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Osaka J. Math.
36 (1999), 259-335

$\mathbb{Q}$-HOMOLOGY PLANES ARE RATIONAL-III

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(Received May 19, 1997)

1. Introduction

This is the last paper in a series of three articles. Recall from part I that a $\mathbb{Q}$ – *homology plane* is a smooth (affine) connected algebraic surface $V over \mathbb{C}$ with $H_i(V;\mathbb{Q}) = (0)$ for $i > 0$ (cf. [13]). In part II, written in collaboration with A. R. Shastri, we have proved that a smooth projective completion of V is not a surface of general type (cf. [5]). In this part III we will complete the proof of the following result which answers affirmatively a question of M. Miyanishi.

Theorem. *A $\mathbb{Q}$-homology plane is rational.*

It has been observed in part I that to complete the proof of the Theorem, it suffices to prove the following.

Proposition 1.1. *A smooth projective completion of a $\mathbb{Q}$-homology plane cannot be an elliptic surface of Kodaira dimension* 1.

We shall continue to use notations and results of part I (cf. [13]). We have the following consequences of the rationality of singular $\mathbb{Q}$-homology planes proved in part I and our Theorem.

Corollary 1. *An algebraic vector bundle on a $\mathbb{Q}$-homology plane is a direct sum of a trivial vector bundle and a line bundle.*

So far there are no general known results about arbitrary $\mathbb{Q}$-homology 3-folds. In this connection, we have the following result.

Corollary 2. *Let W be a smooth (irreducible) affine 3-fold with $H_i(W;\mathbb{Q}) =$ (0) for $i > 0$. If W admits a non-trivial algebraic action of $\mathbb{C}^\star$, then W is rational.*

For proofs of these corollaries, see section 6.

REMARK 1.1. *There is a remarkable example due to J. Winkelmann of a smooth quasi-projective 4-fold W which is diffeomorphic to $\mathbb{C}^4$ but which is not affine (cf. [14]). As mentioned in part I, by a result of T. Fujita any $\mathbb{Q}$-homology plane is affine.*

In [4], $\mathbb{Z}$-homology planes which have smooth projective completions which are elliptic surfaces were considered. The present proof differs considerably from the proof in [4]. As the proof of the Theorem is quite long and involved we give here an overview of the main steps in the proof.

Step 1. We assumed that a smooth projective completion X of our homology plane V is non-rational. Hence X is a simply connected surface with $p_g(X) = 0$. By classification of surfaces, X is then either an elliptic surface of Kodaira dimension 1 or a surface of general type. We assumed that $\Delta = X \setminus V$ is a minimal normal crossing divisor. For this pair (X, Δ) we applied Kobayashi's inequality and deduced an auxiliary inequality which involves various integral invariants of the exceptional divisor of the blow down to the smooth minimal model of X.

Step 2. In [12] the auxiliary inequality proved in [4] along with a detailed knowledge of Zariski-Fujita decomposition of $K_X + \Delta$ was used to show that there are atmost eight possible dual graphs for Δ, in case X is of general type. For this we developed a *Mathematica* program. Representation of K_X as a divisor supported on Δ and properties of *bark of* Δ were used to deduce this list.

Step 3. In case X is an elliptic surface, we proved that there are exactly two multiple fibres of multiplicities 2 and 3. Also, there are at least two horizontal components of Δ for the elliptic fibration. In this part III, we will show that no full fibre of the elliptic fibration is contained in Δ. In [12] a rough list of possible dual graphs of Δ was obtained. This is included here as proposition 2.1. In this part, by a study similar to step 2, we have reduced the number of possibilities for Δ to at most 37.

Step 4. Next, we use the rational equivalence $nK \sim \sum x_i D_i$ where D_is are the irreducible components of Δ. An interesting observation at this stage is that the smallest integer n with this property is *strictly* bigger than 1. Hence we can construct a cyclic ramified cover $f : \overline{X} \to X$ with Galois group $\mathbb{Z}/n\mathbb{Z}$. Since Δ has normal crossings $\overline{X}$ has at most cyclic quotient singularities. We can easily calculate invariants of the minimal resolution $\tilde{X}$ of $\overline{X}$ like $(K_{\tilde{X}})^2$, Euler characteristic and $\chi(\tilde{X}, \mathcal{O})$. In part II, we easily get a contradiction to Noether's formula in case X is of general type (cf. [5]).

Step 5 In this part III, we use the Lefschetz fixed point theorem of Atiyah-

Singer for the automorphism group $\boldsymbol{Z}/n\boldsymbol{Z}$ of $\tilde{X}$. Some results of T. Petrie concerning finite cyclic group actions on trees are used to express the right hand side of the fixed point formula in a computable manner. This gives a contradiction in almost all of the 37 trees. We complete the proof of the Theorem by giving some ad hoc arguments to eliminate the remaining trees.

We are thankful to R. R. Simha for showing us an argument which was very helpful in an earlier version of our proof. We are also thankful to A. R. Shastri for the discussions we had with him.

1.1. Preliminaries

Unless otherwise stated, all varieties are defined over the field of complex numbers $\mathbb{C}$. For a smooth projective surface Y, a $(-n)$-curve $C \subset Y$ is a smooth (irreducible) rational curve C with $(C)^2 = -n$.

In this section we will collect together several auxiliary results which will be used crucially in the proof.

1.1.1. Cyclic quotient singularities

We recall some standard results about resolutions of cyclic quotient singularities of surfaces due to Jung-Hirzebruch. For this, see [2].

Let $f : Y \to Z$ be a finite ramified covering with Z a smooth surface and Y a normal surface. Assume that the branch locus $B \subset Z$ is a divisor with normal crossings. It is well-known that in this case Y has at worst quotient singularities and they lie over the singularities of B. Suppose that Γ is a divisor on Z such that there is a linear equivalence $n\Gamma \sim \Sigma_1^r a_i B_i$ with $B_1, B_2, \ldots, B_r$ the irreducible components of B and $a_i \geq 0$. We assume that n is the smallest integer > 1 (for Γ) with this property and f is the corresponding ramified cover. Clearly, we can assume that $a_i < n$ for all i.

Let $p \in B$ be a singular point B, say $p \in B_1 \cap B_2$. Then there exist local holomorphic coordinates z_1, z_2 in a neighborhood U of p such that $f^{-1}(U)$ is the normalization of the surface $\{w^n = z_1^{a_1} z_2^{a_2}\}$ in a small neighborhood of the origin in $\mathbb{C}^3$. Here, locally near p, $B_i = \{z_i = 0\}$.

(a) Let $d = g.c.d.(n, a_1, a_2)$, $n = \nu.d$, $a_i = \alpha_i.d$ for $i = 1, 2$. Then $f^{-1}(U)$ is a disjoint union of d open sets, each isomorphic to the normalization of $\{w^\nu = z_1^{\alpha_1} z_2^{\alpha_2}\}$.

(b) Now assume that $d = 1$ and let $d_i = g.c.d.(n, a_i)$, $a_i = \alpha_i.d_i$, $n = \nu.d_1.d_2$. Let W be the germ of the surface in $\mathbb{C}^3$ defined by

$$W = \{u^\nu = y_1^{\alpha_1}.y_2^{\alpha_2}\}.$$

Then the map $W \to f^{-1}(U)$ given by $w = u, z_1 = y_1^{d_2}, z_2 = y_2^{d_1}$ is finite and birational. Therefore the induced map on their normalizations $\widetilde{W} \to \widetilde{f^{-1}(U)}$ is an isomorphism. In particular, the germs of $\widetilde{W}$ and $\widetilde{f^{-1}(U)}$ at their singular points are isomorphic.

(c) Finally, let $g.c.d.(n, a_1) = g.c.d.(n, a_2) = 1$. Define the integer q, $0 < q < n$, by $a_1 q \equiv -a_2 \; mod \; (n)$, say $a_1 q = rn - a_2$ with $0 < r \leq a_1$. Then $g.c.d.(q, n) = 1$ and $\widetilde{W}$ is isomorphic to the normalization of the surface $\{\nu^n = z_1 z_2^{n-q}\}$.

The minimal resolution of singularity of $\widetilde{W}$ has exceptional curve $C_1 + \cdots + C_l$ with linear dual graph, where $(C_i)^2 = -e_i$, $n/q = e_1 - \frac{1}{e_2 - \frac{1}{\cdots - \frac{1}{e_l}}}$ and each C_i is a smooth rational curve. The proper transform of $\{z_1 = 0\}$ meets C_1 transversally in one point and no other C_i and the proper transform of $\{z_2 = 0\}$ meets C_l transversally in one point and no other C_i.

1.1.2. Intersection theory on Y

Since Y has at most quotient singularities, some multiple of every Weil divisor is Cartier. Therefore for any two irreducible curves (not necessarily distinct) C_1, C_2 on Y, $(C_1.C_2)$ makes sense as a rational number. Let the ramification index for an irreducible component of $f^{-1}(B_i)$ be m_i. Then $f^\star(B_i) = m_i \overline{B_i}$, where $\overline{B_i}$ is the reduced inverse image of B_i in Y. Then $(f^\star B_i)^2 = nB_i^2 = m_i^2(\overline{B_i})^2$.

From the local description of the singularities of Y discussed in section 1.1.1, it follows that $\overline{B_i}$ is a disjoint union of irreducible curves, say $B_{i1} + B_{i2} + \cdots + B_{il_i}$. Let p_{ij} be a singular point of Y lying on B_{ij}. Let $\sigma : \widetilde{Y} \to Y$ be a minimal resolution of singularities of Y. Then $(\sigma^\star B_{ij})^2 = B_{ij}^2$. If E_{ijk} are all the irreducible curves contracting to points in B_{ij}, then $\sigma^\star B_{ij} = B'_{ij} + \Sigma \lambda_{ijk} E_{ijk}$, where B'_{ij} is the proper transform of B_{ij} in $\widetilde{Y}$ and the positive rational numbers λ_{ijk} are determined by the conditions $(\sigma^\star B_{ij}.E_{ijk}) = 0$ for all k. The dual graph of each connected component of $\cup_k E_{ijk}$ is linear and B'_{ij} meets one of the end irreducible components for each connected component according to the description above.

Consider again the equation $\{w^n = z_1^{a_1} z_2^{a_2}\}$, where $B_i = \{z_i = 0\}$ near $B_1 \cap B_2$. The ramification index of any irreducible component of $f^{-1}(B_i)$ lying over B_i is given by $n/g.c.d.(n, a_i)$. Using all these observations, we can find out the weighted dual graph of $(\sigma \circ f)^{-1}(B)$.

We now recall the ramification formula which relates K_Y, the canonical divisor of Y, K_Z, the canonical divisor of Z and the ramification divisor B. It states that

$$f^\star(K_Z) = K_Y + \sum_i (m_i - 1)\overline{B}_i.$$

1.1.3. Lefschetz fixed point formula of Atiyah-Singer

We will recall the special case of Lefschetz fixed point formula for compact complex surfaces with a holomorphic action of a finite cyclic group proved by Atiyah-Singer (cf. page 567, [1]).

Let Z be a connected compact complex manifold of dimension 2 and $G = (g)$ a finite cyclic group acting on Z by holomorphic automorphisms. We assume that the action of G is faithful. Then the fixed point set Z^g is a disjoint union of isolated fixed points P_j and smooth irreducible curves Δ_k. Denote by π_k the genus of Δ_k and let $e^{i\theta_k}$ be the eigenvalue of g on the normal bundle of Δ_k. Clearly this eigenvalue is a primitive n^{th} root of unity, where $n = |G|$. The Lefschetz fixed point formula is the following equality

$$\sum(-1)^l Tr(g|H^l(Z,\mathcal{O})) = \sum_j \frac{1}{det(1 - g|_{T_{P_j}})} + \sum \frac{(1-\pi_k)(1-e^{-i\theta_k}) - \Delta_k^2.e^{-i\theta_k}}{(1-e^{-i\theta_k})^2}$$

(1)

Here, T_{P_j} is the tangent space to Z at P_j.

1.1.4. Some results of T. Petrie on G-actions on trees

In order to evaluate the right hand side of the fixed point formula for certain ramified covers of the surface X, we use some useful results of T. Petrie (cf. section 3 and 4, [11]). Let Z be a smooth projective surface and D a connected normal crossing divisor on Z. Assume that a finite cyclic group $G = (g)$ of automorphisms of Z acts satisfying the following conditions.

(1) The dual graph of D is a tree of smooth rational curves and there is at least one irreducible component of D which is pointwise fixed by G.

(2) Each irreducible component of D is stable under G and $Z^G \subset D$.

Let T be the weighted dual graph of D. We will denote the vertices of T by the letters $u, v, w, \ldots$. The weight of a vertex of T is the self-intersection of the corresponding irreducible component of D. Next we introduce certain polynomials L_k for $k = 0, 1, 2, \ldots$. L_k is a polynomial in variables $z_1, z_2, \ldots, z_{k-1}$ with integer coefficients and are defined inductively as follows.

$$L_0 = 0,\ L_1 = 1,\ L_{k+1} = -z_k L_k - L_{k-1}.$$

Thus $L_2 = -z_1$, $L_3 = z_1 z_2 - 1$, etc. If T has a branch point, then from the assumption (2) above we see that the corresponding irreducible component of D is contained in Z^G. Now we fix once for all a vertex b of T such that the corresponding curve in D is in Z^G. By assumption (1), such a vertex b exists. For vertices u, v in T, let $\langle u, v\rangle$ be the linear subtree of T with end points u and v. We define $u < v$ if $\langle b, u\rangle$ has strictly less number of vertices than $\langle b, v\rangle$.
Define a function L from T to $\boldsymbol{Z}$ as follows.

$$L(v) = L_k(z_1, \ldots, z_{k-1}),$$

where $bv_1 \ldots v_{k-1}v$ is the linear subtree with end points b and v and $z_i = v_i.v_i$ is the weight at v_i. Hence $L(b) = 0$. It follows easily from the definitions that if $b \leq u < v < w$ and uv, vw are links in T, then $L(w) = -L(u) - w_v.L(v)$, where w_v is the weight at v. In fact, L is defined by this equality and the conditions that $L(b) = 0$ and $L(\widetilde{b}) = 1$, where $\widetilde{b}$ is any vertex linked to b.

Let $|G| = n$. We identify g with $\omega := e^{2\pi i/n}$. This fixes an isomorphism between G and the group of n^{th} roots of unity. The complex 1-dimensional representation of G such that an element of G acts by multiplication by its a^{th} power is denoted by t^a.

Let $P \cong I\!P^1$ be an irreducible curve on Z such that G acts by automorphisms on P. We do not assume this action to be faithful. If $p \in P$ is a fixed point then the tangent space to P at p is $t^{a(p)}$ for some integer $a(p)\ mod\, n$. Clearly, $a(p) \cong 0\ mod\, n$ iff the action of G on P is trivial. If the action on P is not trivial then there are exactly two fixed points, say p, q. Then we see easily that $a(p) = -a(q)$. For any vertex u of T let P_u denote the irreducible component of D corresponding to u.

Let $P = \Delta_i$ be some irreducible component of Δ. By assumption, P is G-stable and so G acts on the total space of the normal bundle of P in Z. If $p \in P$

is a fixed point then the fiber N_p of the normal bundle is $t^{n(p)}$ for some integer $n(p)$ *mod* n. The next result of Petrie is quite useful for our purpose.

