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ON VALUE FUNCTION OF STOCHASTIC DIFFERENTIAL GAMES IN INFINITE DIMENSIONS AND ITS APPLICATION TO SENSITIVE CONTROL

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1. Introduction

We will consider stochastic differential games for the system governed by stochastic partial differential equation (1.1),

$$(1.1)\quad dX(t) = (AX(t) + \beta(X(t), Y(t), Z(t)))dt + dM(t), \quad 0 < t < T,$$

$$\textit{with initial condition} \quad X(0) = \eta \ \ (\in H),$$

where A is a uniformly elliptic differential operator, M a noise and Y and Z are admissible controls of players. The pay-off J is given by (1.2),

$$(1.2)\ \ J(t, \eta; Y, Z) = E\int_0^t h(X(s), Y(s), Z(s))ds + q(X(t)), \quad 0 < t < T.$$

In our game, player I controls Y and wishes to maximize J and player II controls Z and tries to minimize J. Using upper and lower semi-discrete approximations, we showed in [7] that their limit functions provided the unique viscosity solutions of associated Isaacs equations respectively. But it was a problem whether these limit functions coincide with the upper and lower value functions of game respectively. The aim of this paper is to prove that the value functions are also unique viscosity solutions of associated Isaacs equations (see Theorem 4.2). So the upper (resp. lower) value function coincides with the upper (resp. lower) limit function.

Let $W_k,\ k = 1, 2, \ldots$, be independent 1-dimensional Brownian motions. D denotes a bounded and convex open domain of R^n with smooth boundary. Let $\mathbf{Y}$ and $\mathbf{Z}$ be compact convex subsets of $L^2(D, R^L)$ and $L^2(D, R^M)$ respectively. A process taking values in $\mathbf{Y}$ (resp.$\mathbf{Z}$) is called an admissible control of player I (resp. II), if it is F_t-progressively measurable and right-continuous paths with left limits, where F_t is the σ-field generated by $\{ W_k(s),\ s \le t,\ k = 1, 2, \cdots \}$. Let us put $H^k = H_0^k(D)$, $\|\cdot\|_k =$ its norm and $H = H^0(= L^2(D))$, $\|\cdot\| = \|\cdot\|_0$ for simplicity.

When players I and II apply admissible controls Y and Z respectively, the system X evolves according to the stochastic differential equation (1.1) on H and the pay-off J is given by (1.2). We assume

$$A\zeta = \sum_{ij=1}^{n} \frac{\partial}{\partial x_i}(a^{ij}(x)\frac{\partial \zeta}{\partial x_j}) + \sum_{i=1}^{n} r^i(x)\frac{\partial \zeta}{\partial x_i} - c(x)\zeta, \qquad \zeta \in H^1,$$

$\beta : H \times \mathbf{Y} \times \mathbf{Z} \to H$, and $dM(t)$ is an H valued colored noise having the form, $dM(t) = \sum_{k=1}^{\infty} \sqrt{m_k} e_k dW_k(t)$, with $\sum m_k (= m \text{ put}) < \infty$ and an orthonormal base e_k $(\in C_0^\infty(D))$, $k = 1, 2, \cdots$. Precise formulations and assumptions are given in Section 2.

$\mathcal{Y}$ (resp.$\mathcal{Z}$) denotes the set of admissible controls of player I (resp. II). We call a non-anticipative mapping $\alpha : \mathcal{Z} \to \mathcal{Y}$ (resp.$\gamma : \mathcal{Y} \to \mathcal{Z}$) an admissible strategy of player I (resp. II). Denoting by $\mathcal{A}$ (resp. $\mathcal{R}$) the set of admissible strategies of player I (resp. II), we define upper and lower value functions (in Elliott-Kalton sense) as follows,

(1.3) *upper value function* : $U(t, \eta) = \sup_{\alpha \in \mathcal{A}} \inf_{Z \in \mathcal{Z}} J(t, \eta\, ; \alpha Z, Z)$,

(1.4) *lower value function* : $u(t, \eta) = \inf_{\gamma \in \mathcal{R}} \sup_{Y \in \mathcal{Y}} J(t, \eta\, ; Y, \gamma Y)$.

Thus our main step is to show that these value functions are unique viscosity solutions of the associated Isaacs equations (1.5) and (1.6) respectively (Theorem 4.2), employing similar arguments as [4], with B-norm (see (2.1)).

$$\frac{\partial U}{\partial t}(t, \eta) - \langle A^* \partial U(t, \eta), \eta \rangle - \inf_{z \in \mathbf{Z}} \sup_{y \in \mathbf{Y}} (\langle \partial U(t, \eta), \beta(\eta, y, z) \rangle + h(\eta, y, z))$$

(1.5)
$$- \frac{1}{2} trace\ S \partial^2 U(t, \eta) = 0, \qquad 0 < t < T, \quad \eta \in H,$$

with initial condition $U(0) = q$

$$\frac{\partial u}{\partial t}(t, \eta) - \langle A^* \partial u(t, \eta), \eta \rangle - \sup_{y \in \mathbf{Y}} \inf_{z \in \mathbf{Z}} (\langle \partial u(t, \eta), \beta(\eta, y, z) \rangle + h(\eta, y, z))$$

(1.6)
$$- \frac{1}{2} trace\ S \partial^2 u(t, \eta) = 0, \qquad 0 < t < T, \quad \eta \in H,$$

with initial condition $u(0) = q$

where $A^*=$ adjoint operator of A , $S=$ linear operator on H defined by $Se_k = m_k e_k$, $k = 1, 2, \cdots$, $\partial =$ Fréchet derivative on H and $\langle\cdot,\cdot\rangle =$ duality pair between H^{-1} and H^1.

As an application of our results, we will study small noise asymptotics of value functions of risk sensitive control. Regarding a controller as player II, we consider the following system ξ^ε and the exponential criterion $\mathcal{J}^\varepsilon$,

$$d\xi^\varepsilon(t) = (A\xi^\varepsilon(t) + \delta(\xi^\varepsilon(t), Z(t)))dt + \sqrt{\varepsilon}dM(t), \quad 0 < t < T,$$

$$\textit{with initial condition} \qquad \xi^\varepsilon(0) = \eta$$

and

$$\mathcal{J}^\varepsilon(t,\eta\,; Z) = E\exp(\frac{1}{\varepsilon}\int_0^t f(\xi^\varepsilon(s))\,ds).$$

