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A BASE POINT FREE THEOREM FOR LOG CANONICAL SURFACES

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0. Introduction

Let (X, Δ) be a complete, log canonical algebraic surface defined over the field of complex numbers $\mathbf{C}$. A nef and big Cartier divisor D on X is *nef and log big* on (X, Δ) by definition if $\deg(D|_C) > 0$ for all irreducible components C of the reduced part $\lfloor \Delta \rfloor$ of Δ.

We follow the notation and terminology of [5].

In [6] Miles Reid introduced the notion of "log big" and gave the statement as follows:

Let (M, Γ) be a complete, log canonical algebraic variety over $\mathbf{C}$ and L a nef Cartier divisor on M. Suppose that $aL - (K_M + \Gamma)$ is nef and log big on (M, Γ) for some $a \in \mathbf{N}$. Then the linear system $|mL|$ is free from base points for every $m \gg 0$.

In this paper we give a proof to this statement in the surface case.

Main Theorem. *Let H be a nef Cartier divisor on X such that $aH - (K_X + \Delta)$ is nef and log big on (X, Δ) for some $a \in \mathbf{N}$. Then the complete linear system $|mH|$ is free from base points for every sufficiently large integer m.*

REMARK 1. In the case where (X, Δ) is a weakly kawamata log terminal projective surface, we gave a proof to the theorem above in [1].

REMARK 2. Under the assumption that $aH - (K_X + \Delta)$ is not nef and log big on (X, Δ) but nef and big, there exists a counterexample due to Zariski in which the theorem is not valid ([3], remark 3-1-2).

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1. Preliminaries

First we collect some well known results concerning normal surfaces, which will be required for the proof of Main Theorem.

Let $\mu : V \to W$ be a birational morphism between complete, $\mathbf{Q}$-factorial, normal algebraic surfaces over $\mathbf{C}$.

Lemma 1 (Projection Formula). *For $\mathbf{Q}$-divisors D on V and G on W, $(D, \mu^* G) = (\mu_* D, G)$.*

Lemma 2. *If D is a nef $\mathbf{Q}$-divisor on V, $\mu_* D$ is also nef on W.*

Proof. For all irreducible curves C on W, $(\mu_* D, C) = (D, \mu^* C) \geq 0$ from Lemma 1.

Lemma 3. *If D is a big $\mathbf{Q}$-divisor on V, $\mu_* D$ is also big on W.*

Proof. For a Cartier divisor A on V, $H^0(V, \mathcal{O}_V(A)) \hookrightarrow H^0(W, \mathcal{O}_W(\mu_* A))$, because V and W are normal.

Lemma 4. *Let A be a non μ-exceptional prime divisor and B a nef $\mathbf{Q}$-divisor on V. $(\mu_* A, \mu_* B) \geq (A, B)$.*

Proof. From Lemma 1, $(\mu_* A, \mu_* B) = (\mu^* \mu_* A, B)$. Here $\mu^* \mu_* A \geq A$, because A is not μ-exceptional. Thus $(\mu^* \mu_* A, B) \geq (A, B)$.

Lemma 5. *Every complete, $\mathbf{Q}$-factorial, normal algebraic surface over $\mathbf{C}$ is projective.*

Proof. Assume that μ is a resolution of singularities of W and A an ample divisor on V. Then $\mu_* A$ is an ample $\mathbf{Q}$-divisor on W from Lemma 3 and 4 and the Nakai-Moishezon criterion.

Next we note a well known result concerning surface singularities, which will be used without mentioning it throughout this paper. For the convenience of the reader we indicate a proof, which relies on the log minimal model program.

Proposition 0. *If (X, Δ) is weakly kawamata log terminal, then X is $\mathbf{Q}$-factorial.*

Proof. Let $f : M \to X$ be a log resolution of (X, Δ) such that $K_M + f_*^{-1}\Delta + F = f^*(K_X + \Delta) + E$ with $E \geq 0$ and Supp(E)=Exc(f), where $F = \Sigma\{F_i;\quad F_i$ is an f-exceptional prime divisor $\}$.

Apply the relative log minimal model program to $f : (M, f_*^{-1}\Delta + F) \to X$.