Lemma 1.1. *If p, q are distinct fixed points in P then $n(q) = n(p) - a(p).(P)^2$.*

Since a fixed point $p \in \Delta$ may lie on two curves Δ_i, Δ_j, we write $a(p, v_i)$ when describing the action on $T_pP_{v_i}$ as $t^{a(p,v_i)}$. Similarly, $n(p, v_i)$ is defined. If uv is a link in T then $q(u, v) := P_u \cap P_v$ is a fixed point.

Lemma 1.2. *In the notations above, we have, $n(q(u, v), u) = a(q(u, v), v)$.*

From lemmas 1.1 and 1.2 we get the next result.

Lemma 1.3. *Let uv and vw be links in T. Then*

$$a(q(v, w), w) = a(q(u, v), u) - (P_v)^2.a(q(u, v), v).$$

Define $a(b) = 0$ and for any vertex $v \neq b$ define $a(v) = a(q(u, v), v)$, where uv is a link in T and $u < v$. Since $P_b \subset Z^G$ and for any vertex $\widetilde{b}$ linked to b the integer $a(\widetilde{b})$ describes the action of G on any fiber of the normal bundle to P_b, we have equality $a(\widetilde{b}_1) = a(\widetilde{b}_2)$ for any vertices $\widetilde{b}_1, \widetilde{b}_2$ linked to b. The next result explains the role of the function L defined earlier.

Lemma 1.4. *For a vertex $v \in T$, $a(v) = L(v).a(\widetilde{b})$, where $\widetilde{b}$ is any vertex linked to b. Further, $a(\widetilde{b})$ is a unit mod n.*

This result enables us to calculate the integers $a(p, v_i)$ for all vertices v_i. If the curve $P_{v_i} \subset Z^G$, then using lemma 1.2 we can find the eigenvalue for the action of G on the normal bundle of P_{v_i}. Similarly, for calculation of the term *det* $(1 - g|_{T_p})$ for an isolated fixed point $p \in \Delta$, we need to calculate the eigenvalue for the action on the fiber N_p of the normal bundle to a suitable P_u passing through p. This is done using lemmas 1.1 and 1.2. Thus we are able to calculate the right hand side of the fixed point formula.

REMARK 1.2. *In some applications of (1) in section 5, there is no curve in Δ which is pointwise fixed by G. Even in this case, lemmas 1.1, 1.2 and 1.3 are enough for our purpose.*

1.1.5. Fixed point formula revisited

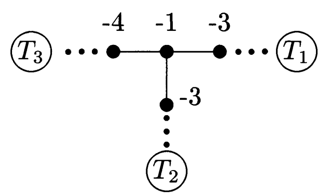
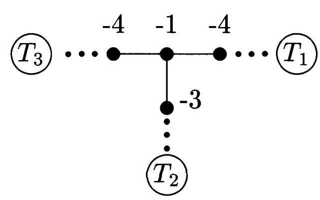
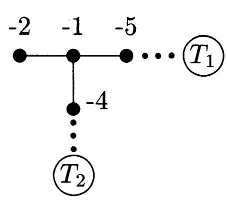
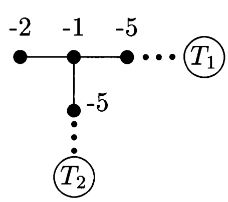
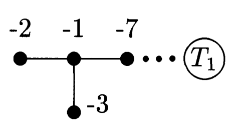
Since each irreducible component of Δ is rational, $\pi_k = 0$ in the notation of section 1.1.3. Using Hodge theory, we have a conjugate linear isomorphism between $H^i(Z, \mathcal{O})$ and $H^0(Z, \Omega^i)$. In our case *dim* $Z = 2$. Suppose that $H^1(Z, \mathcal{O}) = (0)$. Then the left hand side of the fixed point formula becomes $1 + \overline{Tr\ (g|H^0(Z, \Omega^2))}$. The trace term is the sum of the eigenvalues of the action of g on $H^0(Z, \Omega^2)$, which are all suitable n^{th} roots of unity.

2. Analysis of the auxiliary inequality

Our aim in this section is to prove the following.

Proposition 2.1. *Let V be a $\mathbb{Q}$-homology plane with $\overline{\kappa}(V) = 2$, and let (X, Δ) be a smooth projetive completion of V with Δ as MNC divisor and let $\kappa(X) = 1$. Then the dual graph T of Δ necessarily falls into one of the cases listed in the following table where T_is and Weight Set are as described in Table 1 of [13].*

Table 3

Sl.No.	β_2	Lemma	Weight Set	Subtree	θ
1	10	2.11	-3,-3		2
2	10	2.11	-4,-3		3
3	10	2.11	-3,-3,-3		3
4	11	2.15	-5,-3		3
5	11	2.10	-4,-3,-3,-3,-1		3
6	11	2.10	-4,-4,-3,-1		3
7	12	2.10	-5,-4,-3,-1		3
8	12	2.10	-5,-5,-1		3
9	13	2.10	-7,-3,-3,-1		3

Lemma 2.1. *If Δ does not contain any (-1)-curve then there is no fibre F of ϕ contained in Δ and exactly one component from each fibre of ϕ is not contained*

in Δ.

Proof. If possible let F denote a fibre of ϕ contained in Δ and F'' the image of F under π. Absence of (-1)-curves in Δ implies that $F \approx F''$. Also, let H (resp. H'') be a horizontal component in Δ (resp. Δ''). The fact that Δ is simply connected implies that F is also simply connected. Hence F cannot be of the type $m\mathrm{I}_b$, $b \geq 0$. The fact that Δ is a MNC curve free of (-1)-curves implies that F cannot be of the type II, III or IV.

The fact that Δ is simply connected implies that $F+H$ is also simply connected and hence H'' intersects F'' at exactly one point. Since Δ is a MNC curve and does not contain any (-1)-curve, this intersection is a transverse intersection. But then $(F''.H'') \geq 6$ implies that F'' has a component of multiplicity at least 6 and hence equal to 6. Thus F'' cannot be of the type I_b^*, $b \geq 0, \mathrm{III}^*$ or IV^*. Thus we assume that F'' and hence F is of the type II^*.

By lemma 4.3 of [13], we know that there are at least two horizontal components in Δ. Therefore $\beta_2'' \geq 2 + b_2(F) = 11$ and hence $\mathcal{E} \setminus \Delta \neq \emptyset$. Since Δ is free from (-1)-curves, by lemma 3.2 of [13], it follows that $e_1 + \sigma \geq 1$. Also we observe that the maximal twigs of F remain as maximal twigs in Δ and hence $\nu = bk(\Delta) < -2$. Since $\lambda \geq 2$ this contradicts (17) of [13]. This proves that Δ does not contain any fibre of ϕ.

Let $\{F_i\}_{i=1}^k$ be the singular fibres of ϕ. Since Δ contains at least two horizontal components

$$b_2(\Delta) \leq \sum((b_2(F_i) - 1) + 2 \leq rk(Pic(X)) = b_2(\Delta).$$

Hence the equality holds everywhere. Thus, exactly one component from each fibre is missing from Δ. This completes the proof of the lemma. ♠

Following lemma, proved in [4] is helpful in our study.

Lemma 2.2. *Suppose F'' is a fibre of ϕ'' contained in Δ'' and is not of type* II^*. *Assume that*

(a) *there is at most one point $x \in F''$ which is worse than an ordinary double point singularity of Δ'' and*

(b) *if x exists, then F'' is of type $m\mathrm{I}_1$,* II, III *or* IV *with $x \in F''$ being the singularity of F'' and at most one horizontal component of Δ'' passes through x. Then $b_1(\Delta'') \geq 2$.*

Proof. First assume that x exists. By lemma 4.3 of [13] we see that there exists a horizontal component H'' not passing through x. If F is of type $m\mathrm{I}_1$, then $m = 2$ or 3, $6|(F''.H'')$ and hence $F'' \cap H''$ consists of at least two points. Since $b_1(m\mathrm{I}_1) = 1$, we have $b_1(\Delta'') \geq b_1(F'' \cup H'') \geq 2$. If F is of type II, III or IV, then it follows that $F'' \cap H''$ consists of at least six points and hence $b_1(\Delta'') \geq b_1(F'' \cup H'') \geq 5$.

Next assume that x does not exist. Then F'' is of type mI_b, (m=2 or 3, $b \geq 1$), $I_b^\star$, $III^\star$ or $IV^\star$. Hence for every component C of F'', the multiplicity of C, $\mu(C) \leq 4$, $F'' \cap H_i''$ consists of points which are all ordinary double points of Δ''. Hence it follows that $b_1(F'' \cup H_i'') \geq 1$ for each horizontal component of Δ''. Hence $b_1(\Delta'') \geq b_1(F'' \cup H_1'' \cup H_2'') \geq 2$. This completes the proof of the lemma. ♠

By (18) of [13] we know that

$$\theta = \lambda + \tau + \sigma + e_1 + u + r_3 + 2r_4 \leq 4.$$

Since each parameter appearing in the above is non-negative we exhaust all possibilities by explicitly considering all possible values for these parameters. Also by lemma 4.3 of [13] we know that Δ contains at least two horizontal components and $\lambda \geq 2$.

Lemma 2.3. *The case $\lambda = 2$, $r_4 = 1$ does not occur.*

Proof. Since $e_1 = r_3 = \sigma = 0$ we have $\mathcal{E} \subset \Delta$. Let $\{D_0\} = R_4$. Since $r_3 = 0$ we see that we reach X'' after contracting D_0. Since Δ has four maximal twigs it is easy to see that $\nu < -1$ and hence this case does not occur. ♠

Lemma 2.4. *The case $\lambda = 4$ does not occur.*

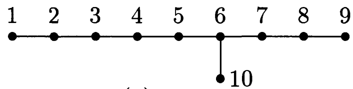
Proof. If $\lambda = 4$, we see that $\theta = 4$ and $\beta_2 = \beta_2'' = 10$. But then $\nu = bk(\Delta) \geq -1$. Since there are at least two horizontal components in Δ, possible weight set for Δ are the following:

(a) $\{-3, -3, -3, -3, -2, \ldots, -2\}$
(b) $\{-4, -3, -3, -2, \ldots, -2\}$
(c) $\{-5, -3, -2, \ldots, -2\}$ or
(d) $\{-4, -4, -2, \ldots, -2\}$.

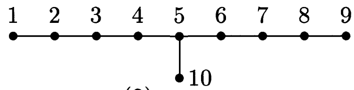
Since Δ has at least three tips, in each of these cases we see that in the worst case the weight set of the tips are $\{-3, -3, -3\}$, $\{-4, -3, -3\}$, $\{-5, -3, -2\}$ and $\{-4, -4, -2\}$ respectively. In all cases except in the case of $\{-4, -3, -3, -2, \ldots, -2\}$, we see that $\nu < -1$. In case the weight set is $\{-4,-3,-3,-2,\ldots,-2\}$ if Δ has four (or more) tips or if Δ has a (-2)-tip, we see that $\nu < -1$. Hence we need to consider only ten vertex trees with exactly three tips and whose weight set of the tips is $\{-4,-3,-3\}$. If every maximal twig has at least two components, then $\nu < -\frac{2}{5} - \frac{2}{5} - \frac{2}{7} < -1$ and hence not possible. Hence at least one of the maximal twigs contain exactly one irreducible component. Such trees arise from partitions of 9 into exactly three parts with at least one of the summands equal to 1. Following are such partitions:

$$\begin{aligned} 9 &= 1+1+7 \\ &= 1+2+6 \\ &= 1+3+5 \\ &= 1+4+4 \end{aligned}$$

Since Δ is free from (-1)-curves, trees corresponding to the partitions 1+1+7 and 1+2+6 are negative definite and hence cannot occur. Following are the trees corresponding to the remaining partitions.



(1)



(2)

We study each of these trees individually and eliminate them.

Tree 1: If $w_1 = -4$, $w_9 = -3$ and $w_{10} = -3$, then $\nu = -\frac{5}{16} - \frac{3}{7} - \frac{1}{3} = -\frac{361}{336} < -1$ and hence this combination cannot occur.
If $w_1 = -3$, $w_9 = -3$ and $w_{10} = -4$, then the tree is negative definite and hence this combination cannot occur.
If $w_1 = -3$, $w_9 = -4$ and $w_{10} = -3$, then $\nu = -\frac{5}{11} - \frac{3}{10} - \frac{1}{3} = -\frac{359}{330} < -1$ and hence this combination cannot occur.

Tree 2: If $w_1 = -4$, $w_9 = -3$ and $w_{10} = -3$, then $\nu = -\frac{4}{13} - \frac{4}{9} - \frac{1}{3} = -\frac{127}{117} < -1$ and hence this combination cannot occur.
If $w_1 = -3$, $w_9 = -3$ and $w_{10} = -4$, then $\nu = -\frac{4}{9} - \frac{4}{9} - \frac{1}{4} = -\frac{41}{36} < -1$ and hence this combination cannot occur.

This completes the proof of the lemma. ♠

2.1. The case $e_1 \neq 0$:

In this section we prove that $e_1 = 0$. We start with the following lemma.

Lemma 2.5. *If $e_1 \neq 0$, we have $\theta \leq 3$.*

Proof. By (18) of [13], we know that $\theta \leq 4$. If possible let $\theta = 4$. But then by (17) of [13] we have $\nu \geq -1$. Since $e_1 \neq 0$, we know that there exits a component $L_0 \not\subset \mathcal{E}_1 \setminus \Delta$. By lemma 3.4(b) of [13] we see that $L_0 \subset R_3$ and hence $r_3 \neq 0$. By lemma 4.3 of [13] we see that $\lambda \geq 2$ and hence we have $\lambda = 2, e_1 = 1$ and $r_3 = 1$. Let D_1, D_2 and D_3 be the three components of Δ such that $(L_0.D_i) = 1,\ i = 1,2,3$.

First we note that if one of $(D_i)^2 = -2$ for $i = 1,2,3$ – say $(D_3)^2 = -2$ – then $(D_1)^2,\ (D_2)^2 \leq -3$. We need to contract $\phi_1(D_3)$ after contracting L_0. Observe that if $(D_3.D_1) = (D_3.D_2) = 0$, then the fact that Δ is connected implies that D_3 intersects some other component of Δ. But then $D_3 \in R_3$, which is a contradiction as $R_3 = \{L_0\}$. If $(D_3.D_1) = (D_3.D_2) = 1$, then after contracting $\phi_1(D_3)$, we see that both D_1'' and D_2'' are singular curves. This is a contradiction to the fact that $\tau = \sum m_{t,i} - 2n_2 = 0$. Thus we may assume that $(D_3.D_1) = 1$ and $(D_3.D_2) = 0$. But then clearly $u \neq 0$ which is a contradiction.

Thus $(D_i)^2 \leq -3$ for $i = 1,2,3$. In this case we reach X'' after blowing down L_0. Thus $n_1 = 1$ and $n_2 = 0$ which implies that $\beta_2 = \beta_2'' + n_1 + n_2 = 11$. Since all the irreducible components of D'' are non-singular rational curves and $\lambda = 2$, we see that D'' consists of exactly two (-3)-curves and nine (-2)-curves. Clearly, $(D_i)^2 = (D_i'')^2 - 1$ for $i = 1,2,3$.

If D_1'', D_2'' and D_3'' are all (-2)-curves, then Δ consists of five (-3)-curves and six (-2)-curves. Since Δ has at least three tips, by lemma 2.3 of [13], we have $\nu < -1$. Hence at least one of D_i'', $i = 1,2,3$ is a (-3)-curve.