The value function W^ε and its logarithmic transformation ν^ε are defined by

$$W^\varepsilon(t,\eta) = \inf_{Z\in\mathcal{Z}} \mathcal{J}^\varepsilon(t,\eta\,; Z),$$

and

$$\nu^\varepsilon(t,\eta) = \varepsilon\log W^\varepsilon(t,\eta).$$

Then, ν^ε is the unique viscosity solution of Isaacs equation (1.7), by the Legendre transformation,

$$\frac{\partial\nu}{\partial t}(t,\eta) - \langle A^*\partial\nu(t,\eta),\eta\rangle - \inf_{z\in\mathbf{Z}}\langle\partial\nu(t,\eta),\delta(\eta,z)\rangle - f(\eta)$$

$$- \sup_{\zeta\in H}(\langle\partial\nu(t,\eta), S\zeta\rangle - \frac{1}{2}\langle S\zeta,\zeta\rangle) - \frac{\varepsilon}{2} trace\ S\partial^2\nu(t,\eta) = 0,$$

$$(1.7) \qquad 0 < t < T, \quad \eta\in H,$$

$$\textit{with initial condition} \qquad \nu(0) = 0.$$

In [6] we proved that the small noise limit of ν^ε exists and its limit ν becomes the unique viscosity solution of (1.7) with ε=0. Moreover it coincides with the value function of deterministic differential game on H. In this paper, we will construct the associated game of (1.7) without passing to small noise limit, using our results. So we can characterize ν^ε as the upper value of the stochastic differential game on H (Theorem 5.2). This yields the speed of convergence of ν^ε, as ε tends to 0,(Theorem 5.3),

$$|\,\nu^\varepsilon(t,\eta) - \nu(t,\eta)\,| \le const.\sqrt{\varepsilon}.$$

The paper is organized as follows. Section 2 is preliminaries, where we give precise formulations and assumptions and also recall some results on stochastic differential equations on H, for the later use. Sectin 3 is devoted to study the properties of value functions. The relations between value functions and Isaacs equations are investigated in Section 4, using the notion of viscosity solution. Section 5 deals with risk sensitive control from the point of view of stochastic differential games.

2. Preliminaries

Let (Ω, F, F_θ, P) be a canonical coodinate space with a standard Wiener measure P, namely Ω is the path space $\{\omega \in C([0,T],\ R^{\mathbf{N}}), \omega(0) = 0\}$ endowed with the usual product topology, where $\mathbf{N}$ denotes the set of natural numbers. Hence it follows that the coordinate functions $W_k(t,\omega) = \omega_k(t),\ \ k = 1, 2, \cdots,$ are independent 1-dimensional Brownian motions on Ω. F_θ denotes the σ-field generated by $\{\omega_k(s),\ s \le \theta,\ k = 1, 2, \cdots\}$ and $F = F_T$. Ocasionally we use the probability space $(\Omega_t, F_t, F_\theta, P_t)$, replacing T by t. Using the stopped path $\omega_t^-(s) = \omega(s),\ \ s \in [0,t]$, and the shifted path $\omega_t^+(s) = \omega(s+t) - \omega(t), s \in [0, T-t]$, we can identify $\Omega = \Omega_t \times \Omega_{T-t}$ and $P = P_t \times P_{T-t}$, by the mapping $\Pi_t : \Omega \to \Omega_t \times \Omega_{T-t},\ \Pi_t(\omega) = (\omega_t^-, \omega_t^+)$.

Let us assume the conditions (A1)$\sim$(A3) on A,

(A1). a^{ij} and r^i are bounded and continuous up to third derivatives

(A2). $n \times n$ matrix $(a^{ij}(x))$ is uniformly positive definite, say

$$\sum_{ij=1}^{n} a^{ij}(x) t_i t_j \ge \lambda_0 |\,t\,|^2 \qquad for \quad t = (t_1, \cdots t_n), \quad with\ \lambda_0 > 0$$

(A3). $c(\cdot)$ is non-negative and continuous.

Then, from (A2) and (A3), it follows that $-A$ is coercive, say

$$\langle -A\zeta, \zeta\rangle \ge \lambda ||\zeta||_1^2 - r||\zeta||^2, \quad with \ \ a\ positive\ \lambda.$$

The operator $B : H \to H^2$ defined by

$$B = [\ I - (A - \sum_{i=1}^{n} r^i \frac{\partial}{\partial x_i})\ ]^{-1} \quad with\ boundary\ value\ 0, \tag{2.1}$$

is a compact operator on H and satisfies the structural condition

$$\langle -A^* B\phi, \phi\rangle \ \ge \ \frac{1}{2} ||\phi||^2 - p|\,\phi\,|_B^2$$

with a constant $p \ge 0$, where $|\cdot|_B$ is called B-norm given by $|\,\phi\,|_B^2 = \langle B\phi, \phi\rangle$. When H carries B-norm, we denote H by H_B. We will prove the strucural condition. Putting

$$L = A - \sum_{i=1}^{n} r^i \frac{\partial}{\partial x_i} \quad \textit{and} \quad \psi = B\phi,$$

we have

$$\langle -A^* B\phi, \phi \rangle \ = ||\phi||^2 - |\,\phi\,|_B^2 + \sum_{i=1}^{n} \langle \frac{\partial}{\partial x_i}(r^i\psi), \phi \rangle$$

and

$$|\,\textit{the 3rd term}\,| \le \frac{1}{2}||\phi||^2 + k||\psi||_1^2, \quad \textit{with a constant } k.$$

Since (A2) and (A3) derive

$$|\,\phi\,|_B^2 = \langle \psi, (I-L)\psi \rangle \ge ||\psi||^2 + \lambda_0 ||\partial\psi||^2 \ge min.(1, \lambda_0)||\psi||_1^2,$$

we can conclude the structural condition.

Moreover we assume (A4) $\sim$ (A6), besides (A1) $\sim$ (A3), putting $|\cdot|_1$ = norm of $\mathbf{Y}$ and $|\cdot|_2$ = norm of $\mathbf{Z}$.

(A4). β is bounded and Lipshitz continuous, say

$$\hat{\beta} = \sup_{\zeta y z} ||\beta(\zeta, y, z)|| \ \textit{ and } \ ||\beta(\tilde{\zeta}, \tilde{y}, \tilde{z}) - \beta(\zeta, y, z)|| \le \ell(||\tilde{\zeta} - \zeta|| + |\,\tilde{y} - y\,|_1 + |\,\tilde{z} - z\,|_2)$$

(A5). h is bounded and Lipshitz continuous, say

$$\hat{h} = \sup_{\zeta y z} |\,h(\zeta, y, z)\,| \ \textit{ and } \ |\,h(\tilde{\zeta}, \tilde{y}, \tilde{z}) - h(\zeta, y, z)\,| \le \tilde{\ell}(||\tilde{\zeta} - \zeta|| + |\,\tilde{y} - y\,|_1 + |\,\tilde{z} - z\,|_2)$$

(A6). q is bounded and B-Lipshitz continuous, say

$$\hat{q} = \sup_{\zeta} |\,q(\zeta)\,| \ \textit{ and } \ |\,q(\tilde{\zeta}) - q(\zeta)\,| \le \tilde{\ell}||\tilde{\zeta} - \zeta||_B.$$

Denoting by $M^2(0,T;H^1)$ the subset of $L^2((0,T) \times \Omega; H^1)$ consisting of F_t -progressively measurable processes, we will define a solution of (1.1).