We end up with a $\mathbf{Q}$-factorial weakly kawamata log terminal surface (over X) $g : (Y, g_*^{-1}\Delta + (F)_Y) \to X$ such that $K_Y + g_*^{-1}\Delta + (F)_Y$ is g-nef.

Because $(E)_Y$ is a g-exceptional g-nef divisor, $(E)_Y = 0$. Thus Exc(g)= $\emptyset$. Therefore g is an isomorphism from Zariski's Main Theorem.

Lastly we mention variations of results by Kawamata and Keel-Matsuki-McKernan.

Proposition 1. *Suppose that (X, Δ) is a weakly kawamata log terminal projective surface. Let R be a $(K_X + \Delta)$-extremal ray. Then there exists a rational curve $C \in R$ such that $(-(K_X + \Delta), C) \leq 4$.*

Proof. For some $r \geq 1$, $r(K_X + \Delta)$ is Cartier. Let R be a $(K_X + \Delta)$-extremal ray. Put $s := \min\{(\Delta, E);\quad E$ is an irreducible component of $\Delta\}$. For $0 < \varepsilon \ll \frac{1}{(|s|+1)r}$, $K_X+(1-\varepsilon)\Delta$ is kawamata log terminal and R is a $(K_X+(1-\varepsilon)\Delta)$-extremal ray.

Thus, from [2], there exists a rational curve $C \in R$ such that $-(K_X + (1-\varepsilon)\Delta).C \leq 4$.

If Δ does not include C, then $(\Delta, C) \geq 0$. Hence $-(K_X + \Delta).C \leq -(K_X + (1-\varepsilon)\Delta).C \leq 4$.

If Δ includes C, then $s \leq (\Delta, C)$. Hence $-(K_X + \Delta).C = -(K_X + (1-\varepsilon)\Delta).C - \varepsilon(\Delta, C) \leq 4 + (-s)\varepsilon$. By the choice of ε, $-(K_X + \Delta).C \leq 4$.

Proposition 2. *Suppose that (X, Δ) is a weakly kawamata log terminal projective surface. D is a nef $\mathbf{Q}$-divisor, but $K_X + \Delta$ is not nef. Set $\lambda := \sup\{\lambda \in \mathbf{Q};\quad D + \lambda(K_X + \Delta)\quad is\quad nef\}$.*

Then λ is a rational number and moreover there is a $(K_X + \Delta)$-extremal ray R such that $(D + \lambda(K_X + \Delta)).R = 0$.

Proof. From Proposition 1 and [4], the proof of 2.1, the assertion follows.

2. Proof of the main theorem

From a result by Shokurov ([5], 17.10) (cf. [7], 9.1) and Lemma 5, we may find a weakly kawamata log terminal projective surface $(Y, S + B)$ and a birational morphism $g : Y \to X$ such that $K_Y + S + B = g^*(K_X + \Delta)$, where S is the reduced part of $S + B$. We note that $g^*(aH - (K_X + \Delta))$ is nef and log big on $(Y, g_*^{-1}\lfloor\Delta\rfloor + B)$. In the case where $S = g_*^{-1}\lfloor\Delta\rfloor$, [1] implies the assertion. Thus we may assume that $S - g_*^{-1}\lfloor\Delta\rfloor > 0$.

We consider the following exact sequence for $m \in \mathbf{N}$:

$$0 \to \mathcal{O}_Y(mg^*H - (S - g_*^{-1}\lfloor\Delta\rfloor)) \to \mathcal{O}_Y(mg^*H) \to \mathcal{O}_{S-g_*^{-1}\lfloor\Delta\rfloor}(mg^*H) \to 0$$

Here $mg^*H - (S - g_*^{-1}\lfloor\Delta\rfloor) - (K_Y + g_*^{-1}\lfloor\Delta\rfloor + B) = mg^*H - (K_Y + S + B) = g^*(mH - (K_X + \Delta)) = g^*(m-a)H + g^*(aH - (K_X + \Delta))$ is nef and log big on