Let exactly one of D_i'', $i = 1,2,3$ be a (-3)-curve. Then the weight set of Δ is $\{-4,-3,-3,-3,-2,\ldots,-2\}$. If Δ has four (or more) maximal twigs, then by lemma 2.3 of [13], we have $\nu < -1$. Thus Δ has exactly three maximal twigs. If one of the tips has weight (-2), then it is easy to see that $\nu < -1$. Also for the same reason all three tips cannot be (-3)-tips. Thus we need to consider all the eleven vertex trees with exactly three maximal twigs and the weight set of the tips $\{-4,-3,-3\}$. Let the three maximal twigs be denoted by M_1, M_2 and M_3. If all the three maximal twigs have at least two irreducible components each, we see that they are the following (or twigs with bark *less* than these): [3,3], [3,2] and [4,2] or

[3,2], [3,2] and [4,3]. In the former case $\nu \leq -\frac{3}{8} - \frac{2}{5} - \frac{2}{7} = -\frac{297}{280} < -1$. In the latter case $\nu \leq -\frac{2}{5} - \frac{2}{5} - \frac{3}{11} = -\frac{59}{55} < -1$. Hence these cases cannot occur. Thus we see that at least one of the maximal twigs – say M_1 – consists of exactly one component.

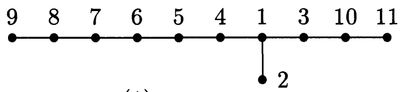
If possible let M_1 be [4] twig. Now if both M_2 and M_3 have at least two components each, then $\nu \leq -\frac{1}{4} - \frac{3}{8} - \frac{2}{5} = -\frac{41}{40} < -1$ which is a contradiction. Hence one of them – say M_2 – has exactly one component. But then, since Δ is free from (-1)-curves we easily see that this tree has no positive eigen value. Thus M_1 cannot be a [4] twig.

If possible let M_1 be a [3] twig. If both M_2 and M_3 have at least two irreducible components each, then M_2 and M_3 have twigs of the form [4,3] and [3,2] (resp) or [4,2] and [3,3] (resp). In the former case $\nu \leq -\frac{1}{3} - \frac{3}{11} - \frac{2}{5} = -\frac{166}{165} < -1$. In the latter case twigs have to be either [3], $[4,2,2]$ and $[3,3]$ in which case $\nu \leq -\frac{1}{3} - \frac{3}{10} - \frac{3}{8} = -\frac{242}{240} < -1$ or [3], $[4,2]$ and $[3,3,2]$ in which case $\nu \leq -\frac{1}{3} - \frac{2}{7} - \frac{5}{13} = -\frac{274}{271} < -1$. Hence one of M_2 or M_3 has exactly one component. Thus we need to consider a eleven-vertex tree with exactly three tips, two of which contain only one component each and the weight of tips are –4, –3 and –3. But then, absence of (-1)-curve in Δ implies that this tree does not have any positive eigen value and hence this case cannot occur.

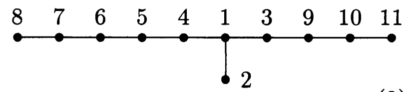
Let two of D_1'', D_2'', D_3'' be (-3)-curves. Then the weight set of Δ is seen to be $\{-4, -4, -3, -2, \ldots, -2\}$. If Δ has four maximal twigs, by lemma 2.3 of [13] we see that $\nu < -1$. Thus we may assume that Δ has three maximal twigs. Such trees arise from partitions of 10 into exactly three parts:

$$\begin{aligned}
10 &= 1+1+8 \\
&= 1+2+7 \\
&= 1+3+6 \\
&= 1+4+5 \\
&= 2+2+6 \\
&= 2+3+5 \\
&= 2+4+4 \\
&= 3+3+4
\end{aligned}$$

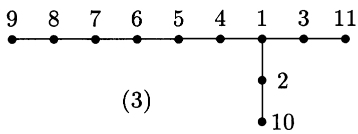
The trees corresponding to the partitions 1+1+8 and 1+2+7 are clearly seen to be negative definite or $\nu < -1$ and hence cannot occur. Trees corresponding to the partitions 2+4+4 and 3+3+4 contain a fibre of ϕ contradicting lemma 2.1 or $\nu < -1$. Hence we need to consider the following four trees.



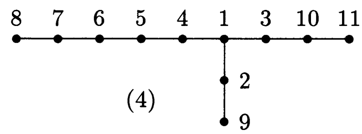
(1)



(2)



(3)



(4)

Tree 1: Clearly if $w_2 = -4$, the tree has no positive eigen value. If $w_2 = -3$, then $w_9 = w_{11} = -4$. But then we see that $K = -\frac{364}{71}D_1 - \frac{145}{71}D_2 - \frac{269}{71}D_3 - \frac{314}{71}D_4 - \frac{264}{71}D_5 - \frac{214}{71}D_6 - \frac{164}{71}D_7 - \frac{114}{71}D_8 - \frac{64}{71}D_9 - \frac{174}{71}D_{10} - \frac{79}{71}D_{11}$ and $(K)^2 = -\frac{431}{71}$ and hence this tree cannot occur.

Tree 2: If $w_2 = w_{11} = -4$ and $w_8 = -3$ then $K = -\frac{142}{31}D_1 - \frac{51}{31}D_2 - \frac{114}{31}D_3 - \frac{119}{31}D_4 - \frac{96}{31}D_5 - \frac{73}{31}D_6 - \frac{50}{31}D_7 - \frac{27}{31}D_8 - \frac{86}{31}D_9 - \frac{58}{31}D_{10} - \frac{30}{31}D_{11}$ and $(K)^2 = -\frac{189}{31}$. Hence this case cannot occur. If $w_2 = w_8 = -4$ and $w_{11} = -3$, then $K = -\frac{106}{23}D_1 - \frac{38}{23}D_2 - \frac{85}{23}D_3 - \frac{89}{23}D_4 - \frac{72}{23}D_5 - \frac{55}{23}D_6 - \frac{38}{23}D_7 - \frac{21}{23}D_8 - \frac{64}{23}D_9 - \frac{43}{23}D_{10} - \frac{22}{23}D_{11}$ and $(K)^2 = -\frac{140}{23}$. Hence this case cannot occur. If $w_2 = -3$ and $w_8 = w_{11} = -4$, then $K = -\frac{382}{53}D_1 - \frac{145}{53}D_2 - \frac{302}{53}D_3 - \frac{317}{53}D_4 - \frac{252}{53}D_5 - \frac{187}{53}D_6 - \frac{122}{53}D_7 - \frac{57}{53}D_8 - \frac{222}{53}D_9 - \frac{142}{53}D_{10} - \frac{62}{53}D_{11}$ and $(K)^2 = -\frac{383}{53}$. Hence this case cannot occur.

Tree 3: If $w_9 = w_{10} = -4$ and $w_{11} = -3$ then $K = \frac{131}{3}D_1 + \frac{74}{3}D_2 + \frac{26}{3}D_3 + \frac{110}{3}D_4 + \frac{89}{3}D_5 + \frac{68}{3}D_6 + \frac{47}{3}D_7 + \frac{26}{3}D_8 + \frac{5}{3}D_9 + \frac{17}{3}D_{10} + \frac{25}{3}D_{11}$ and $(K)^2 = 23$. Hence this case cannot occur. If $w_9 = -3$ and $w_{10} = w_{11} = -4$ then $K = -59D_1 - 34D_2 - 34D_3 - 50D_4 - 41D_5 - 32D_6 - 23D_7 - 14D_8 - 5D_9 - 9D_{10} - 9D_{11}$ and $(K)^2 = -41$. Thus this tree cannot occur.

Tree 4: If $w_8 = w_9 = -4$ and $w_{11} = -3$, then $K = \frac{62}{11}D_1 + \frac{226}{77}D_2 + \frac{299}{77}D_3 + \frac{49}{11}D_4 + \frac{36}{11}D_5 + \frac{23}{11}D_6 + \frac{10}{11}D_7 + -\frac{3}{11}D_8\frac{18}{77}D_9 + \frac{164}{77}D_{10} + \frac{29}{77}D_{11}$ and $(K)^2 = \frac{23}{77}$. hence this case cannot occur. If $w_8 = w_{11} = -4$ and $w_9 = -3$, then $K = \frac{14}{3}D_1 + \frac{13}{5}D_2 + \frac{46}{15}D_3 + \frac{11}{3}D_4 + \frac{8}{3}D_5 + \frac{5}{3}D_6 + \frac{2}{3}D_7 + \frac{-1}{3}D_8 + \frac{8}{15}D_9 + \frac{22}{15}D_{10} - \frac{2}{15}D_{11}$ and $(K)^2 = -\frac{2}{5}$. Thus this case cannot occur. If $w_8 = -3$ and $w_9 = w_{11} = -4$, then $K = \frac{148}{23}D_1 + \frac{78}{23}D_2 + \frac{99}{23}D_3 + \frac{119}{23}D_4 + \frac{90}{23}D_5 + \frac{61}{23}D_6 + \frac{32}{23}D_7 + \frac{3}{23}D_8 + \frac{8}{23}D_9 + \frac{50}{23}D_{10} + \frac{1}{23}D_{11}$ and $(K)^2 = \frac{21}{23}$. Hence this case cannot occur.

This proves the lemma. ♠

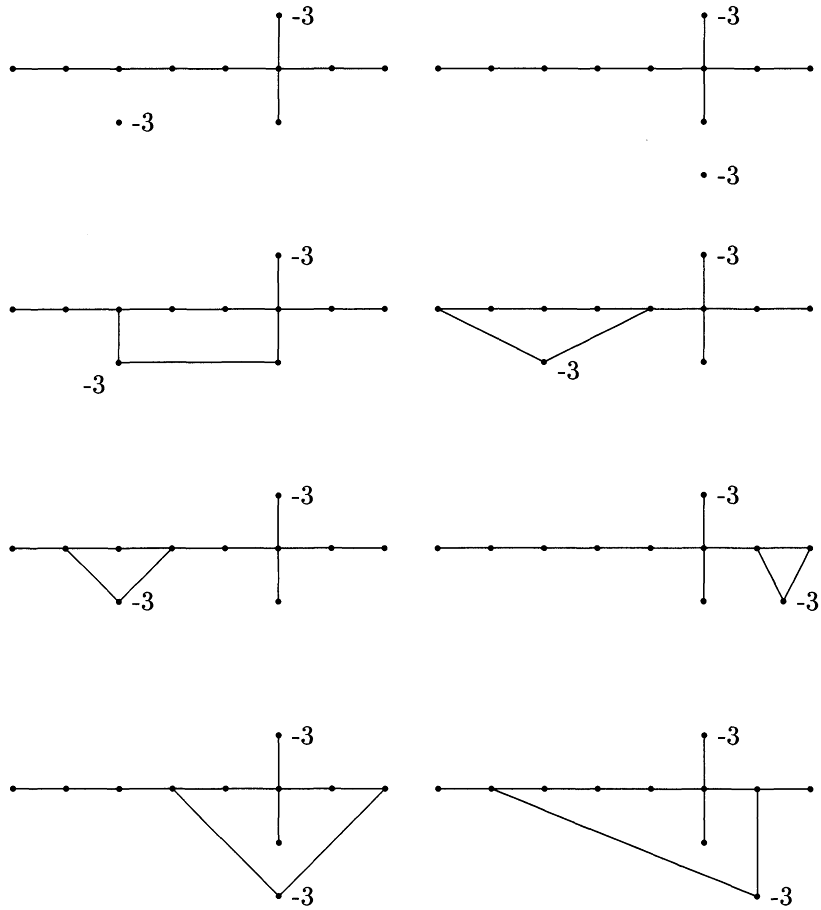
Since $\lambda \geq 2$ and $\theta \leq 3$ we have to deal with the case $\lambda = 2, e_1 = 1$. Next lemma asserts that this case cannot occur.

Lemma 2.6. *The case $\lambda = 2,\ e_1 = 1$ does not occur.*

Proof. Let $\{L_0\} = \mathcal{E}_1 \setminus \Delta$. Clearly $(L_0)^2 = -1$ and $L_0 \in R_2$. Since $\sigma = 0$, by lemma 3.2 of [13], we see that $\mathcal{E}_2 \subset \Delta'$ and $n_2 \leq 1$ and $b_1(\Delta'') = 1$.

Suppose $n_2 = 1$. Let $\mathcal{E}_2 = \{E_1'\},\ (E_1')^2 = -1$. Since E_1' is a component of Δ' we see that there is exactly one component Δ_1' of Δ' such that $(E_1'.\Delta_1') = 2$ and $(E_1'.\Delta_j') = 0,\ j \neq 1$. Also, E_1' has to intersect Δ_1' at two distinct points. Thus, on X'' all components of Δ'' are smooth except for Δ_1'' which has a node at $\pi_2(E_1') = x$ and all other singularities of Δ'' are ordinary double points. Consider the fibre F containing L_0. By lemma 3.2 of [13] and lemma 2.1, it follows that $F \setminus L_0 \subset \Delta$ and the fibre F'' of ϕ'' through x is contained in Δ''. Since no other component of Δ'' passes through x, we conclude that $F''_{red} = \Delta_1''$. But then by lemma 2.2, we conclude that $b_1(\Delta'') \geq 2$ which is absurd. Thus $n_2 = 0$.

Now, we have Δ' to be a NC curve, $b_1(\Delta'') = 1,\ b_2(\Delta'') = 11$. Again by lemma 2.1, we see that the fibre F'' through x of $\phi'(= \phi'')$ is contained in $\Delta'(= \Delta'')$. Since $b_1(\Delta'') = 1$, by lemma 2.2 we conclude that F'' is of the type II*. Let the two horizontal components be H_i'', $i = 1, 2$. Since $(K''.H_i'') = 1$, we see that $(F''.H_i'') = 6$, $i = 1, 2$. Using the facts that Δ' is a NC curve and $b_1(\Delta') = 1$ we easily deduce that one of the horizontal components – say H_1'' - meets Δ_0'' transeversely at one point, where Δ_0'' is the component of F with multiplicity $\mu(\Delta_0'') = 6$. The other horizontal component H_2'' will intersect two (not necessarily distinct) components Δ_i'', Δ_j'' of F'' such that $\mu(\Delta_i'') + \mu(\Delta_j'') = 6$ (Note that if H_2'' intersects the component with multiplicity 6, then $\nu < -2$) . Following eight are all such configurations. We let the unnumbered vertex denote a (-2)-curve.



If $n_1 > 1$, it is easily seen that $\nu < -2$ and hence we consider the case $n_1 = 1$ only. But again in this case it is easy to see that Δ corresponding to each of the above configuration has either $\nu < -2$ or $(K)^2$ of the corresponding surface is not equal to –1. This completes the proof of the lemma. ♠

Thus we have shown that $e_1 = 0$.

2.2. The case $r_3 \neq 0$

In this section we study the cases when $r_3 \neq 0$. Since $e_1 = 0$ by (18) of [13] we have

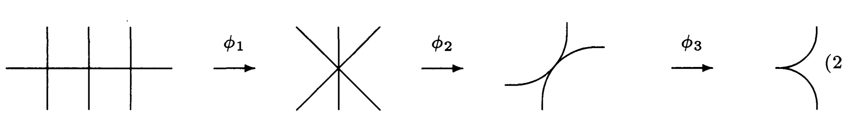
$$\lambda + \tau + \sigma + u + r_3 \leq 4.$$

Since $\lambda \geq 2$, we have $r_3 \leq 2$. First we eliminate the case $r_3 = 2$.