DEFINITION 2.1. $X \in M^2\ (0,T;H^1)$ is called a solution of (1.1), if $X \in C([0,T];\,H)$ a.s. and for any t and smooth function ϕ with support in D,

$$\langle X(t), \phi \rangle = \langle \eta, \phi \rangle + \int_0^t \langle AX(s), \phi \rangle + \langle \beta(X(s), Y(s), Z(s)), \phi \rangle ds + \langle M(t), \phi \rangle,$$

with probability 1.

Now we have

Proposition 2.1. There is a unique solution $X(\cdot\,;\eta,Y,Z)$ of (1.1) having the following properties

$$E\,(\sup_{t\le T}\|X(t\,;\eta,Y,Z)\|^2+\int_0^T\|X(s\,;\eta,Y,Z)\|_1^2ds)\le K_1(\|\eta\|^2+1)$$

and

$$E\,(\sup_{t\le T}|\,X(t\,;\eta,Y,Z)\,|_B^2+\int_0^T\|X(s\,;\eta,Y,Z)\|^2ds)\le K_1(|\,\eta\,|_B^2+1)$$

where K_1 is independent of Y and Z.

Proof. Since we can see the first inequality in [8, Theorem 4 of Section 3], we will only show the second one. Putting $X(t)=X(t;\eta,Y,Z)$, we have, by the structural condition,

$$\begin{aligned}d|\,X(t)\,|_B^2 &= 2\langle BX(t),dX(t)\rangle+|\,dX(t)\,|_B^2\\ &\le (-\|X(t)\|^2+(2p+1)|\,X(t)\,|_B^2+k)dt+2\langle BX(t),dM(t)\rangle,\end{aligned}$$

where $k=m+\hat{\beta}^2$. Hence integrating from 0 to t, we get the following three evaluations,

$$Ee^{-(2p+1)t}|\,X(t)\,|_B^2\le|\,\eta\,|_B^2+kt$$

$$\int_0^t E\|X(s)\|^2ds\le|\,\eta\,|_B^2+kt+(2p+1)\int_0^t E|\,X(s)\,|_B^2ds$$

and

$$\sup_{\theta\le t}|\,X(\theta)\,|_B^2\le|\,\eta\,|_B^2+kt+(2p+1)\int_0^t|\,X(s)\,|_B^2ds+2\sup_{\theta\le t}\int_0^\theta\langle BX(s),dM(s)\rangle.$$

Recalling the definition of dM and noting

$$E\int_0^\theta\langle BX(s),dM(s)\rangle^2\le m\int_0^\theta E\|BX(s)\|^2ds,\le m\int_0^\theta E|\,X(s)\,|_B^2ds,$$

we see, from a martingale inequality

$$E\sup_{\theta\le T}\int_0^\theta\langle BX(s),dM(s)\rangle\le k_1(1+|\,\eta\,|_B)\le k_1(2+|\,\eta\,|_B^2).$$

Combining the above calculations together, we can conclude the second inequality. □

Since the dynamics of $X(t\,;\eta,Y,Z)-X(t\,;\tilde{\eta},\tilde{Y},\tilde{Z})$ is independent of $M(\cdot)$, we see

Proposition 2.2. (see Propositions 2.2 and 2.3 in [7]). With probability 1, (2.2),(2.3) and (2.4) hold,

$$\sup_{t\leq T} \|X(t\,;\eta,Y,Z)-X(t\,;\tilde{\eta},Y,Z)\|^2+\int_0^T \|X(t\,;\eta,Y,Z)-X(t\,;\tilde{\eta},Y,Z)\|_1^2\,ds$$

$$\leq K_2\|\eta-\tilde{\eta}\|^2 \tag{2.2}$$

$$\sup_{t\leq T} |\,X(t\,;\eta,Y,Z)-X(t\,;\tilde{\eta},Y,Z)\,|_B^2+\int_0^T \|X(t\,;\eta,Y,Z)-X(t\,;\tilde{\eta},Y,Z)\|^2\,ds$$

$$\leq K_2|\,\eta-\tilde{\eta}\,|_B^2, \tag{2.3}$$

with K_2 independent of Y, Z and $\omega\in\Omega$,

$$|\,X(t\,;\eta,Y,Z)-X(t\,;\eta,\tilde{Y},\tilde{Z})\,|_B^2\leq \|X(t\,;\eta,Y,Z)-X(t\,;\eta,\tilde{Y},\tilde{Z})\|^2$$

$$\leq K_3\int_0^t (|\,Y(s)-\tilde{Y}(s)\,|_1^2+|\,Z(s)-\tilde{Z}(s)\,|_2^2)\,ds, \tag{2.4}$$

with K_3 independent of η, t and $\omega\in\Omega$.

For the continuity w.r.to time, we need finer calculations using structural condition.

Proposition 2.3.

$$E\,(|\,X(t\,;\eta,Y,Z)-X(s\,;\eta,Y,Z)\,|_B^2)\leq K_4(1+\|\eta\|^2)|\,t-s\,| \tag{2.5}$$

$$E\,(\sup_{t\leq\theta}\|X(t\,;\eta,Y,Z)-\eta\|^4)\leq K_4(\sup_{t\leq\theta}\|e^{tA}\,\eta-\eta\|^4+\theta^2), \tag{2.6}$$

where K_4 is independent of η, Y and Z.

Proof. Since we see (2.5) in Proposition 2.4 in [7], we will only prove (2.6). Putting $X(t)=X(t;\eta,Y,Z)$ and $\xi(t)=X(t)-e^{tA}\eta$, we have

$$d\xi(t)=(A\xi(t)+\beta(X(t),Y(t),Z(t))dt+dM(t),\qquad \textit{for}\ \ 0<t<T,$$

$$\textit{with initial condition}\ \ \xi(0)=0.$$

From the coercive condition, we see

$$\begin{aligned} d\|\xi(t)\|^2 &= 2\langle\xi(t),d\xi(t)\rangle+\|d\xi(t)\|^2\\ &\leq (2(r+1)\|\xi(t)\|^2+k)dt+2\langle\xi(t),dM(t)\rangle, \end{aligned}$$

with a constant k. Hence we get

$$Ee^{-2(r+1)t}||\xi(t)||^2 \le kt$$

and

$$\sup_{t\le\theta} e^{-2(r+1)t}||\xi(t)||^2 \le k\theta + 2\sup_{t\le\theta}\int_0^t e^{-2(r+1)s}\langle\xi(s), dM(s)\rangle.$$

Taking the square of both sides, we obtain

$$E\sup_{t\le\theta} e^{-4(r+1)t}||\xi(t)||^4 \le 2k^2\theta^2 + 8E(\sup_{t\le\theta}\int_0^t e^{-2(r+1)s}\langle\xi(s), dM(s)\rangle)^2.$$