$(Y, g_*^{-1}\lfloor\Delta\rfloor + B)$ for $m \geq a$. Thus from [1],

$$H^1(Y, \mathcal{O}_Y(mg^*H - (S - g_*^{-1}\lfloor\Delta\rfloor))) = 0$$

Therefore the homomorphism

$$H^0(Y, \mathcal{O}_Y(mg^*H)) \to H^0(S - g_*^{-1}\lfloor\Delta\rfloor, \mathcal{O}_{S-g_*^{-1}\lfloor\Delta\rfloor}(mg^*H))$$

is surjective. Because dim $g(S - g_*^{-1}\lfloor\Delta\rfloor) = 0$,

$$(***) \qquad Bs|mg^*H| \cap (S - g_*^{-1}\lfloor\Delta\rfloor) = \emptyset$$

Now run $(K_Y + g_*^{-1}\lfloor\Delta\rfloor + B)$-Minimal Model Program with extremal rays that are g^*H-trivial (cf. [3], lemma 3-2-5 and [4]).

We have three cases:

Case (A). *We obtain the morphism $p : Y \to Z$ such that $g^*H = p^*(p_*g^*H)$, p_*g^*H is Cartier, $(Z, p_*(g_*^{-1}\lfloor\Delta\rfloor + B))$ is a weakly kawamata log terminal projective surface and $K_Z + p_*(g_*^{-1}\lfloor\Delta\rfloor + B)$ gives a non-negative function on $\{C \in \overline{NE}(Z); (p_*g^*H, C) = 0\}$.*

We put $\lambda := \sup\{\lambda \in \mathbf{Q};\quad p_*g^*H + \lambda(K_Z + p_*(g_*^{-1}\lfloor\Delta\rfloor + B))$ is nef $\}$.

If $K_Z + p_*(g_*^{-1}\lfloor\Delta\rfloor + B)$ is nef, then $\lambda = \infty$. If $K_Z + p_*(g_*^{-1}\lfloor\Delta\rfloor + B)$ is not nef and $\lambda = 0$, then there exists a $(K_Z + p_*(g_*^{-1}\lfloor\Delta\rfloor + B))$-extremal ray R such that $(p_*g^*H, R) = 0$ from Proposition 2, but this is a contradiction ! Thus $\lambda > 0$.

We note that $m(p_*g^*H) - p_*(S - g_*^{-1}\lfloor\Delta\rfloor) = p_*(g^*(mH) - (S - g_*^{-1}\lfloor\Delta\rfloor)) = K_Z + p_*(g_*^{-1}\lfloor\Delta\rfloor + B) + p_*(g^*(mH) - (K_Y + S + B)) = K_Z + p_*(g_*^{-1}\lfloor\Delta\rfloor + B) + p_*(g^*(mH - (K_X + \Delta))) = K_Z + p_*(g_*^{-1}\lfloor\Delta\rfloor + B) + p_*(g^*(aH - (K_X + \Delta))) + p_*(g^*(m - a)H)$ is nef for m such that $m - a > \frac{1}{\lambda}$. Here $p_*(g^*(aH - (K_X + \Delta)))$ is nef and log big on $(Z, p_*(g_*^{-1}\lfloor\Delta\rfloor + B))$ from Lemmas 2,3 and 4.

Thus $|m(r(p_*g^*H) - t(p_*(S - g_*^{-1}\lfloor\Delta\rfloor)))|$ is base point free for $m \gg 0$ from [1], where t is a positive natural number such that $t(p_*(S - g_*^{-1}\lfloor\Delta\rfloor))$ is a Cartier divisor and r is a sufficiently large prime number.

Therefore $Bs|m(p_*g^*H)| \subseteq p_*(S - g_*^{-1}\lfloor\Delta\rfloor)$ for every sufficiently large integer m (cf. [3], the proof of theorem 3-1-1). Noting the fact that $g^*H = p^*(p_*g^*H)$, we come to the conclusion that $Bs|m(p_*g^*H)| = \emptyset$ from (***).

Case (B). *We obtain the morphism $p : Y \to Z$, where Z is a smooth curve and $g^*H \sim p^*P$ for some divisor P on Z.*

If $\deg(P) > 0$, then $|mP|$ is base point free for $m \gg 0$. Thus $|mH|$ is base point free.

If $\deg(P) = 0$, then g^*H is numerically trivial. From (***), $|g^*(mH)| \neq \emptyset$ for $m \geq a$. Thus mH is linearly trivial.

Case (C). *We obtain the morphism $p : Y \dashrightarrow Z$, where Z is a point and g^*H is linearly trivial.*

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