Lemma 2.7. *The case $\lambda = 2$, $r_3 = 2$ does not occur.*

Proof. In this case $\theta = 4$ and hence $\nu \geq -1$. Since $e_1 = \sigma = 0$ we see that $\mathcal{E} \subset \Delta$. Let $\{D_1, D_2\} = R_3$ and without loss of generality we may assume that $(D_1)^2 = -1$. We study this under the following two exhaustive cases: $D_1 \cap D_2 = \emptyset$ and $D_1 \cap D_2 \neq \emptyset$.

$\underline{D_1 \cap D_2 = \emptyset}$: Let $D_{i,j}$, $1 \leq j \leq 3$ be the components of Δ such that $(D_i.D_{i,j}) = 1$ for $i = 1, 2$. Note that for a fixed i, $D_{i,j}$s are distinct. Clearly $(D_1)^2 = (D_2)^2 = -1$. Also, for any fibre F'' and horizontal component H'' of ϕ'' we know that $(F''.H'') = 6$. From this it is easy to see that $\pi(\sum D_{i,j})$ does not contain any fibre of ϕ. Maximal sequence of contractions at each D_i is as follows:



Thus we have $2 \leq n_1 \leq 4$ and $0 \leq n_2 \leq 2$. Also if $n_2 > 0$, then $n_1 > 2$ and if $n_2 = 2$, then $n_1 = 4$. In any case we observe that Δ has at least four maximal twigs and if $n_1 > 2$ then Δ has at least one (-2)-tip. We exhaust the possibilities in this case by explicitly considering the possible values for (n_1, n_2) and showing that $\nu < -1$.

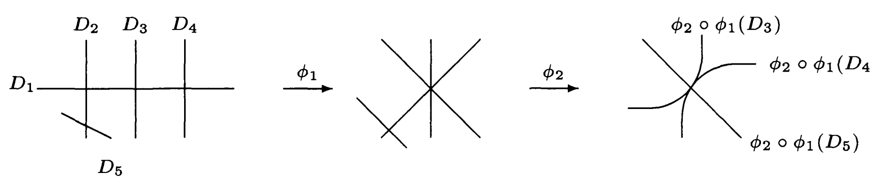
$\underline{(n_1, n_2) = (2, 0)}$: In this case if both the horizontal components are tips they have to be either (-4)-tips or (-3)-tips. Also since there are at least four tips, T has at least two more tips. Weights of these tips cannot be less than (-3). Hence $\nu < -1$.

$\underline{n_1 = 3}$: Irrespective of the value of n_2, we already have a (-2)-tip. Without loss of generality we may assume that this (-2)-tip is adjacent to D_1. If Δ has five (or more) tips it is easy to see that $\nu < -1$. Hence we consider only those trees with exactly four maximal twigs. Since there are exactly two horizontal components, the weights of the tips of the two maximal twigs branching at D_2 cannot be both less than or equal to -4. Now it is easy to see that $\nu < -1$.

$\underline{n_1 = 4}$: In this case irrespective of the value of n_2 we see that there are at least two (-2)-tips in Δ and hence $\nu < -1$.

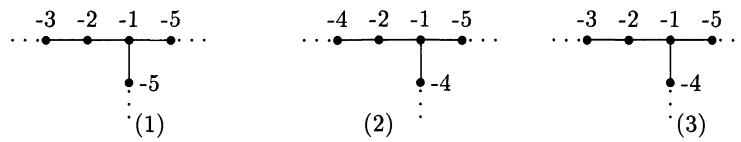
Thus we have shown that D_1 and D_2 are not disjoint.

$\underline{D_1 \cap D_2 \neq \emptyset}$: Without loss of generality we have $(D_1)^2 = -1$ and $(D_2)^2 = -2$. Then we have the following sequence of contractions:

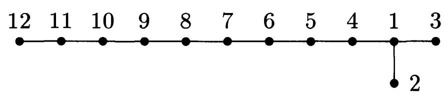


If $(\phi_2 \circ \phi_1(D_5))^2 = -1$, it does not intersect any other component of $\phi_2 \circ \phi_1(D)$ other than $\phi_2 \circ \phi_1(D_3)$ and $\phi_2 \circ \phi_1(D_4)$. Hence we have $[2,2]$ as a maximal twig in D. Note that $\tau = 0$ implies that $m_{t,i} \leq 2$ for all t, i. But then $\mathcal{E}_2 = \emptyset$. Now it is easily seen that $\nu < -1$.

Thus $(\phi_2 \circ \phi_1(D_5))^2 \neq -1$. Since $\sigma = 0$, by lemma 3.2 of [13] we have $\mathcal{E}_2 = S$ is a disjoint union of (-1)-curves. If $\mathcal{E}_2 \neq \emptyset$, clearly $(D'_3)^2$ or $(D'_4)^2$ equals -1. In either case we see that $u = 1$ and hence not possible. Hence $\mathcal{E}_2 = \emptyset$. But then all components of Δ'' are smooth. Also $\beta_2 = 12$. The weight set of Δ'' is $\{-3, -3, -2, \ldots, -2\}$. At least one of D''_3 or D''_4 is a (-3)-curve for otherwise we see that $D''_3 + D''_4$ is a fibre F'' of ϕ'' and $(F''.H'') \neq 6$ for any horizontal component H'' in Δ''. If $(D''_3)^2 = (D''_4)^2 = -3$ and $(D''_5)^2 = -2$ we have the subtree (1) in Δ. If $(D''_4)^2 = (D''_5)^2 = -3$ and $(D''_3)^2 = -2$ we have the the subtree (2) in Δ. If $(D''_4)^2 = (D''_5)^2 = -2$ and $(D''_3)^2 = -3$ we have the subtree (3) in Δ.



In case of (1) and (2) it is easy to see that $\nu < -1$ and hence these cases do not occur. In case (3), we observe that among the components not in the subtree there is exactly one (-3)-curve. In this case if tree has four (or more) maximal twigs then clearly $\nu < -1$. Hence we need to consider only trees with exactly three maximal twigs. In case of three maximal twigs, we see that if it has a (-2)-tip, then $\nu \leq -\frac{3}{5} - \frac{1}{5} - \frac{1}{4} < -1$. If there are two (-3)-tips then $\nu < -\frac{2}{5} - \frac{2}{5} - \frac{1}{5} = -1$ and hence not possible. Thus we are left with the following tree:



For this tree we have $w_1 = -1$, $w_2 = -4$, $w_3 = -5, w_5 = w_{12} = -3$ and $w_j = -2$ for $j \neq 1, 2, 3, 5, 12$. It is easy to see that $K = \frac{418}{101}D_1 + \frac{54}{101}D_2 + \frac{23}{101}D_3 + \frac{240}{101}D_4 + \frac{62}{101}D_5 + \frac{47}{101}D_6 + \frac{32}{101}D_7 + \frac{17}{101}D_8 + \frac{2}{101}D_9 - \frac{13}{101}D_{10} - \frac{28}{101}D_{11} - \frac{43}{101}D_{12}$ which implies that $(K)^2 = -\frac{222}{101}$ which is a contradiction. Hence this tree cannot occur.

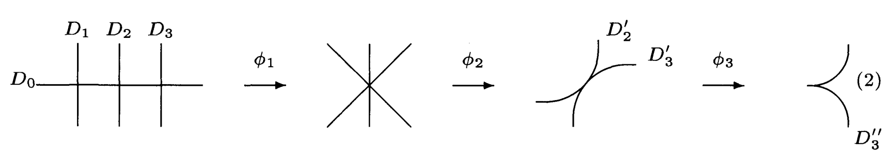
This completes the proof of the lemma. ♠

Thus we have $r_3 = 1$ and hence

$$\lambda + \tau + \sigma + u \leq 3, \ \ \lambda \geq 2.$$

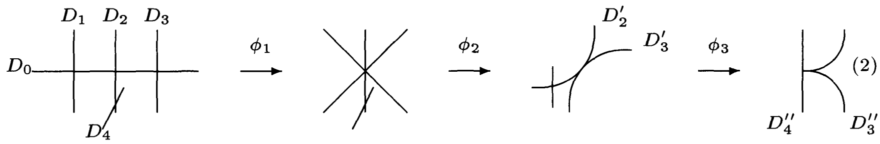
Lemma 2.8. *We have* $\tau + \sigma + u = 0$.

Proof. Clearly we have $\tau + \sigma + u \leq 1$. If $\tau = 1$ we see that $\sigma = u = 0$ and $n_2 \neq 0$. Since $\sigma = 0$, we see that $\mathcal{E} \subset \Delta$ and hence we have the following sequence of contractions on X to reach X'':



But then $\tau = 0$ which is a contradiction.

Now, if $u = 1$ exactly as above we have the following sequence of contractions from X to X'':

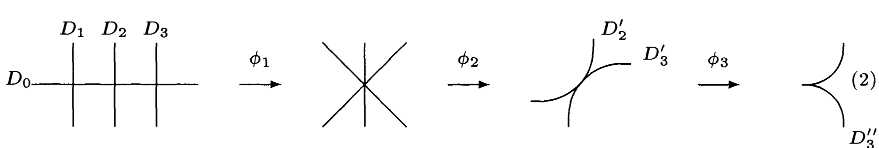


If $(K''.D_3'') = 0$ then $(D_3'')^2 = 0$. But then D_3'' is a fibre of ϕ'' and hence $(D_3''.H'') = 6$ for every horizontal component H'' in Δ''. But as seen above, for all components C'' of Δ'' which intersect D_3'', we have $(C''.D_3'') \le 2$. Hence $(K''.D_3'') = 1$. Then we have $(D_3)^2 = -7$. Also $(D_4)^2 = -4$ or -3 depending on whether it is a horizontal component or not. Also D_1 is a tip. Now, we see that if (-7) is a tip, then we have at least a $[3,2]$ twig. Hence $\nu < -\frac{1}{2} - \frac{2}{5} - \frac{1}{7} < -1$. If (-7) is not a tip the we have one more (-2)-tip and hence $\nu < -1$. This proves that $u = 0$.

Now, we are left with the case $\sigma = 1$. If possible let $\mathcal{E} \not\subset \Delta$. Then by lemma 3.2(c)(i) of [13] we see that $\mathcal{E}_2 \setminus \Delta' = \{E'\}$ and S is disjoint union of (-1)-curves. Since $u = 0$, E' intersects exactly one component D_1' of Δ'. Since $\tau = 0$ we have $(E'.\Delta') = 2$. This contradicts lemma 3.4(b) of [13]. Hence we see that $\mathcal{E} \subset \Delta$. Then clearly $\mathcal{E}_1 = \{D_0, D_1\}$ and $\mathcal{E}_2 = \{D_2', D_4'\}$ where $(D_4)^2 = -2$, $(D_4.D_2) = (D_4'.D_2') = 1$. Also since $u = 0$, D_4 is a tip of Δ. But then $\nu < -1$ which is a contradiction. This completes the proof of the lemma. ♠

Lemma 2.9. *The case $\lambda = 3$ does not occur.*

Proof. In this case $\theta = 4$ and hence by (17) of [13] $\nu \ge -1$. Since $e_1 = \sigma = 0$, we have $\mathcal{E} \subset \Delta$. Then the maximal sequence of contractions is as follows:



For any fibre F'' of ϕ'' contained in Δ'' we need to have $(F''.H'') = 6$ for any horizontal component H'' in Δ''. This shows that $D_0 + D_1 + D_2 + D_3$ does not contain a full fibre of ϕ. But then as in lemma 4.4 of [13] we see that the weight set $W = \{(D_0)^2, (D_1)^2, (D_2)^2, (D_3)^2\}$ is one of the following:

(a) $W = \{-1, (D_1'')^2 - 1, (D_2'')^2 - 1, -\alpha - 3\}$.

(b) $W = \{-1, -2, (D_2'')^2 - 2, -\alpha - 4\}$ and D_1 is an isolated tip in Δ.

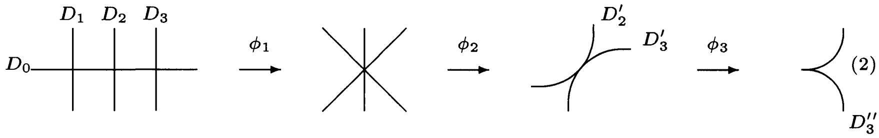
(c) $W = \{-1, -2, -3, -\alpha - 6\}$ and D_1 is an isolated tip.

Here $\alpha = (K''.D_3'')$.

In all these cases if there are four (or more) tips clearly $\nu < -1$ and hence not possible. In case of three tips it is easy to see that either $\nu = bk(\Delta) < -1$ or the tree has negative definite intersection form and hence not possible. ♠

Lemma 2.10. *If $\lambda = 2$ possibilities for Δ arise from the case (5), (6), (7), (8) or (9) of the table 3.*

Proof. Since $e_1 = \sigma = 0$, we see that $\mathcal{E} \subset D$. Clearly the following is the maximal sequence of contractions:



For any fibre F of ϕ and any horizontal component H in Δ we know that $(F.H) = 6$. From this we easily see that $D_0 + D_1 + D_2 + D_3$ does not contain a full fibre of ϕ.

$\underline{(n_1, n_2) = (1, 0)}$: Clearly $\beta_2 = 11$. One of D_1', D_2' or D_3' is a horizontal component. If exactly one of them is a horizontal component then we are in the case (5) of the table 3. If two of them are horizontal components then we are in the case (6) of the Table.

$\underline{(n_1, n_2) = (2, 0)}$: Clearly $\beta_2 = 12$. One of D_2' or D_3' is a horizontal component. If exactly one of them is a horizontal component then we are in the case (7) of the table 3. If both of them are horizontal components then we are in the case (8) of the table 3.

$\underline{(n_1, n_2) = (2, 1)}$: Clearly $\beta_2 = 13$. Then D_3' is a horizontal component and we are in the case (9) of the table 3. ♠

This completes the analysis in case $r_3 \neq 0$.

2.3. The case $r_3 = 0$

In this case we have

$$\lambda + \tau + \sigma + u \leq 4, \quad \lambda \geq 2.$$

Lemma 2.11. *If $\sigma = 0$ possibilities for Δ arise from the cases (1), (2) or (3)* *of the table 3.*

Proof. Note that if $\sigma = 0$, then $\mathcal{E} = \emptyset$. Hence $(X, \Delta) = (X'', \Delta'')$. Thus $\beta_2 = 10$. By lemma 2.4 we see that $\lambda \leq 3$. If $\lambda = 2$, we are in the case (1) of the table 3. If $\lambda = 3$ depending on whether Δ has two or three horizontal components we are in the case (2) or (3) of the table 3. ♠

In view of this lemma and the fact that

$$\lambda + \tau + \sigma + u \leq 4, \quad \lambda \geq 2.$$

we are left with the following possibilities:

1. $\sigma = 2,\ \tau = u = 0$. This is studied in lemma 2.12 below.
2. $\sigma = 1,\ \tau = 1,\ u = 0$. This is studied in lemma 2.13 below.
3. $\sigma = 1,\ \tau = 0,\ u = 1$. This is studied in lemma 2.14 below.
4. $\sigma = 1,\ \tau = u = 0$. This is studied in lemma 2.15 below.