Since a martingale inequality derives

$$E(\sup_{t\le\theta}\int_0^t e^{-2(r+1)s}\langle\xi(s), dM(s)\rangle)^2 \le 4\sum_{i=1}^{\infty} m_i E\int_0^\theta e^{-4(r+1)s}\langle\xi(s), e_i\rangle^2 ds$$

$$\le 4m\int_0^\theta e^{-4(r+1)s}E||\xi(s)||^2 ds,$$

the above calculations yield

$$E\sup_{t\le\theta} e^{-4(r+1)t}||\xi(t)||^4 \le k_1\theta^2, \qquad for \ \ 0\le\theta\le T.$$

Now we complete the proof of (2.6), recalling the definition of $\xi(t)$. □

Setting $\tau(\eta, d)$ = exit time from the ball of radius d centered at η, and fixing small $\hat\theta(\eta, d)$ such that

$$\hat\theta \le d(3\hat\beta\sup_{t\le T}\mathbf{I}e^{tA}\mathbf{I})^{-1} \ \ and \ \ \sup_{t\le\hat\theta}||e^{tA}\eta - \eta|| < \frac{d}{3}, \tag{2.7}$$

where $\mathbf{I}\cdot\mathbf{I}$ means the operator norm, we get (2.8), by (2.6),

$$P(\,\tau(\eta, d) < s\,) \le K_5 s^2 d^{-4} \qquad whenever \quad s \le \hat\theta(\eta, d), \tag{2.8}$$

where K_5 is independent of η, d, Y and Z.

3. Value functions

First of all, we define strategies of players.

DEFINITION 3.1. An admissible strategy α (resp. γ) of player I (resp. II) is a mapping $\alpha: \mathcal{Z}\to\mathcal{Y}$ (resp. $\gamma: \mathcal{Y}\to\mathcal{Z}$), which is $(\mathbf{B}[0,T]\times F, \mathbf{B}(\mathbf{Y}))$–measurable and non-anticipative, namely

$$if \quad P(Z(s) = \tilde Z(s)) = 1 \quad for \quad s < t, \quad then \quad P(\alpha Z(t) = \alpha\tilde Z(t)) = 1.$$

$(resp.\ \ if\ \ P(Y(s) = \tilde{Y}(s)) = 1\ \ for\ \ s < t,\ \ then\ \ P(\gamma Y(t) = \gamma\tilde{Y}(t)) = 1).$

$\mathcal{A}$ (resp. $\mathcal{R}$) denotes the set of admissible strategies of player I (resp. II). Putting $\omega^- = \omega_t^-$ and $\omega^+ = \omega_t^+$ for simplicity and $Z_{\omega^-}(\theta, \omega^+) = Z(\theta + t, (\omega^-, \omega^+))$ for $\theta \in [0, T-t]$, we note that Z_{ω^-} can be regarded as an admissible control of player II on Ω_{T-t}, for almost all $\omega^- \in \Omega_t$. But, it is a ploblem whether $\alpha(Z_{\omega^-})(\theta, \omega^+)$ is measurable w.r.to $(\theta, \omega^-, \omega^+)$, as Fleming and Souganidis pointed out [4]. Therefore we introduce some restrictive class where the measurability holds.

DEFINITION 3.2. ([4]). When $\alpha(\in \mathcal{A})$ satisfies the following additional property (R), we call α an r-strategy of player I.
(R). For any $t \in (0, T)$ and $Z \in \mathcal{Z}$, the mapping: $(\theta, \omega) \to \alpha(Z_{\omega^-})(\theta, \omega^+)$, is $(\mathbf{B}[0, T-t] \times F, \mathbf{B}(\mathbf{Y}))$-measurable.

$\mathbf{A}$ denotes the set of r-strategies of player I. Similarly, we define r-strategy of player II with their collection denoted by $\mathbf{R}$. Replacing $\mathcal{A}$ and $\mathcal{R}$ in the definitions (1.3) and (1.4) by $\mathbf{A}$ and $\mathbf{R}$ respectively, we define r-value functions.

DEFINITION 3.3.
r-upper value function $\mathbf{U}(t,\eta) = \sup_{\alpha \in \mathbf{A}} \inf_{Z \in \mathcal{Z}} J(t, \eta; \alpha, Z)$
r-lower value function $\mathbf{u}(t,\eta) = \inf_{\gamma \in \mathbf{R}} \sup_{Y \in \mathcal{Y}} J(t, \eta; Y, \gamma)$
where $J(t,\eta;\alpha,Z) = J(t,\eta;\alpha Z, Z)$ and $J(t,\eta;Y,\gamma) = J(t,\eta;Y,\gamma Y)$.

From (2.3) and (2.5), we can easily see

Proposition 3.1.

$$| J(t, \eta; Y, Z) | \le \hat{h}T + \hat{q} \tag{3.1}$$

$$| J(t, \eta; Y, Z) - J(s, \zeta; Y, Z) | \le K_6[\, |\eta - \zeta|_B + (1 + \|\eta\|)\sqrt{|t-s|}\,] \tag{3.2}$$

where K_6 is independent of Y and Z.

Hence, both of $\mathbf{U}(t,\eta)$ and $\mathbf{u}(t,\eta)$ also satisfy (3.1) and (3.2).

Proposition 3.2.
Upper-optimality dynamic programming principle

(3.3)

$$\sup_{\alpha \in \mathbf{A}} \inf_{Z \in \mathcal{Z}} E[\int_0^\theta h(X(s;\eta,\alpha,Z), \alpha Z(s), Z(s))\, ds + \mathbf{U}(t-\theta, X(\theta;\eta,\alpha,Z))\,] \le \mathbf{U}(t,\eta)$$

Sub-optimality dynamic programming principle

(3.4)

$$\inf_{\gamma\in\mathbf{R}}\sup_{Y\in\mathcal{Y}} E[\int_0^\theta h(X(s;\eta,Y,\gamma),Y(s),\gamma Y(s))\,ds + \mathbf{u}(t-\theta,X(\theta;\eta,Y,\gamma))\,] \geq \mathbf{u}(t,\eta)$$

Proof. Using B-norm, we can apply the standard method because of condition (R). So, we only give an outline for (3.3), since (3.4) is proved in a similar way.

