. $\mathbf{H}_3$ is clear since $C_\infty(X) \subset \mathbf{B}_0$. Besides, Lemma 6.2 and Assumption $\mathbf{A}_3$ imply $\mathbf{H}_4$ and $\mathbf{H}_5$ respectively. □

Now we present another proof of Theorem 6.2 under conditions imposed in [12]:

$\mathbf{A}_1'$ For all $x \in X$, $p_1(x,y) > 0$ m-a.e. y.

$\mathbf{A}_3'$ $p_1(x,\cdot)$ as a mapping from X to $L^1(X;m)$ is continuous.

$\mathbf{A}_1'$ and $\mathbf{A}_3'$ imply $\mathbf{A}_1$ and $\mathbf{A}_3$ respectively. Of course, $\mathbf{A}_1'$ and $\mathbf{A}_3'$ are fulfilled by the Brownian motions on the nested fractals and the diffusions of [1] on Sierpinski carpets.

Proof of Theorem 6.2. Let ϕ be a function in $\mathcal{D}^+(A)$ such that $\phi^2 m \in O$. Put $\mu = \phi^2 m$. Let U be a neighbourhood of μ such that $U \subset O$. Then, by taking t large enough, we have $\{L_t \in O\} \supset \{L_{t-2}(\theta_1) \in U\}$. Hence,

$$\begin{aligned} & P_x(L_t \in O, t < \zeta) \geq P_x(L_{t-2}(\theta_1) \in U, t < \zeta) \\ = \;& P_x(L_{t-2}(\theta_1) \in U, 1 < \zeta, t-2 < \zeta(\theta_1), 1 < \zeta(\theta_{t-2} \circ \theta_1)) \\ = \;& E_x \left(E_{X_1} \left(P_{X_{t-2}}(1 < \zeta); L_{t-2} \in U, t-2 < \zeta \right); 1 < \zeta \right) \\ \geq \;& \int_X p_1(x,dy) P_y(L_{t-2} \in U, X_{t-2} \in K) \cdot W, \end{aligned} \tag{6.7}$$

where K is any compact set of X with $m(K) > 0$ and $W = \inf_{x\in K} P_x(1 < \zeta) > 0$.

Set

$$S(t,\epsilon) = \left\{\omega \in \Omega : \left| \int_X \frac{A\phi}{\phi}(x) L_t(\omega, dx) - \int_X \phi A\phi dm \right| < \epsilon \right\}$$

and

$$\Lambda_t = S(t,\epsilon) \cap \{L_t \in U\}.$$

Since

$$\begin{aligned} P_x(L_t \in U, X_t \in K) &= E_x^{\phi}((N_t^{\phi,0})^{-1}; L_t \in U, X_t \in K) \\ \geq\ & \exp\left(t\left(\int_X \phi A\phi dm - \epsilon\right)\right) E_x^{\phi}\left(\frac{\phi(X_0)}{\phi(X_t)}; \Lambda_t, X_t \in K\right) \end{aligned}$$

and $\int_X \phi A\phi dm = -I_{\mathcal{E}}(\mu)$, the right hand side of (6.7) is greater than

$$\exp\left((t-2)(-I_{\mathcal{E}}(\mu) - \epsilon)\right) \int_X p_1(x,dy) E_y^{\phi}\left(\frac{\phi(X_0)}{\phi(X_{t-2})}; \Lambda_{t-2}, X_{t-2} \in K\right) \cdot W.$$

Noting that

$$E_y^{\phi}\left(\frac{\phi(X_0)}{\phi(X_{t-2})}; \Lambda_{t-2}, X_{t-2} \in K\right) \geq \frac{1}{\sup_{x\in K}\phi(x)} \phi(y) P_y^{\phi}(\Lambda_{t-2}, X_{t-2} \in K),$$

and that, by the ergodicity of $P^{\phi}_{\phi^2 m}$ and Theorem 2 in [14]

$$\begin{aligned} \lim_{t\to\infty} P_y^{\phi}(\Lambda_{t-2}, X_{t-2} \in K) &= \lim_{t\to\infty} \left(P_y^{\phi}(X_{t-2} \in K) - P_y^{\phi}(\Lambda_{t-2}^c, X_{t-2} \in K)\right) \\ &= \mu(K) \qquad \mu\text{-a.e.}, \end{aligned}$$

we obtain

$$\liminf_{t\to\infty} \int_X E_y^{\phi}(\frac{\phi(X_0)}{\phi(X_{t-2})}; \Lambda_{t-2}, X_{t-2} \in K) d\mu(y) \geq \frac{\mu(K)}{\sup_{x\in K}\phi(x)} \int_X \phi d\mu > 0$$

and, by lemma 6.3 below,

$$\liminf_{t\to\infty} \inf_{x\in F} \int_X p_1(x,dy) E_y^{\phi}\left(\frac{\phi(X_0)}{\phi(X_{t-2})}; \Lambda_{t-2}, X_{t-2} \in K\right) > 0.$$

Therefore, we can conclude that

$$\liminf_{t\to\infty} \frac{1}{t} \log \inf_{x\in F} P_x(L_t \in O, t < \zeta) \geq -I(\mu).$$

Noting that for $f \in \mathcal{F} \cap C_\infty^+(X)$ with $f \not\equiv 0$

$$\frac{\alpha R_\alpha f}{\|\alpha R_\alpha f\|_{L^2}} \in \mathcal{D}^+(A) \longrightarrow \frac{f}{\|f\|_{L^2}} \quad \text{in } \mathcal{E}_1 \quad \text{as } \alpha \to \infty$$

and that the space $\mathcal{F} \cap C_\infty^+(X)$ is dense in $\mathcal{F}^+$, the space of non-negative functions in $\mathcal{F}$, we arrive at the theorem. □

Lemma 6.3 ([12,Lemma 5.3]). *Assume* $\mathbf{A}_1'$ *and* $\mathbf{A}_3'$. *Let* $\mu = \phi^2 m \in \mathcal{M}_1$, $\phi \in \mathcal{F}$, *and let* ϕ_n, $n = 1, 2, \ldots$ *such that* $0 \leq \phi_n \leq 1$ *and*

$$\int_X \phi_n(y) d\mu(y) \geq c_1 > 0, \quad n = 1, 2, \ldots.$$

Then, for any compact set $F \subset X$, *there exists* $c_2 > 0$ *such that*

$$\inf_{x\in F} \int_X \phi_n(y) p_1(x, dy) \geq c_2 > 0, \quad n = 1, 2, \ldots.$$

A connected open set $D \subset X$ is called a regular domain if every boundary point of D is a regular point of $X \setminus D$ with respect to $\mathbf{M}$. By Proposition 1 in [29] and Theorem in [5], we obtain

Proposition 6.2. *If* D *is a regular domain of* X, *the part of the process* $\mathbf{M}$ *on* D *satisfies* $\mathbf{A}_1$, $\mathbf{A}_2$ *and* $\mathbf{A}_3$.

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