Lemma 2.12. *The case $\lambda = 2,\ \sigma = 2$ does not occur.*

Proof. By lemma 3.4(b) of [13] we see that $\mathcal{E}_1 = \emptyset$. Also since $\mathcal{E} \neq \emptyset$, we see that $\mathcal{E}_2 \setminus \Delta \neq \emptyset$. Let $E' \in \mathcal{E}_2 \setminus \Delta$ and $(E')^2 = -1$. Since $\tau = u = 0$, we see that E' intersects Δ at exactly two distinct points transversally contradicting lemma 3.4(b) of [13]. Hence this case does not occur. ♠

Lemma 2.13. *The case $\lambda = 2,\ \sigma = 1,\ \tau = 1$ does not occur.*

Proof. Since $e_1 = r_3 = r_4 = 0$, we see that $\mathcal{E}_1 = \emptyset$. By lemma 3.2 of [13] we see that $\mathcal{E} = \mathcal{E}_2 = \{E'\} \not\subset \Delta$. Since $\tau = 1$ and $u = 0$ we see that E' intersects exactly one component of Δ - say D_1 and $(E'.D_1) = 3$. Clearly D_1'' is a horizontal component and hence $(K''.D_1'') = 1$. But then the weight set of Δ is $\{-6, -3, -2, \ldots, -2\}$. Clearly then $\nu < -1$ and hence this case does not occur. ♠

Lemma 2.14. *The case $\lambda = 2,\ \sigma = u = 1$ does not occur.*

Proof. Since $e_1 = r_3 = r_4 = 0$, we see that $\mathcal{E}_1 = \emptyset$. Thus by lemma 3.2 of [13] we have $\mathcal{E} = \mathcal{E}_2 = \{E'\} \not\subset \Delta$. Since $\tau = 0$ there exists a component D_1 in Δ such that $(E'.D_1) = 2$. Since $u = 1$, there exists a component D_2 in Δ such that $(E'.D_2) = 1$.

If $(K''.D_1'') = 0$, then D_1'' is a fibre of ϕ''. Let its multiplicity be m. Let H_1'' and H_2'' be the two horizontal components in Δ''. Since Δ is a MNC curve free of (-1)-curves, we see that it is not possible for both H_1'' and H_2'' to intersect D_1'' such that $(mD_1.H_1'') = (mD_1.H_2'') = 6$. This is a contradiction. Hence $(K''.D_1'') \neq 0$.

Now, let $(K''.D_1'') = 1$. Then $(D_1'')^2 = -1$ and hence $(D_1)^2 = -5$. Then depending on whether D_2 is a horizontal component or not, the weight set of Δ is $\{-5, -4, -2, \ldots, -2\}$ or $\{-5, -3, -3, -2, \ldots, -2\}$. In the former case clearly $\nu < -1$ and hence not possible. In case the weight is $\{-5, -3, -3, -2, \ldots, -2\}$ and if there are four (or more) tips then $\nu < -1$ and hence we need to consider only trees with three maximal twigs. If there is a (-2)-tip in any such tree, then $\nu < -1$ and hence there are no (-2)-tips. Further, if there are two $[3, 2]$ twigs then $\nu < 2(-\frac{2}{5}) - \frac{1}{5} = -1$ and hence one of the maximal twigs is the twig $[3]$. Clearly such trees correspond to partitions of 10 into exactly three parts with one of the summands equal to 1. We have considered such partitions in lemma 2.5. Trees corresponding to the partitions 1+1+8 and 1+2+7 are negative definite and hence cannot occur. Trees corresponding to the other two partitions are listed as tree number (1) and (2) in the lemma 2.5.

Tree 1: If $w_9 = -3$ and $w_{11} = -5$, then $\nu = bk(\Delta) = -\frac{40}{39} < -1$ and hence not possible. If $w_9 = -5$ and $w_{11} = -3$ then $\nu = -\frac{526}{525} < -1$ and hence not possible.

Tree 2: If $w_8 = -5$ and $w_{11} = -3$ then $bk(\Delta) = -\frac{147}{144} < -1$ and hence this case cannot occur. If $w_8 = -3$ and $w_{11} = -5$ then $bk(\Delta) = -\frac{573}{561} < -1$ this case cannot occur. ♠

Lemma 2.15. *If $\sigma = 1$ possibilities for Δ arise from the case 4 of the table 3.*

Proof. Since $\lambda = 2$ and $\sigma = 1$ we have $e_1 = r_3 = 0$ and hence $\mathcal{E}_1 = \emptyset$. Since $\sigma = 1$, we have $\mathcal{E}_2 = \{E'\} \not\subset \Delta'$. Thus $\beta_2 = 11$. Since $\tau = u = 0$ we see that E' intersects exactly one component of Δ – say D_1 – at two distinct points. As before it is easy to see that D_1'' cannot be a fibre of ϕ''. Hence $(K''.D_1'') = 1$. But then the weight set of Δ is $\{-5, -3, -2, \ldots, -2\}$. This is the case 4 mentioned in the table 3. ♠

Since we have exhausted all possible values for the parameters appearing in the expression for θ, we have completed the proof of proposition 2.1.

3. Listing of Trees

Case 1. In this case we have $\theta = 2$. If T has seven (or more) maximal twigs, then the tips are at worst $2 \times [3]$, $5 \times [2]$. But then $bk(T) \leq 2(-\frac{1}{3}) + 5(-\frac{1}{2}) = -\frac{19}{6} < -3$ which contradicts (17) of [13]. Thus T has at most six maximal twigs. Using the list in [6] we list all these trees. They are the trees numbered 1-88 in the list given below.

Case 2. In this case we have $\theta = 3$. If T has five (or more) maximal twigs, then the tips are at worst [4], [3], $3 \times [2]$. But then $bk(T) \leq -\frac{25}{12} < -2$ which contradicts (17) of [13]. Thus T has at most four maximal twigs. These are the trees numbered 59-88 in the list given below.

Case 3. In this case we have $\theta = 3$. If T has six (or more) maximal twigs, then the tips are at worst $3 \times [3]$, $3 \times [2]$. But then $bk(T) \leq -\frac{5}{2} < -2$ which contradicts (17) of [13]. Thus T has at most five maximal twigs. If T has exactly five maximal twigs and if it has three (or more) [2] tips then, $bk(T) \leq -\frac{13}{6} < -2$ which again contradicts (17) of [13]. Hence T has exactly three [3] tips and two [2] tips. We observe that if one of the maximal twigs has at least two irreducible components, then $bk(T) \leq -\frac{31}{15} < -2$ which contradicts (17) of [13]. Hence, in this case each maximal twig is a tip. These are trees numbered 32, 57 and 58 in the list below. Trees with at most four maximal twigs are listed as numbers 59-88 in the list below.

Case 4. In this case we have $\theta = 3$. If T has five (or more) maximal twigs, then the tips are at worst [5], [3] and $3 \times [2]$. But then $bk(T) \leq 3(-\frac{1}{2}) - \frac{1}{3} - \frac{1}{5} = -\frac{61}{30} < -2$ which contradicts (17) of [13]. Thus we need to consider eleven vertex trees with at most four maximal twigs. First, we construct those with exactly four maximal twigs. These arise from partitions of integers 4 to 10 into at exactly four parts. Partition of 4 into exactly four parts correspond to tree number (38) in the list of eleven vertex trees given below. The intersection form of this tree for all possibilities of weights is negative definite. To see this we use the singular fibre $\mathrm{I}_b^\star$ of a minimal elliptic fibration. The intersection form on $\mathrm{I}_b^\star$ is negative semi-definite and on any proper subset of $\mathrm{I}_b^\star$ it is negative-definite. Since in our case at least one vertex has weight less than (-2), we see that the intersection form is negative definite. This contradicts lemma 3.3(d) of [13]. Thus this tree does not arise. Partition of 5 gives rise to the tree number (39). Two partitions of 6 into exactly four parts are:

$$\begin{aligned} 6 &= 3+1+1+1 \\ &= 2+2+1+1 \end{aligned}$$

These give rise to trees numbered (40), (42) and (50). Three partitions of 7 into exactly four parts are:

$$\begin{aligned} 7 &= 4+1+1+1 \\ &= 3+2+1+1 \\ &= 2+2+2+1 \end{aligned}$$

These give rise to the trees numbered (37), (41), (49) and (51). Five partitions of 8 into exactly four parts are :

$$\begin{aligned} 8 &= 5+1+1+1 \\ &= 4+2+1+1 \\ &= 3+3+1+1 \\ &= 3+2+2+1 \\ &= 2+2+2+2 \end{aligned}$$

These give rise to the trees numbered (34), (35), (36), (46), (47), (55), (56) and (62). Six partitions of 9 into exactly four parts are:

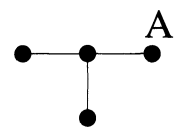
$$\begin{aligned} 9 &= 6+1+1+1 \\ &= 5+2+1+1 \\ &= 4+3+1+1 \\ &= 4+2+2+1 \\ &= 3+3+2+1 \\ &= 3+2+2+2 \end{aligned}$$

These give rise to the trees numbered (31), (32), (33), (43), (44), (45), (52), (53), (54), (59) and (60). There are nine partitions of 10 into exactly four parts and the nine trees corresponding to these are listed as trees numbered (89) to (97) in the list below. Trees with exactly three maximal twigs arise from partition of 10 into exactly three parts. There are eight such partitions and they give rise to trees numbered (81) to (88) in the list below.

Case 5. In this case we have $\theta = 3$. If T has six (or more) maximal twigs, then the tips are at worst [4], [3], [3] and $3 \times [2]$. But then $bk(T) \leq -\frac{29}{12} < -2$ which contradicts (17) of [13]. Hence T has at most five maximal twigs.

Consider trees with exactly five tips. Clearly at least two of them are [2] tips. If at least three of them are [2] tips, the other two can be at worst [4] and [3] tips. But then $bk(T) \leq -\frac{25}{12} < -2$ contradicting (17) of [13]. Hence T has exactly two

[2] tips. Among the other three all of them may be [3] tips or two of them [3] tips and one [4] tip. In any case two of the tips are adjacent to the [1] vertex. Thus we get the required tree by attaching G_7 (seven vertex tree) to the vertex A in the following configuration:



Since the resulting tree has exactly five tips, G_7 may have at most three connected components. If G_7 has three connected components, all of them have to be linear twigs. These arise from partitions of 7 into exactly three parts:

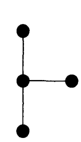
$$\begin{aligned} 7 &= 5+1+1 \\ &= 4+2+1 \\ &= 3+3+1 \\ &= 3+2+2 \end{aligned}$$

It is easy to see that $bk(T) < -2$ in all these cases. Let G_7 have exactly two connected components. These arise from partition of 7 into exactly two parts:

$$\begin{aligned} 7 &= 6+1 \\ &= 5+2 \\ &= 4+3 \end{aligned}$$

Using the list in [6] we construct these trees. Trees corresponding to 6+1 are numbered (1) to (6) in the list. Trees corresponding to 5+2 are numbered (7) to (10) below and those corresponding to 4+3 are numbered (11) to (13) below. If G_7 is connected we need to consider trees numbered (14) to (30) below.

Now, we consider trees with exactly four tips. These arise by attaching (not necessarily connected) seven vertex tree to the tips of the following configuration:

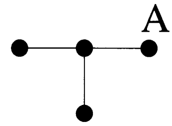


These correspond to partitions of 7 into at most three parts. They are

$$\begin{aligned} 7 &= 7+0+0 \\ &= 6+1+0 \end{aligned}$$

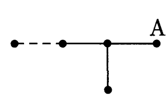
$$\begin{aligned}
&= 5+2+0 \\
&= 5+1+1 \\
&= 4+3+0 \\
&= 4+2+1 \\
&= 3+3+1 \\
&= 3+2+2
\end{aligned}$$

7+0+0: These are obtained by attaching G_7 to the vertex A in the following configuration:



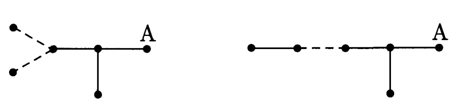
Clearly, G_7 has at most two connected components. If it does have two connected components, both are linear twigs and they correspond to trees numbered (31) to (33) below. If G_7 is connected, then we have trees numbered (34) to (42) below.

6+1+0: These are obtained by attaching G_6 to the vertex A in the following configuration:



As above, G_6 has at most two connected components. If G_6 has exactly two connected components, they correspond to trees numbered (43) to (45). If G_6 is connected, we have trees numbered (46) to (51) below.

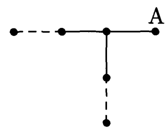
5+2+0: These are obtained by attaching G_5 to the vertex A to one of the following configurations:



(a) (b)

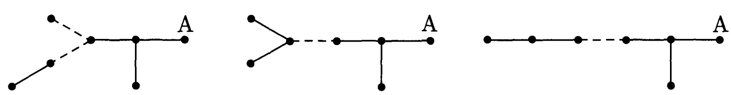
In case (a), clearly G_5 is a linear twig and thus the corresponding tree is numbered (52) below. In case (b), G_5 has at most two connected components. If it has exactly two connected components we need to consider trees numbered (53) and (54) and if G_5 is connected, we need to consider trees numbered (55) to (58).

<u>5+1+1:</u> These are obtained by attaching G_5 to the vertex A in the following configuration:



By considerations similar to the above we need to consider trees numbered (59) to (64).

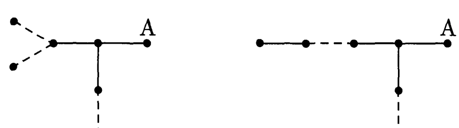
<u>4+3+0:</u> These are obtained by attaching G_4 to the vertex A in one of the following configurations:



(a) (b) (c)

In these cases we have trees numbered (65) to (70) below.

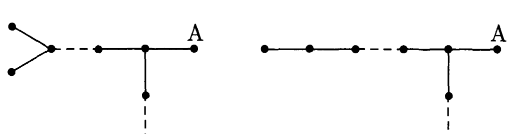
<u>4+2+1:</u> These are obtained by attaching G_4 to the vertex A in one of the following configurations:



(a) (b)

These are trees numbered (71) to (75) below.

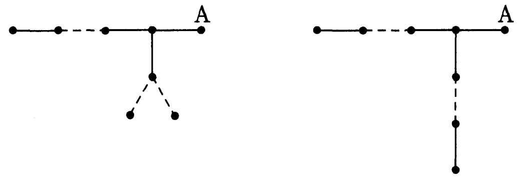
<u>3+3+1:</u> These are obtained by attaching G_3 to the vertex A in the following configuration:



(a) (b)

These are the trees numbered (76) and (77).

3+2+2: These are obtained by attaching G_3 to the vertex A in one of the following configurations:



(a) (b)

These are the trees numbered (78) to (80) below.

Now, consider trees with exactly three maximal twigs. These correspond to partition of 7 into at most three parts:

$$\begin{aligned}
7 &= 7+0+0 \\
&= 6+1+0 \\
&= 5+2+0 \\
&= 5+1+1 \\
&= 4+3+0 \\
&= 4+2+1 \\
&= 3+3+1 \\
&= 3+2+2
\end{aligned}$$

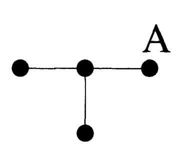
These are numbered (81) to (88) below.