We set $W(t,\eta)=$ the right hand side of (3.3). For $\varepsilon>0$, there is $\hat\alpha\in\mathbf{A}$ such that

$$W(t,\eta)\leq E[\int_0^\theta h(X(s;\eta,\hat\alpha,Z),\hat\alpha Z(s),Z(s))\,ds + \mathbf{U}(t-\theta,X(\theta;\eta,\hat\alpha,Z))\,]+\varepsilon$$

(3.5) $\qquad$ *for any* $Z\in\mathcal{Z}$,

On the other hand, there is $\alpha_\zeta\in\mathbf{A}$ such that

$$\mathbf{U}(t-\theta,\zeta)\leq \inf_{Z\in\mathcal{Z}} J(t-\theta,\zeta;\alpha_\zeta,Z)+\varepsilon.$$

Dividing $H=\bigcup_{j=1}^\infty A_j$ with B-diam.$(A_j)<\frac{\varepsilon}{K_6}$ and choosing $\zeta_j\in A_j$ arbitrarily, we define α^* by

$$\alpha^*(Z)(s,\omega)=\hat\alpha(Z)(s,\omega)I_{[0,\theta)}(s)+\sum_{j=1}^\infty I_{A_j}(X(\theta;\eta,\hat\alpha,Z,\omega^-))\alpha_j(Z_{\omega^-})(s-\theta,\omega^+)$$

where $\alpha_j=\alpha_{\zeta_j}$, $I_A=$ indicator of set A and $\omega^-=\omega_\theta^-$, $\omega^+=\omega_\theta^+$. Since $\hat\alpha$ and α_j are r-strategies, α^* is also r-strategy. Moreover, (3.2) yields

(3.6) $\qquad \mathbf{U}(t-\theta,\xi)\leq \inf_{Z\in\mathcal{Z}} J(t-\theta,\xi;\alpha_j,Z)+3\varepsilon \qquad$ *for* $\xi\in A_j$,

Hence, from (3.5) and (3.6), we see for $Z\in\mathcal{Z}$

$$W(t,\eta)<E[\int_0^\theta h(X(s;\eta,\alpha^*,Z),\alpha^*Z(s),Z(s))\,ds$$

$$+\sum_{j=1}^\infty I_{A_j}(X(\theta;\eta,\alpha^*,Z))J(t-\theta,X(\theta;\eta,\alpha^*,Z);\alpha^*,Z)\,]+5\varepsilon$$

$$=J(t,\eta;\alpha^*,Z)+5\varepsilon.$$

Since Z is arbitrary,

$$W(t,\eta)\leq\inf_{Z\in\mathcal{Z}} J(t,\eta;\alpha^*,Z)+5\varepsilon\leq \mathbf{U}(t,\eta)+5\varepsilon.$$

This completes the proof of (3.3). □

4. Isaacs equations

We recall the definition of viscosity solution of Isaacs equations [2], putting

$$F^{+}(\eta,p,Q) = -\inf_{z\in\mathbf{Z}}\sup_{y\in\mathbf{Y}}[\,\langle p,\beta(\eta,y,z)\rangle + h(\eta,y,z)\,] - \frac{1}{2}trace(SQ)$$

and

$$F^{-}(\eta,p,Q) = -\sup_{y\in\mathbf{Y}}\inf_{z\in\mathbf{Z}}[\,\langle p,\beta(\eta,y,z)\rangle + h(\eta,y,z)\,] - \frac{1}{2}trace(SQ)$$

where $p \in H$ and $Q \in L(H)$ (=the Banach space of bounded linear operators equipped with the operator norm $\mathbf{I}\cdot\mathbf{I}$).

$\Phi \in C^{12}((0,T)\times H)$ is called a test function, if

(i). Φ is weakly lower semi-continuous and bounded from below, and

(ii). $\partial\Phi(t,\eta) \in H^2$ and both of $\partial\Phi$ and $A^*\partial\Phi$ are continuous.

$g \in C^2(H)$ is called radial, if $g(\eta) = \tilde{g}(||\eta||)$ with $\tilde{g} \in C^2[0,\infty)$ increasing from 0 to ∞.

By virtue of (A1)~(A3), there is a constant $\mu \geq 0$ such that

$$\langle -A\zeta,\zeta\rangle + \mu||\zeta||^2 \geq 0 \qquad for \quad \zeta \in H^1.$$

Hence, $-\tilde{A} = -A + \mu I$ is dissipative. Putting $\tilde{\beta}(\eta,y,z) = \beta(\eta,y,z) + \mu\eta$, we can replace A and β in the Isaacs equations (1.5) and (1.6) by $\tilde{A}$ and $\tilde{\beta}$ respectively. Moreover noting

$$-\langle \tilde{A}^*\partial\Phi(t,\eta),\eta\rangle - \inf_{z\in\mathbf{Z}}\sup_{y\in\mathbf{Y}}\langle \partial(\Phi+g)(t,\eta),\tilde{\beta}(\eta,y,z)\rangle$$

$$= -\langle A^*\partial\Phi(t,\eta),\eta\rangle - \inf_{z\in\mathbf{Z}}\sup_{y\in\mathbf{Y}}\langle \partial(\Phi+g)(t,\eta),\beta(\eta,y,z)\rangle - \mu\langle\partial g(\eta),\eta\rangle,$$

we have the definition 4.1, according to [2].

DEFINITION 4.1. $V \in C([0,T]\times H)$ is called a sub-solution (resp. super-solution) of (1.5), if $V(0,\eta) = q(\eta)$ and the following condition (1) (resp.(2)) holds for any test function Φ and radial function g,

(1). If $V - \Phi - g$ has a local maximum at $(\hat{t},\hat{\eta}) \in (0,T)\times H$, then

$$\frac{\partial\Phi}{\partial t}(\hat{t},\hat{\eta}) - \langle A^*\partial\Phi(\hat{t},\hat{\eta}),\hat{\eta}\rangle + F^{+}(\hat{\eta},\partial(\Phi+g)(\hat{t},\hat{\eta}),\partial^2(\Phi+g)(\hat{t},\hat{\eta})) \leq \mu\tilde{g}'(||\hat{\eta}||)||\eta||.$$

(2). If $V + \Phi + g$ has a local minimum at $(\hat{t},\hat{\eta}) \in (0,T)\times H$, then

$$-\frac{\partial\Phi}{\partial t}(\hat{t},\hat{\eta}) + \langle A^*\partial\Phi(\hat{t},\hat{\eta}),\hat{\eta}\rangle + F^{+}(\hat{\eta},-\partial(\Phi+g)(\hat{t},\hat{\eta}),-\partial^2(\Phi+g)(\hat{t},\hat{\eta})) \geq -\mu\tilde{g}'(||\hat{\eta}||)||\eta||.$$

V is called a viscosity solution, if it is both a sub- and super-solution.

Replacing F^+ by F^-, we define a viscosity solution of (1.6). Since our value functions are B-continuous, local maximum (resp.minimum) can be replaced by strictly local maximum (resp. minimum) in Definition 4.1,[3].

Theorem 4.1.
(i). $\mathbf{U}$ is a super-solution of (1.5), and (ii). $\mathbf{u}$ is a sub-solution of (1.6).

Proof. We only prove (i), because (ii) follows in a similar way.

Appealing to the super-optimality (3.3), we employ a routine method. So, we only show the outline of proof.

Supose $\mathbf{U}+\mathbf{\Phi}+\mathbf{g}$ has a local minimum at $(\hat{t},\hat{\eta})$, say

(4.1)
$$\mathbf{U}(\hat{t},\hat{\eta})+\Phi(\hat{t},\hat{\eta})+g(\hat{\eta}) \le \mathbf{U}(t,\eta)+\Phi(t,\eta)+g(\eta), \quad for \quad |t-\hat{t}|, \|\eta-\hat{\eta}\| \le \tilde{\delta}.$$

For $\hat{\varepsilon}>0$, there is $\hat{\delta}>0$, such that if $|t-\hat{t}|<\hat{\delta}$ and $\|\eta-\hat{\eta}\|<\hat{\delta}$ then

$$|f(t,\eta)-f(\hat{t},\hat{\eta})|<\hat{\varepsilon}, \tag{4.2}$$

where $f=\Phi, g, \frac{\partial\Phi}{\partial t}, \partial\Phi, \partial g, A^*\partial\Phi, \partial^2\Phi, \partial^2 g$ and $|\cdot|$ means their own norms. Let us set $\delta=\min(\tilde{\delta},\hat{\delta})$ and τ =exit time form the closed ball of radius δ centered at $\hat{\eta}$. Putting

$$\lambda=\frac{\partial\Phi}{\partial t}(\hat{t},\hat{\eta})-\langle A^*\partial\Phi(\hat{t},\hat{\eta}),\hat{\eta}\rangle-\frac{1}{2}trace\ S\partial^2(\Phi+g)(\hat{t},\hat{\eta})-\mu\langle\partial g(\hat{\eta}),\hat{\eta}\rangle$$

and using (2.8), (4.1), (4.2) and Itô's formula, we obtain

$$E[\,\mathbf{U}(\hat{t}-\theta,X(\theta;\hat{\eta},\alpha,Z))-\mathbf{U}(\hat{t},\hat{\eta});\ \tau\ge\theta\,]$$

(4.3)
$$\ge\lambda\theta-E[\int_0^\theta\langle\partial(\Phi+g)(\hat{t},\hat{\eta}),\beta(\hat{\eta},\alpha Z(s),Z(s))\rangle ds\,]-k_1\sqrt{\theta}^3\delta^{-2}-k_2(\hat{\varepsilon}+\delta)\theta,$$

for $\theta<\hat{\theta}(\hat{\eta},\delta)$, (see (2.7)), where k_1 and k_2 are independent of α and Z. On the other hand, (2.8) yields

(4.4)
$$E[\,\mathbf{U}(\hat{t}-\theta,X(\theta;\hat{\eta},\alpha,Z))-\mathbf{U}(\hat{t},\hat{\eta});\tau<\theta\,]\ge-2(\hat{h}T+\hat{q})P(\tau<\theta)\ge-k_3\theta^2\delta^{-4}.$$

Now, (4.3) and (4.4) together with the super-optimality dynamic programming principle yield

(4.5)
$$0\ge\sup_{\alpha\in\mathbf{A}}\inf_{Z\in\mathcal{Z}}E[\int_0^\theta F(\alpha Z(s),Z(s))\,ds\,]+\lambda\theta-k_4\sqrt{\theta}^3\delta^{-2}-k_5(\hat{\varepsilon}+\delta)\theta,$$

where

$$F(y,z) = h(\hat{\eta}, y, z) - \langle \partial(\Phi + g)(\hat{t}, \hat{\eta}), \beta(\hat{\eta}, y, z) \rangle.$$

Assume that there is a positive c such that

$$\lambda + \inf_{z \in \mathbf{Z}} \sup_{y \in \mathbf{Y}} F(y,z) > c.$$

From Lipshitz continuity of h and β, it follows that there is $\Delta > 0$, such that

$$\sup_{y \in \mathbf{Y}} |\, F(y,z) - F(y,\hat{z}) \,| < \frac{c}{2}, \quad if \ \ |\, z - \hat{z} \,| < \Delta.$$

Thus dividing $\mathbf{Z} = \cup_{j=1}^{N} \mathbf{Z}_j$ with diam.$(\mathbf{Z}_j) < \Delta$ and fixing $z_j \in \mathbf{Z}_j$ arbitrarily, we can take y_j such that

$$F(y_j, z_j) > c - \lambda, \qquad j = 1, 2, \cdots N.$$

Let us define $\hat{\alpha} : \mathcal{Z} \to \mathcal{Y}$ by

$$\hat{\alpha} Z(t,\omega) = \sum_{j=1}^{N} y_j I_{\mathbf{Z}_j}(Z(t,\omega)).$$

Then, $\hat{\alpha} \in \mathbf{A}$ and

$$\inf_{Z \in \mathcal{Z}} E[\int_0^\theta F(\hat{\alpha} Z(s), Z(s))\, ds\,] \geq (\frac{c}{2} - \lambda)\,\theta.$$

Noting (4.5), we thus get

$$0 \geq \frac{c}{2} - k_4 \sqrt{\theta} \delta^{-2} - k_5(\hat{\varepsilon} + \delta). \tag{4.8}$$

For $\hat{\varepsilon}$, $\delta < \frac{\varepsilon}{8K_5}$ and small θ, (4.7) contradicts to $c > 0$. Hence Theorem 4.1,(i) holds. □

In [7], we constructed the unique viscosity solutions V (resp. v) of (1.5) (resp. (1.6)), as follows. Putting $\Delta = 2^{-N} T$, $N = 1, 2, \cdots$, we call Z($\in \mathcal{Z}$) a Δ-step control, if $Z(t) = z$ for $t \in [0, \Delta)$ and $Z(t) = Z(k\Delta)$ for $t \in [k\Delta, (k+1)\Delta)$. $\mathcal{Z}_N$ denotes the set of Δ-step controls of player II. $\gamma(\in \mathcal{R})$ is called Δ-step, if $\gamma Y \in \mathcal{Z}_N$ and $\gamma Y(t)$, $t \in [0, \Delta)$, does not depend on Y. $\mathcal{R}_N$ denotes their collection. Let us define

$$V_N(t,\eta) = \inf_{\gamma \in \mathcal{R}_N} \sup_{Y \in \mathcal{Y}} J(t, \eta; Y, \gamma).$$

Then, V_N is decreasing and satisfies the evaluations (3.1) and (3.2). Moreover, the limit function, $V(t,\eta) = \lim_{N \to \infty} V_N(t,\eta)$, is the unique viscosity solution of (1.5). Therefore from the comparison theorem [9], it follows that

$$U(t,\eta) \geq \mathbf{U}(t,\eta) \geq V(t,\eta).$$

Next we will show the opposite inequality, $U(t,\eta) \leq V(t,\eta)$. Since, for any $\alpha \in \mathcal{A}$ and $\gamma \in \mathcal{R}_N$, there exist $\hat{Y} \in \mathcal{Y}$ and $\hat{Z} \in \mathcal{Z}_N$ such that $\alpha\hat{Z} = \hat{Y}$ and $\gamma\hat{Y} = \hat{Z}$, (see (2.5) in [4]), we get

$$\sup_{Y\in\mathcal{Y}} J(t,\eta;Y,\gamma) \geq J(t,\eta;\hat{Y},\gamma) = J(t,\eta;\alpha,\hat{Z}) \geq \inf_{Z\in\mathcal{Z}} J(t,\eta;\alpha,Z).$$

Hence, for any $\alpha \in \mathcal{A}$, we have

$$V_N(t,\eta) \geq \inf_{Z\in\mathcal{Z}} J(t,\eta;\alpha,Z).$$

Taking supremum w.r.to α and letting N tend to ∞, we get the opposite inequality $U(t,\eta) \leq V(t,\eta)$, which yields $U(t,\eta) = V(t,\eta)$. □

Consequently, we obtain the main theorem,

Theorem 4.2. The upper value function U (resp. lower value function u) is the unique viscosity solution of (1.5) (resp. (1.6)), in $C_b([0,T] \times H_W)$ (= the set of bounded weakly continuous functions).