Case 6. In this case we have $\theta = 3$. If T has six (or more) maximal twigs, then they are at worst $2 \times [4]$ and $4 \times [2]$. But then $bk(T) \leq -\frac{5}{2} < -2$ contradicting (17) of [13]. Hence T has at most five maximal twigs. First consider the trees with exactly five maximal twigs. Then they are at worst $2 \times [4]$ and $3 \times [2]$ which implies that $bk(T) \leq -2$. But then, clearly the five tips are actually the ones listed above and they are all maximal twigs. Thus T_1 and T_3 are empty. In the listing of trees for case 5, we have listed all the eleven vertex trees with five maximal twigs. We see that among them, there are exactly five trees which have only one component in each of its maximal twigs. Such trees are numbered (3), (14), (24), (28) and (30) in the list of eleven vertex trees given below. Observe that among these tree numbers (14), (24), (28) and (30) contain a fibre and $(F''.H'') < 6$ which is a contradiction and hence they do not occur. Thus we need to consider only tree number (3) in the list below. Now, let us consider trees with exactly four maximal twigs. We note that not all four tips can be [2] tips as otherwise $bk(T) < -2$ contradicting (17) of [13]. Hence at least one of T_1, T_2 and T_3 is necessarily empty. In case 5 we have listed all eleven vertex trees with exactly four maximal twigs. Among these, the only trees which can occur in this case are numbered (31) to (58) and (65) to (70). Among these tree (42) has a fibre and $(F''.H'') < 6$ which is a contradiction and hence cannot occur. Among the remaining trees (43) to (47), (49), (51) to (56), (58) and (65) to (70) have $bk(T) < -2$ and hence cannot occur. Thus we are left with trees (31) to (41), (48), (50) and (57). Trees with 3 tips are numbered (81) to (88), as in case 5.

Case 7. In this case we have $\theta = 3$. If T has five (or more) tips they are at worst [5], [3] and $3 \times [2]$. But then $bk(T) \leq -\frac{61}{60} < -2$ contradicting (17) of [13]. Hence T has at most four maximal twigs. First we construct the possible four tip trees arising in this case. Let the four maximal twigs be denoted by M_1, M_2, M_3 and M_4. These must necessarily be the following (or twigs with bark less than these): $M_1 = [2]$, $M_2 = [n \times 2]$, $M_3 = [m \times 2, 5]$ and $M_4 = [3, l \times 2]$ where $n \geq 1$ and l, $m \geq 0$. We see that $bk(M_1) = -\frac{1}{2}$, $bk(M_2) = -\frac{n}{n+1}$, $bk(M_3) = -\frac{4m+1}{4m+5}$ and $bk(M_4) = -\frac{l+1}{2l+3}$. If $m \geq 2$, then $bk(T) \leq -\frac{1}{2} - \frac{1}{2} - \frac{9}{13} - \frac{1}{3} = -\frac{79}{39} < -2$ which contradicts (17) of [13]. Thus $m \leq 1$.

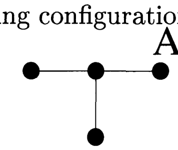
Let $m = 1$. If $n \geq 2$, we have $bk(T) \leq -\frac{1}{2} - \frac{2}{3} - \frac{5}{9} - \frac{1}{3} = -\frac{37}{18} < -2$ which contradicts (17) of [13]. Thus $n = 1$. Now, if $l \geq 4$, we have $bk(T) \leq -\frac{1}{2} - \frac{1}{2} - \frac{5}{9} - \frac{5}{11} = -\frac{199}{99} < -2$ which contradicts (17) of [13]. Thus if $m = 1$, then $n = 1$ and $l \leq 3$. These are trees numbered (1) to (4) given in the list of twelve vertex trees.

Let $m = 0$. Trees in this case are obtained by attaching G_8 to the vertex A in the following configuration:



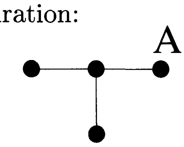
These are numbered (5) to (20) in the list below. Trees with three tips are numbered (21) to (25).

Case 8. In this case we have $\theta = 3$. If T has five (or more) maximal twigs, then they are at worst [5] and $4 \times [2]$. But then $bk(T) < -2$ which contradicts (17) of [13]. Hence T has at most four maximal twigs. Let T have four tips. Observe that all four of them cannot be [2] tips as otherwise we have $bk(T) < -2$ which is a contradiction to (17) of [13]. Thus the required trees are obtained by attaching G_8 to the vertex A in the following configuration:



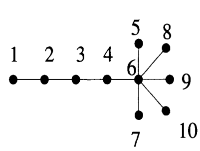
These are listed as trees numbered (5) to (20) in the list below. Among these, clearly trees numbered (17) to (20) contain a fibre of ϕ and $(F''.H'') < 6$ which is a contradiction. Also for trees numbered (5) to (11), (13) and (16) have $bk(T) < -2$ contradicting (17) of [13]. Thus we need to consider trees numbered (12), (14) and (15) only. Trees with exactly three tips are listed as trees numbered (21) to (25).

Case 9. In this case we have $\theta = 3$. If T has five (or more) maximal twigs, then they are at worst $3 \times [2]$ and $2 \times [3]$. But then $bk(T) < -2$ contradicting (17) of [13]. Hence T has at most four tips. These are obtained by attaching G_9 to the vertex A in the following configuration:

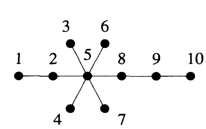


These are numbered (1) to (20) in the list of thirteen vertex trees in the list given below. The lone three tip tree is listed as tree number (21) in the list.

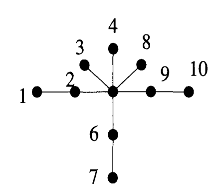
TEN-VERTEX TREES WITH EXACTLY SIX MAXIMAL TWIGS:



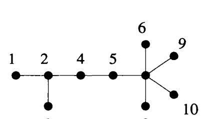
(1)



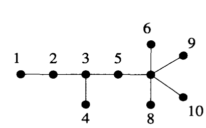
(2)



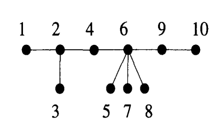
(3)



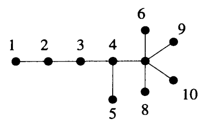
(4)



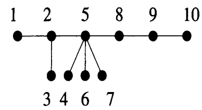
(5)



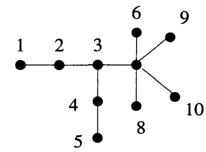
(6)



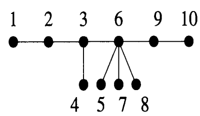
(7)



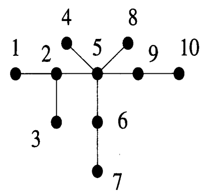
(8)



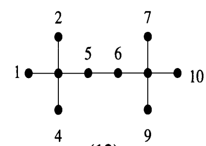
(9)



(10)



(11)



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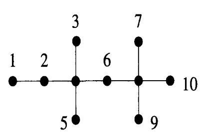
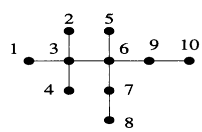
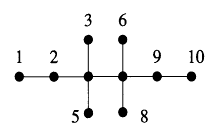


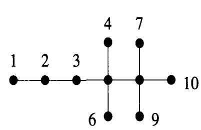
(13)



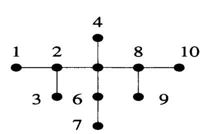
(14)



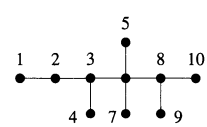
(15)



(16)



(17)



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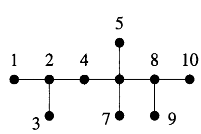

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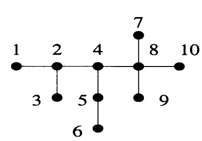

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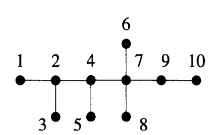

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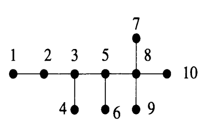

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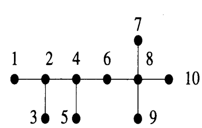

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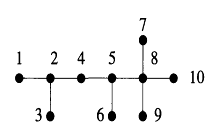

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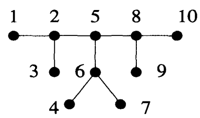

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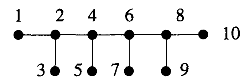
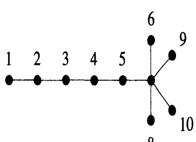


(26)

TEN-VERTEX TREES WITH EXACTLY FIVE MAXIMAL TWIGS:



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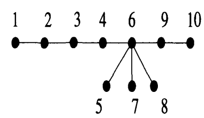

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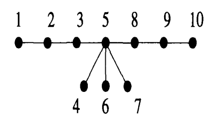

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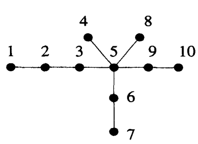

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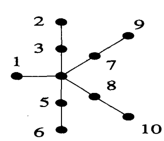

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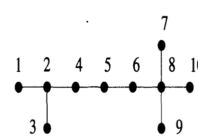

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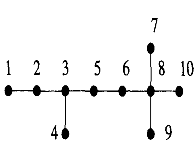

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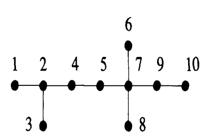

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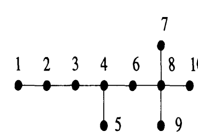

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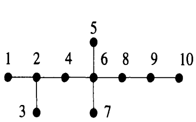

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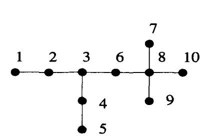

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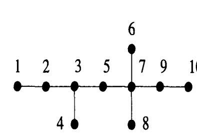

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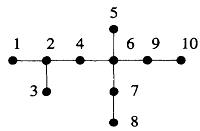

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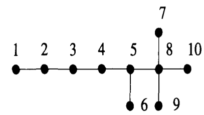

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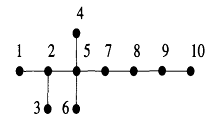

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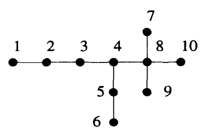

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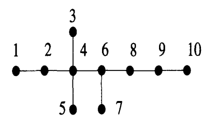

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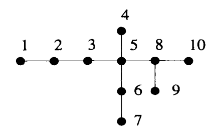

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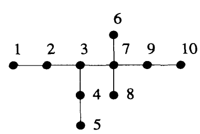

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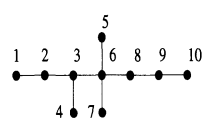

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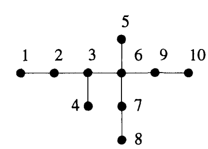

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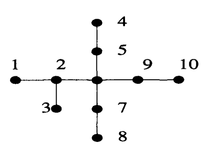

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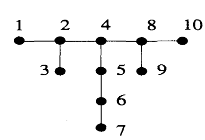

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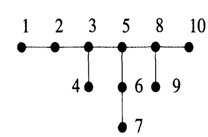

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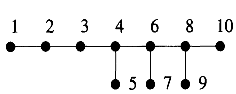

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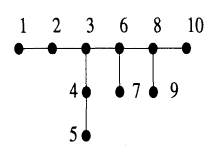

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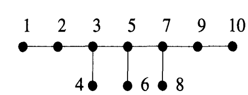

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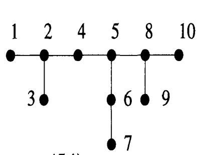

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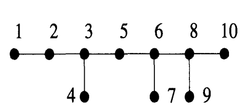

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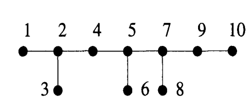

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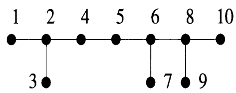

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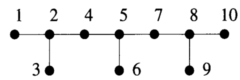
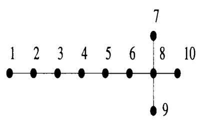
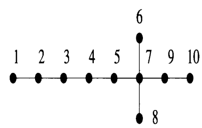


(58)

TEN-VERTEX TREES WITH EXACTLY FOUR MAXIMAL TWIGS:



(59)



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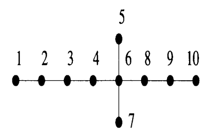

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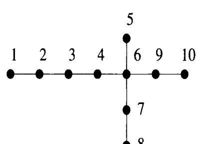

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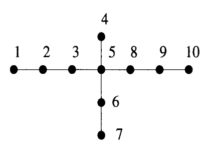

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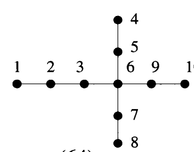

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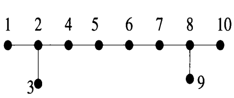

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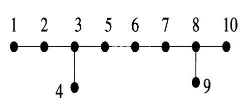

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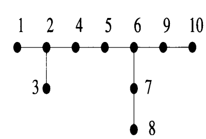

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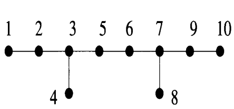

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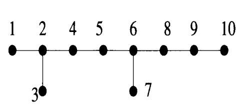

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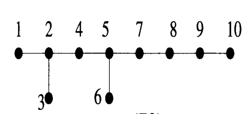

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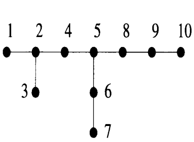

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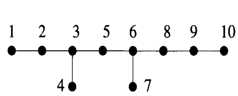

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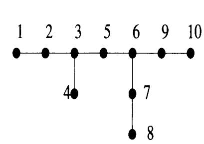

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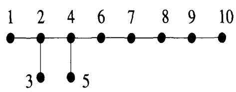

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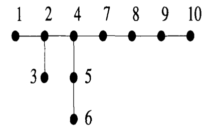

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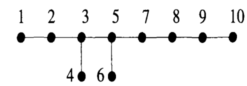

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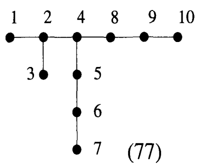

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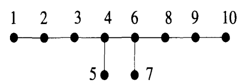

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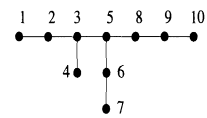

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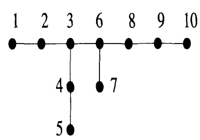

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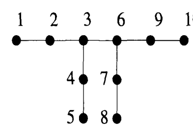

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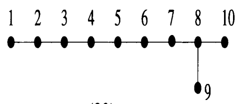
TEN-VERTEX TREES WITH EXACTLY THREE MAXIMAL TWIGS:



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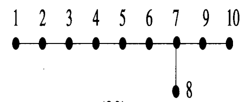

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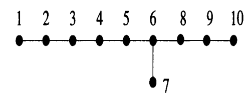

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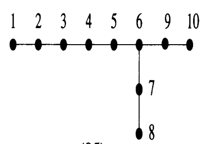

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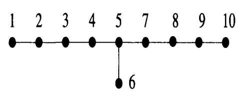

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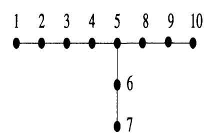

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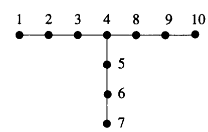

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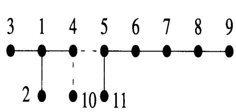

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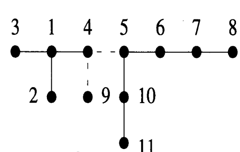

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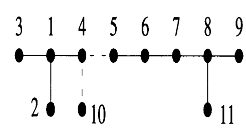

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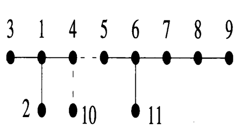

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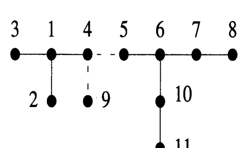

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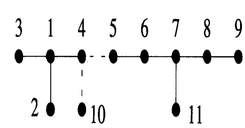