Collary. Under Isaacs condition, our stochastic differential game has the value.

Recalling the definitions of value functions, we see

$$U(t,\eta) = \lim_{N\to\infty} V_N(t,\eta) = \inf_{\gamma\in\cup\mathcal{R}_N} \sup_{Y\in\mathcal{Y}} J(t,\eta;Y,\gamma) \geq u(t,\eta).$$

Hence, if $U(t,\eta) - u(t,\eta)$ (= c put) > 0, then for any step strategy $\gamma(\in \cup_{N=1}^{\infty}\mathcal{R}_N)$,

$$\sup_{Y\in\mathcal{Y}} J(t,\eta;Y,\gamma) \geq u(t,\eta) + c.$$

Namely, γ can not be nearly optimal.

5. Application to sensitive control

Regarding a controller as player II, we will consider the following stochastic control. For $\varepsilon > 0$, when a controller applies an admissible control Z, the system ξ^ε and the pay-off $\mathcal{J}^\varepsilon$ are given by (5.1) and (5.2) respectively,

(5.1) $$d\xi^\varepsilon(t) = (A\xi^\varepsilon(t) + \delta(\xi^\varepsilon(t), Z(t)))\,dt + \sqrt{\varepsilon}dM(t), \qquad 0 < t < T,$$

$$\textit{with initial condition} \qquad \xi^\varepsilon(0) = \eta \ \ (\in H),$$

and

$$\mathcal{J}^{\varepsilon}(t,\eta;Z) = E(\exp\frac{1}{\varepsilon}\int_0^t f(\xi^{\varepsilon}(s))\,ds) \tag{5.2}$$

where δ and f are bounded and Lipshitz continuous.

Let us define the value vunction W^{ε} by

$$W^{\varepsilon}(t,\eta) = \inf_{Z\in\mathcal{Z}} \mathcal{J}^{\varepsilon}(t,\eta;Z).$$

Then W^{ε} is the unique viscosity solution of Hamilton-Jacobi-Bellman equation (5.3),

$$\begin{aligned}\frac{\partial W}{\partial t}(t,\eta) - \langle A^*\partial W(t,\eta),\eta\rangle - \inf_{z\in\mathbf{Z}}\langle \partial W(t,\eta),\delta(\eta,z)\rangle - \frac{1}{\varepsilon}f(\eta)W(t,\eta)\\ -\frac{\varepsilon}{2}trace\ S\partial^2 W(t,\eta) = 0, \qquad 0<t<T,\ \ \eta\in H,\end{aligned} \tag{5.3}$$

$$with\ \ initial\ \ condition \qquad W(0)=1.$$

Hence its logarithmic transformation ν^{ε}

$$\nu^{\varepsilon}(t,\eta) = \varepsilon\log W^{\varepsilon}(t,\eta)$$

is the unique viscosity solution of (5.4) in $C_b([0,T]\times H_W)$,

$$\begin{aligned}\frac{\partial \nu}{\partial t}(t,\eta) - \langle A^*\partial \nu(t,\eta),\eta\rangle - \inf_{z\in\mathbf{Z}}\langle \partial \nu(t,\eta),\delta(\eta,z)\rangle - f(\eta)\\ -\frac{1}{2}\langle S\partial\nu(t,\eta),\partial\nu(t,\eta)\rangle - \frac{\varepsilon}{2}trace\ S\partial^2\nu(t,\eta) = 0, \qquad 0<t<T,\ \ \eta\in H,\end{aligned} \tag{5.4}$$

$$with\ \ initial\ \ condition \qquad \nu(0)=0.$$

But, (5.4) turns out to be the Isaacs equation (1.7) by Legendre transformation. Moreover, we showed [6] that the small noise limit of ν^{ε}, say ν, exists and turns out to be the unique viscosity solution of (1.7) with $\varepsilon = 0$, which coincides with the value function of deterministic differential game on H.

Since H is not compact, we will introduce admissible controls and strategies of stochastic differential game associated with (1.7) as follows. Putting $\Lambda = \sqrt{S}$, we set $\mathbf{Y}_N = \{\Lambda\zeta \in H;\ \|\zeta\| \le N\}$. Then $\mathbf{Y}_N$ is compact.

Replacing $\mathbf{Y}$ in previous sections by $\mathbf{Y}_N$, we denote the set of admissible controls and strategies by $\mathcal{Y}_N$ and $\mathcal{A}_N$ respectively. Let us set

$$\mathbf{Y} = \bigcup_{N=1}^{\infty}\mathbf{Y}_N, \qquad \mathcal{Y} = \bigcup_{N=1}^{\infty}\mathcal{Y}_N, \quad \mathcal{A} = \bigcup_{N=1}^{\infty}\mathcal{A}_N,$$

$$\beta(\eta, y, z) = \delta(\eta, z) + \Lambda y, \qquad h(\eta, y) = f(\eta) - \frac{1}{2}||y||^2.$$

When players I and II apply admissible controls Y and Z respectively, the system X^ε evolves according to stochastic differential equation,

$$dX^\varepsilon(t) = (AX^\varepsilon(t) + \beta(X^\varepsilon(t), Y(t), Z(t)))\,dt + \sqrt{\varepsilon}dM(t), \qquad 0 < t < T,$$

$$\textit{with initial condition} \qquad X^\varepsilon(0) = \eta \ (\in H),$$

and the pay-off J^ε is defined by

$$J^\varepsilon(t, \eta; Y, Z) = E\int_0^t h(X^\varepsilon(s; \eta, Y, Z), Y(s))\,ds.$$

Using similar notations as before, we define

$$U_N^\varepsilon(t, \eta) = \sup_{\alpha\in\mathcal{A}_N} \inf_{Z\in\mathcal{Z}} J^\varepsilon(t, \eta; \alpha, Z)$$

$$u_N^\varepsilon(t, \eta) = \inf_{\gamma\in\mathcal{R}} \sup_{Y\in\mathcal{Y}_N} J^\varepsilon(t, \eta; Y, \gamma)$$

Then, we have

$$U^\varepsilon(t, \eta) = \sup_{\alpha\in\mathcal{A}} \inf_{Z\in\mathcal{Z}} J^\varepsilon(t, \eta; \alpha, Z) = \lim_{N\to\infty} U_N^\varepsilon(t, \eta)$$

$$u^\varepsilon(t, \eta) = \inf_{\gamma\in\mathcal{R}} \sup_{Y\in\mathcal{Y}} J^\varepsilon(t, \eta; Y, \gamma) \geq \lim_{N\to\infty} u_N^\varepsilon(t, \eta).$$