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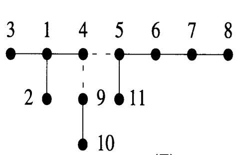

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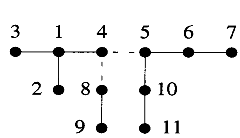

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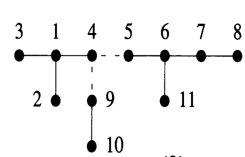

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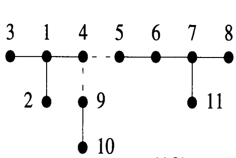

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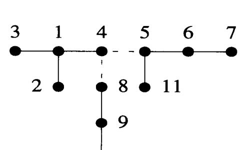

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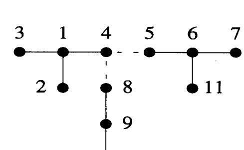

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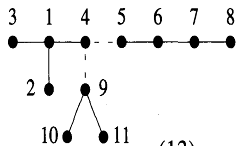

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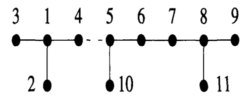

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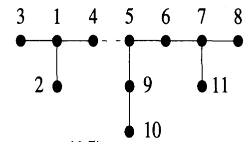

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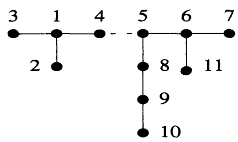

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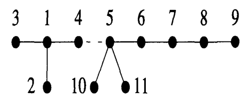

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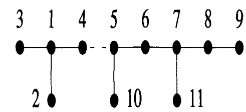

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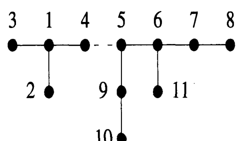

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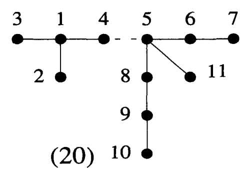

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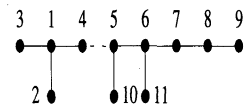

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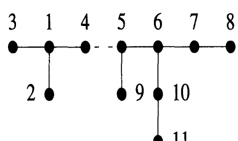

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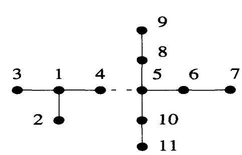

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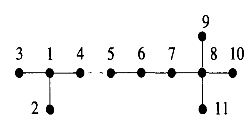

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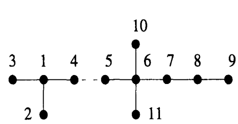

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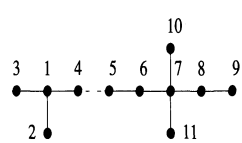

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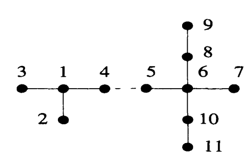

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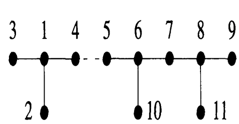

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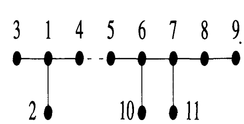

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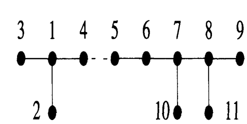

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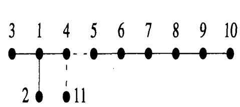

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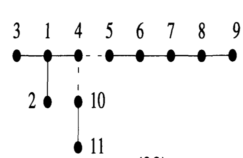

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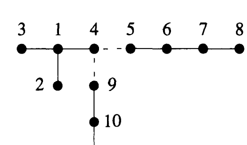

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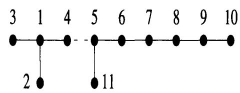

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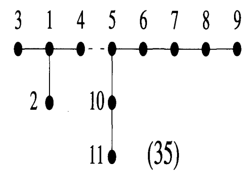

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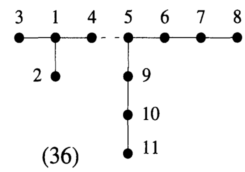

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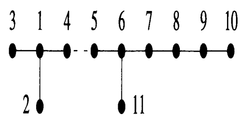

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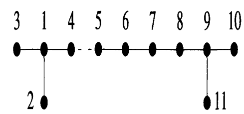

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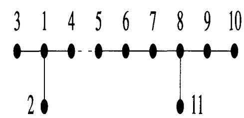

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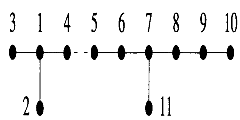

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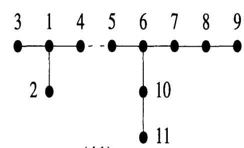

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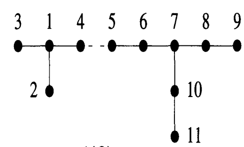

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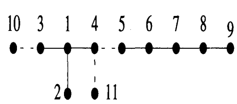

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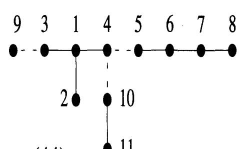

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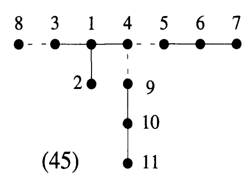

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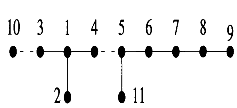

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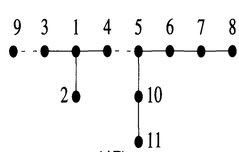

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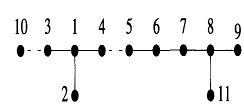

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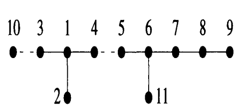

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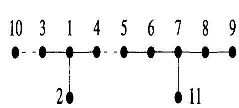

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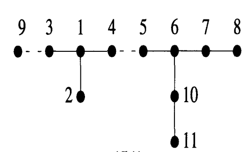

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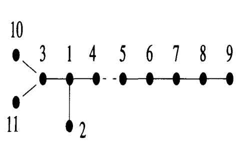

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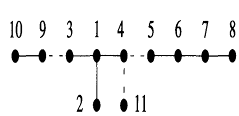

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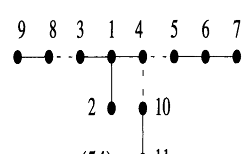

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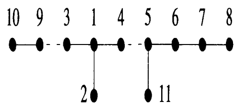

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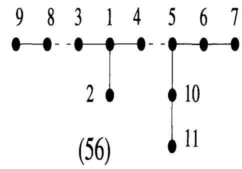

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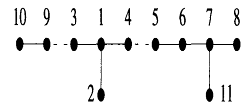

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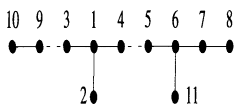

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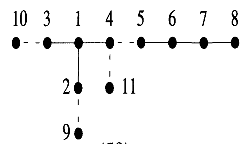

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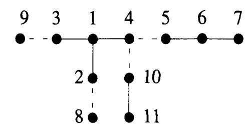

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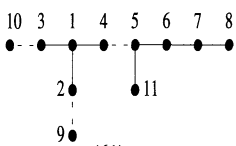

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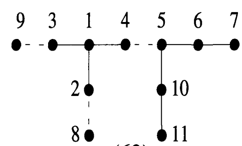

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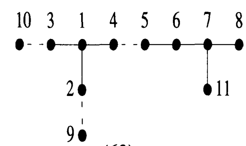

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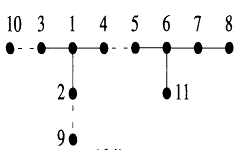

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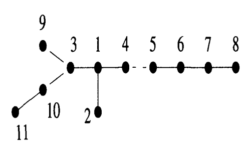

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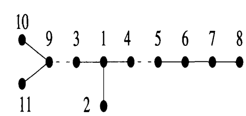

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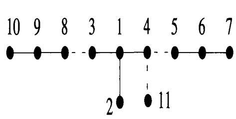

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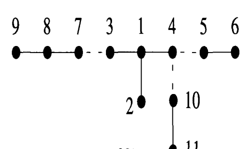

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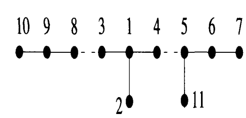

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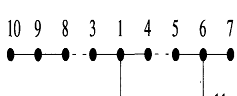

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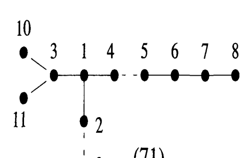

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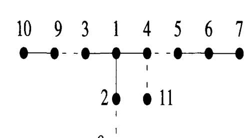

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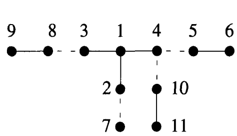

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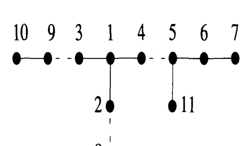

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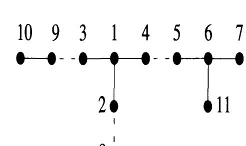

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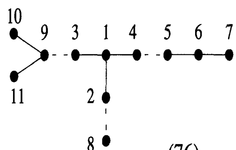

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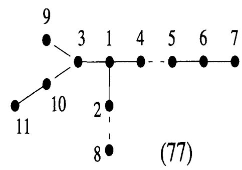

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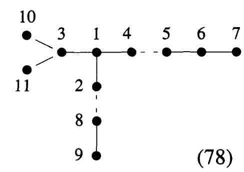

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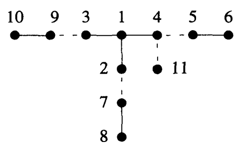

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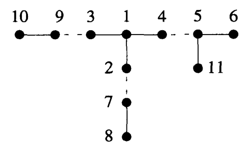

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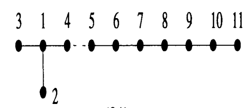

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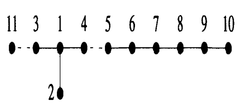

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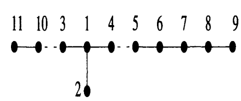

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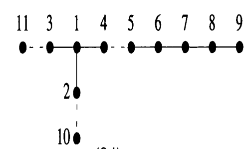

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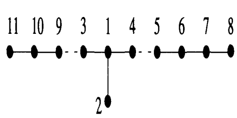

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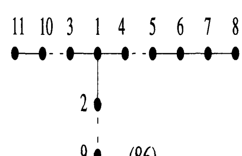

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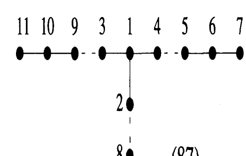

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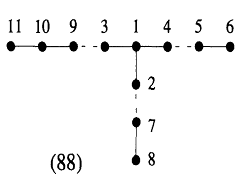

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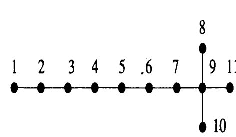

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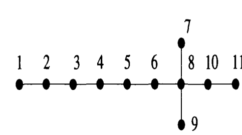

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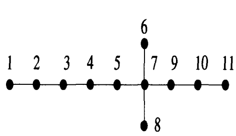

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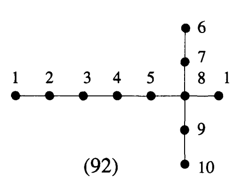

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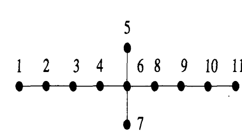

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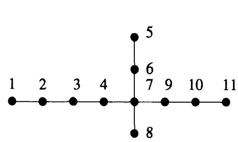

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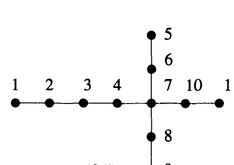

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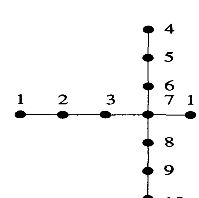

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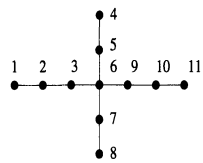

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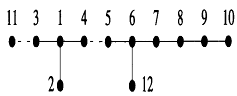

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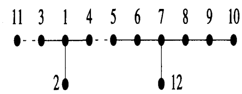

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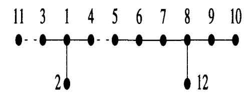

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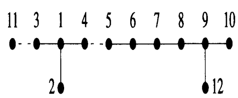

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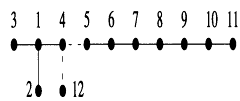

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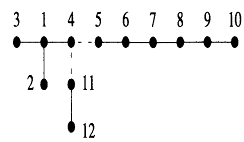

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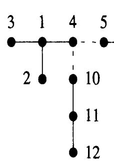

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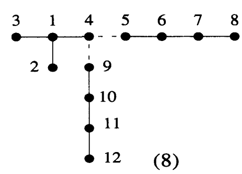

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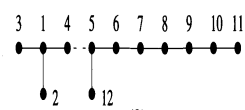

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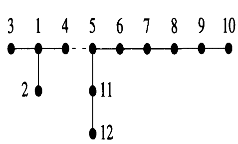

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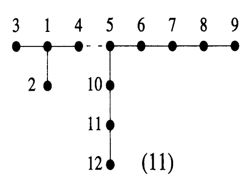

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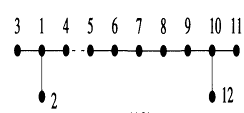

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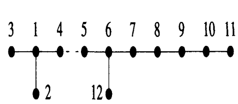

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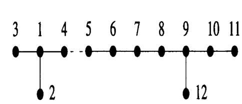

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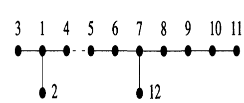

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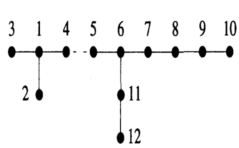

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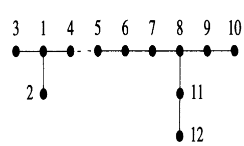

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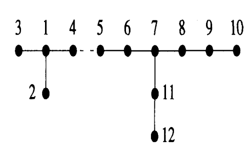

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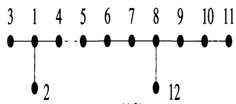

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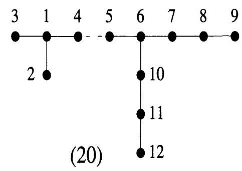

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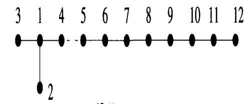

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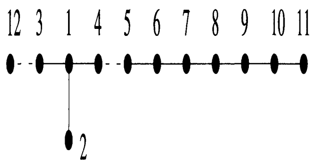

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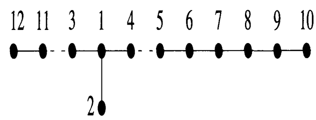

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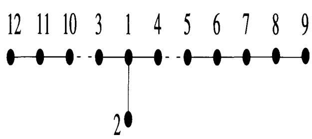

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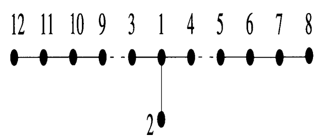

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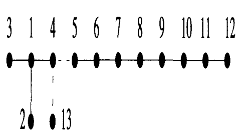

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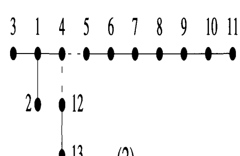

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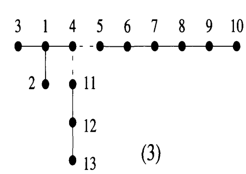