From Theorem 4.2, it follows that $U_N^\varepsilon = u_N^\varepsilon$ and they are unique viscosity solutions of the following Isaacs equation,

$$\frac{\partial U}{\partial t}(t, \eta) - \langle A^*\partial U(t, \eta), \eta\rangle - \inf_{z\in\mathbf{Z}}\langle \partial U(t, \eta), \delta(\eta, z)\rangle - f(\eta)$$

$$-\frac{\varepsilon}{2}\textit{trace } S\partial^2 U(t, \eta) - \sup_{\zeta\in Y_N}(\langle\partial U(t, \eta), \Lambda\zeta\rangle - \frac{1}{2}||\zeta||^2) = 0, \qquad 0 < t < T,$$

$$\textit{with initial condition} \qquad U(0) = 0.$$

Proposition 5.1. Putting $\hat{f} = \sup_{\zeta\in H}|f(\zeta)|$, we have $|U_N^\varepsilon(t, \eta)| \leq \hat{f}t$.

Proof. Since the strategy 0 belongs to $\mathcal{A}_N$,

$$U_N^\varepsilon(t, \eta) \geq \inf_{Z\in\mathcal{Z}} J^\varepsilon(t, \eta; 0, Z) \geq -\hat{f}t$$

holds. Noting $h(\cdot) \leq \hat{f}$, we complete the proof. □

Proposition 5.2.

$$| U_N^\varepsilon(t,\eta) - U_N^\varepsilon(s,\zeta) | \leq c| \eta - \zeta |_B + \hat{f} | t - s |$$

with a constant c, independent of N and ε.

Proof. In a similar way as (2.3), we get

$$\int_0^T \|X^\varepsilon(t;\eta,Y,Z) - X^\varepsilon(t;\tilde{\eta},Y,Z)\|^2 \, ds \leq c_1 | \eta - \tilde{\eta} |_B^2$$

with c_1 independent of Y, Z, N, ε and $\omega \in \Omega$. Hence we have

$$\sup_{t \leq T} | U_N^\varepsilon(t,\eta) - U_N^\varepsilon(t,\zeta) | \leq c| \eta - \zeta |_B. \tag{5.5}$$

Next we evaluate the continuity w.r.to t, using similar arguments as in [1]. For $\tilde{\varepsilon} > 0$, taking $\alpha^* = \alpha^*(s,\eta,\tilde{\varepsilon}) \in \mathcal{A}_N$ such that

$$U_N^\varepsilon(s,\eta) < \inf_{Z \in \mathcal{Z}} J^\varepsilon(s,\eta;\alpha^*,Z) + \tilde{\varepsilon},$$

and defining $\hat{\alpha} \in \mathcal{A}_N$ by

$$\hat{\alpha}Z(\theta) = \alpha^* Z(\theta) \qquad for \quad \theta \in [0,s), \qquad = 0 \qquad for \qquad \theta \in [s,T],$$

we have

(5.6)

$$U_N^\varepsilon(t,\eta) - U_N^\varepsilon(s,\eta) \geq \inf_{Z \in \mathcal{Z}} J^\varepsilon(t,\eta;\hat{\alpha},Z) - \inf_{Z \in \mathcal{Z}} J^\varepsilon(s,\eta;\hat{\alpha},Z) - \tilde{\varepsilon} \geq -\hat{f}(t-s) - \tilde{\varepsilon}.$$

Choosing $\tilde{\alpha} = \tilde{\alpha}(t,\eta,\tilde{\varepsilon}) \in \mathcal{A}_N$ and $\tilde{Z} \in \mathcal{Z}$ such that

$$U_N^\varepsilon(t,\eta) \leq \inf_{Z \in \mathcal{Z}} J^\varepsilon(t,\eta;\tilde{\alpha},Z) + \tilde{\varepsilon},$$

and

$$\inf_{Z \in \mathcal{Z}} J^\varepsilon(t,\eta;\tilde{\alpha},Z) \geq J^\varepsilon(t,\eta;\tilde{\alpha},\tilde{Z}) - \tilde{\varepsilon},$$

we have

$$U_N^\varepsilon(t,\eta) - U_N^\varepsilon(s,\eta) \leq \inf_{Z \in \mathcal{Z}} J^\varepsilon(t,\eta;\tilde{\alpha},Z) - \inf_{Z \in \mathcal{Z}} J^\varepsilon(s,\eta;\tilde{\alpha},Z) + \tilde{\varepsilon}$$

$$\leq J^\varepsilon(t,\eta;\tilde{\alpha},\tilde{Z}) - J^\varepsilon(s,\eta;\tilde{\alpha},\tilde{Z}) + 2\tilde{\varepsilon} \leq \hat{f}(t-s) + 2\tilde{\varepsilon}. \tag{5.7}$$

Now, Proposition 5.2 follows from (5.5),(5.6) and (5.7). □

Since $U_N^\varepsilon(t,\eta)$ is increasing to $U^\varepsilon(t,\eta)$, as $N\to\infty$, Propositions 5.1 and 5.2 yield the following theorem,

Theorem 5.1. As $N\to\infty$, U_N^ε is increasing to U^ε uniformly on any bounded set of $[0,T]\times H$. Moreover, U^ε is bounded and B-coninuous and the unique viscosity solution of (1.7) in $C_b([0,T]\times H_W)$.

Recalling that ν^ε is the unique viscosity solution of (1.7), we have

Theorem 5.2. ν^ε has a min-max expression

$$\nu^\varepsilon(t,\eta)=\sup_{\alpha\in\mathcal{A}}\inf_{Z\in\mathcal{Z}}J^\varepsilon(t,\eta;\alpha,Z)\quad(=U^\varepsilon(t,\eta)).$$

For $\varepsilon=0$, we define X^0, J^0 and U^0 in a similar way. Then we get by standard arguments

$$\sup_{t\le T}E(\|X^\varepsilon(t;\eta,Y,Z)-X^0(t;\eta,Y,Z)\|^2)\le c_3\varepsilon$$

with a constant c_3 independent of η, Y and Z. So, we have

$$|\,U^\varepsilon(t,\eta)-U^0(t,\eta)\,|\le c_4\sqrt{\varepsilon}\tag{5.8}$$

with a constant c_4 independent of t and η. Therefore U^ε converges to U^0 uniformly, as $\varepsilon\to 0$. Form this fact, it follows that U^0 is the unique viscosity solution of (1.7) with $\varepsilon=0$. Consequently $U^0=\nu$. Now, (5.8) yields the speed of convergence of ν^ε.

Theorem 5.3. There is a constant c independent of t and η, such that

$$|\,\nu^\varepsilon(t,\eta)-\nu(t,\eta)\,|\le c\sqrt{\varepsilon}.$$

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