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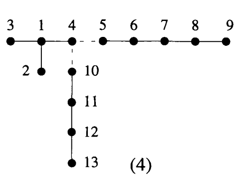

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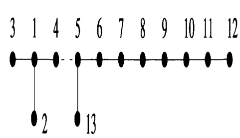

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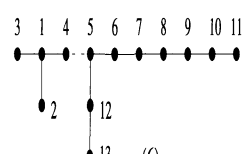

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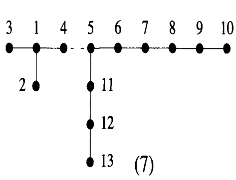

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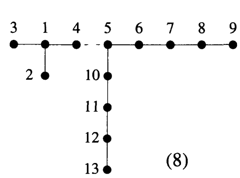

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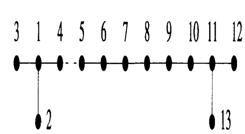

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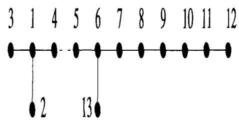

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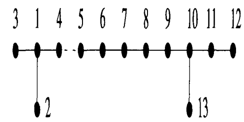

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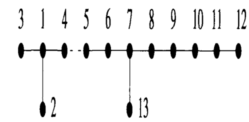

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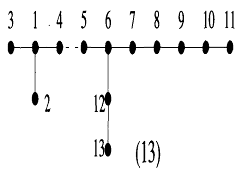

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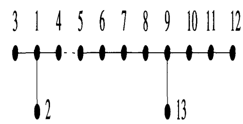

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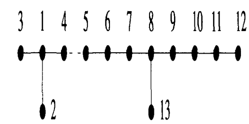

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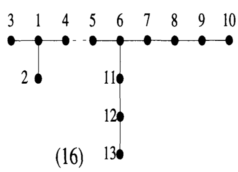

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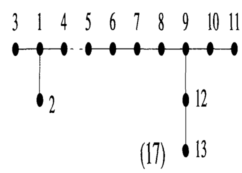

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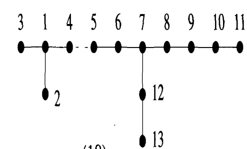

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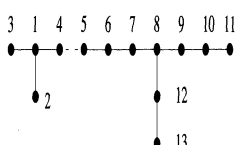

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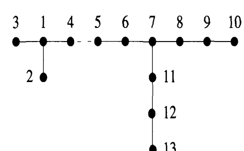

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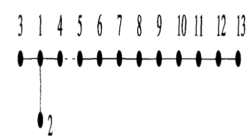


(21)

4. Reduction of possibilities for Δ

In this section, we reduce the possibilities for Δ to at most 37. This is done by explicitly computing the representation of canonical divisor in terms of irreducible components of Δ. We have developed a *Mathematica* program *kay.m* to carry out the necessary computations. We also use (15) of part I which may be stated as

$$-bk(\Delta) \leq 5 - \theta.$$

Further, we make use of lemma 2.1 which states that there is no fibre of ϕ contained in Δ, if Δ is free of (-1)-curves.

In case 1 computation of $(K)^2$ eliminates all but the trees listed in the table below with the following conventions. Column number 1 gives the tree number. Column 2 gives the two irreducible components whose self-intersection numbers are equal to -3. All the other components of Δ have self-intersection number equal to -2. Column 3 lists the vector (x_i) where the canonical divisor $K = \sum x_i D_i$. In the column 4, the fibre of ϕ contained in Δ (if any) is listed. Since we have already seen that there can be no fibre of ϕ contained in Δ, we need to further study only those trees for which the column 4 is blank.

Tree No.	$D_j,\ D_k$	K	Fibre
1	$D_4,\ D_5$	$\frac{1}{2}(0,0,0,0,0,2,1,1,1,1)$	$\sum_6^{10} D_i$
2	$D_2,\ D_8$	$\frac{1}{2}(0,0,1,1,2,1,1,0,0,0)$	$\sum_4^7 D_i$
4	$D_6,\ D_8$	$\frac{1}{2}(1,2,1,2,2,0,2,0,1,1)$	$\sum_{i\neq 6,8} D_i$
6	$D_5,\ D_9$	$\frac{1}{2}(1,2,1,2,0,2,1,1,0,0)$	$\sum_{i\neq 5,9,10} D_i$
8	$D_4,\ D_8$	$\frac{1}{2}(1,2,1,0,2,1,1,0,0,0)$	$\sum_{i\neq 4,8,9,10} D_i$
11	$D_6,\ D_9$	$\frac{1}{2}(1,1,1,1,2,0,0,1,0,0)$	$\sum_{i\neq 6,7,9,10} D_i$
12	$D_1,\ D_7$	$\frac{1}{2}(0,1,1,1,2,2,0,2,1,1)$	$\sum_{i\neq 1,7} D_i$
13	$D_2,\ D_7$	$\frac{1}{2}(0,0,1,2,1,2,0,2,1,1)$	$\sum_{i\neq 1,2,7} D_i$
15	$D_2,\ D_9$	$\frac{1}{2}(0,0,1,1,1,1,2,1,0,0)$	$\sum_3^8 D_i$
16	$D_3,\ D_7$	$\frac{1}{2}(0,0,0,1,2,1,0,2,1,1)$	$\sum_{i\neq 1,2,3,7} D_i$
17	$D_4,\ D_6$	$\frac{1}{2}(1,1,1,0,2,0,0,2,1,1)$	$\sum_{i\neq 4,6,7} D_i$
19	$D_5,\ D_7$	$\frac{1}{2}(1,2,1,2,0,2,0,2,1,1)$	$\sum_{i\neq 5,7} D_i$
20	$D_5,\ D_7$	$\frac{1}{2}(1,2,1,2,0,0,0,2,1,1)$	$\sum_{i\neq 5,6,7} D_i$
21	$D_5,\ D_9$	$\frac{1}{2}(1,2,1,2,0,1,2,1,0,0)$	$\sum_{i\neq 5,9,10} D_i$
23	$D_5,\ D_7$	$\frac{1}{2}(1,2,1,2,0,2,0,2,1,1)$	$\sum_{i\neq 5,7} D_i$
24	$D_6,\ D_7$	$\frac{1}{2}(1,2,1,2,2,0,0,2,1,1)$	$\sum_{i\neq 6,7} D_i$
26	$D_5,\ D_7$	$\frac{1}{2}(1,2,1,2,0,2,0,2,1,1)$	$\sum_{i\neq 5,7} D_i$

Tree No.	D_j, D_k	K	Remark
29	D_4, D_6	$\frac{1}{4}(1,2,3,0,4,0,2,3,2,1)$	$\sum_{i\neq 4,6} D_i$
31	D_1, D_3	$\frac{1}{3}(0,0,0,3,2,1,2,2,1,1)$	$\sum_4^{10} D_i$
35	D_7, D_9	$\frac{1}{2}(1,2,3,4,2,3,0,2,0,1)$	$\sum_{i\neq 7,9} D_i$
38	D_4, D_6	$\frac{1}{9}(2,4,6,-1,9,1,12,6,8,4)$	
39	D_1, D_4	$\frac{1}{3}(-1,0,0,1,3,6,4,2,4,2)$	
43	D_3, D_5	$\frac{1}{3}(1,2,0,3,0,4,2,3,2,1)$	$\sum_{i\neq 3,5} D_i$
45	D_6, D_9	$\frac{1}{2}(1,2,3,2,1,0,2,1,0,0)$	$\sum_{i\neq 6,9,10} D_i$
50	D_2, D_9	$\frac{1}{2}(0,0,2,1,3,2,1,2,0,1)$	$\sum_{i\neq 1,2,9} D_i$
50	D_8, D_9	$\frac{1}{8}(6,12,18,9,15,10,5,2,-2,1)$	
53	D_4, D_8	$\frac{1}{3}(1,2,3,0,4,2,3,0,2,1)$	$\sum_{i\neq 4,8} D_i$
56	D_1, D_6	$\frac{1}{7}(-1,4,2,7,10,1,12,6,8,4)$	
58	D_1, D_9	$\frac{1}{2}(0,2,1,3,4,2,3,2,0,1)$	$\sum_{i\neq 1,9} D_i$
71	D_1, D_7	$\frac{1}{2}(0,2,1,3,4,2,0,3,2,1)$	$\sum_{i\neq 1,7} D_i$
72	D_7, D_{10}	$\frac{1}{5}(4,8,12,6,10,8,1,5,2,-1)$	
77	D_1, D_3	$\frac{1}{2}(0,2,0,4,3,2,1,3,2,1)$	$\sum_{i\neq 1,3} D_i$
78	D_1, D_3	$\frac{1}{3}(-1,0,1,6,3,8,4,6,4,2)$	
79	D_8, D_9	$\frac{1}{7}(6,12,18,9,15,10,5,2,-2,-1)$	
87	D_9, D_{10}	$\frac{1}{4}(3,6,9,12,15,10,5,8,1,-1)$	

In case 2 computation of $(K)^2$ eliminates all but the trees listed in the table below with the conventions similar to the above one. Column 2 gives the two irreducible components whose self-intersection numbers are equal to -4 and -3 respectively. In the column 4 is given the value of $bk(\Delta)$ if it is less than -2 or a fibre of ϕ supported on Δ. Since we have already seen that $bk(\Delta) \geq -2$ and Δ contains no fibre of ϕ, we see that we need to further study only those trees for which the column 4 is blank.

Tree No.	D_j, D_k	K	$bk(T)$
63	D_4, D_7	$\frac{1}{2}(1,2,3,0,4,2,0,3,2,1)$	$\sum_{i\neq 4,7} D_i$
75	D_6, D_5	$\frac{1}{3}(5,10,5,10,2,-1,8,6,4,2)$	$-\frac{114}{55}$
79	D_9, D_4	$\frac{1}{9}(7,14,21,4,24,16,8,11,-2,-1)$	$-\frac{9}{4}$
79	D_6, D_7	$\frac{1}{7}(8,16,24,12,20,1,-2,15,10,5)$	$-\frac{301}{132}$
80	D_1, D_4	$\frac{1}{4}(-1,4,9,2,1,12,6,9,6,3)$	$-\frac{299}{140}$
87	D_{10}, D_7	$\frac{1}{5}(4,8,12,16,20,11,2,13,6,-1)$	

In case 3 computation of $(K)^2$ eliminates all but the trees listed in the table below with the conventions similar to the above one. Column 2 gives the three irreducible components whose self-intersection numbers are equal to -3. We need to further study only those trees for which the column 4 is blank.

Tree No.	D_j, D_k, D_l	K	$bk(T)$
64	D_1, D_3, D_4	$\frac{1}{3}(-1,0,1,0,3,6,4,2,4,2)$	$-\frac{43}{20}$
77	D_1, D_6, D_7	$\frac{1}{3}(1,6,3,8,4,0,-1,6,4,2)$	
79	D_6, D_7, D_{10}	$\frac{1}{9}(8,16,24,12,20,3,-2,13,6,-1)$	

In case 4 computation of $(K)^2$ eliminates all but the trees listed in the table below with the above conventions. Column number 1 gives the tree number. Column 2 gives the two irreducible components whose self-intersection numbers are equal to -5 and -3 respectively. In the column 4 is given the value of $bk(\Delta)$ if it is less than -2 or a fibre of ϕ supported on Δ. Since we have already seen that $bk(\Delta) \geq -2$ and Δ contains no fibre of ϕ, we see that this case does not occur.

Tree No.	$D_j,\ D_k,\ D_l$	K	Remarks
32	$D_2,\ D_8$	$\frac{1}{11}(18,-3,9,30,22,14,6,-2,-1,20,10)$	$\sum_{i\neq 2,7,8,9} D_i$
32	$D_{11},\ D_{10}$	$\frac{1}{2}(6,3,3,6,5,4,3,2,1,1,-1)$	$-\frac{43}{21}$
54	$D_6,\ D_2$	$\frac{1}{13}(28,5,21,30,12,-6,-3,14,7,20,10)$	$-\frac{37}{16}$
54	$D_{10},\ D_{11}$	$\frac{1}{5}(16,8,12,12,9,6,3,8,4,-1,-2)$	$\sum_{i\neq 6,7,10,11} D_i$
59	$D_{10},\ D_2$	$\frac{1}{11}(21,4,8,30,24,18,12,6,2,-5,15)$	$-\frac{191}{90}$
60	$D_7,\ D_9$	$\frac{1}{13}(21,-4,10,36,27,18,9,-2,-1,24,12)$	$\sum_{i\neq 2,7,8,9} D_i$
86	$D_9,\ D_8$	$\frac{1}{7}(24,11,18,19,14,9,4,-1,-2,12,6)$	$\sum_{i\neq 7,8,9} D_i$
86	$D_{11},\ D_9$	$\frac{1}{7}(24,13,15,20,16,12,8,4,2,6,-3)$	$\sum_{i\neq 9,11}$

In case 5 computation of $(K)^2$ eliminates all but the trees listed in the table below with the conventions similar to the above. Column number 1 gives the tree number. Column 2 gives the four irreducible components whose self-intersection numbers are equal to -4, -3, -3 and -3 respectively. $(D_1)^2 = -1$ and all the other components of D have self-intersection number equal to -2. We need to further study only those trees for which the column 4 is blank.

Tree No.	$D_j,\ D_k,\ D_l,\ D_m$	K	$bk(T)$
4	$D_2,\ D_3,\ D_4,\ D_{11}$	$\frac{1}{10}(28,2,6,10,7,4,3,2,1,5,-2)$	$-\frac{13}{6}$
7	$D_2,\ D_3,\ D_4,\ D_6$	$\frac{1}{15}(42,3,9,15,8,-3,-2,-1,10,5,4)$	$-\frac{69}{28}$
12	$D_2,\ D_3,\ D_4,\ D_6$	$\frac{1}{7}(22,2,5,8,3,-2,-1,6,4,2,-1)$	$-\frac{7}{3}$
21	$D_2,\ D_3,\ D_4,\ D_{10}$	$\frac{1}{3}(6,0,1,2,3,4,3,2,1,0,2)$	$-\frac{13}{6}$
23	$D_2,\ D_3,\ D_4,\ D_6$	$\frac{1}{3}(6,0,1,2,3,0,0,2,1,2,1)$	$-\frac{151}{60}$
27	$D_4,\ D_2,\ D_3,\ D_5$	$\frac{1}{2}(-1,-1,-1,-1,1,6,3,4,2,4,2)$	$-\frac{5}{2}$
29	$D_2,\ D_3,\ D_4,\ D_{11}$	$\frac{1}{3}(6,0,1,2,3,4,3,2,1,2,0)$	$-\frac{25}{12}$
31	$D_2,\ D_3,\ D_4,\ D_7$	$\frac{1}{11}(38,4,9,14,8,2,-4,-3,-2,-1,7)$	
34	$D_2,\ D_3,\ D_4,\ D_8$	$\frac{1}{9}(30,3,7,11,12,7,2,-3,-2,-1,6)$	
36	$D_2,\ D_3,\ D_4,\ D_7$	$\frac{1}{10}(28,2,6,10,12,5,-2,-1,9,6,3)$	
37	$D_2,\ D_3,\ D_4,\ D_9$	$\frac{1}{7}(22,2,5,8,9,10,6,2,-2,-1,5)$	
40	$D_2,\ D_3,\ D_4,\ D_{10}$	$\frac{1}{5}(14,1,3,5,6,7,8,5,2,-1,4)$	
43	$D_2,\ D_3,\ D_4,\ D_6$	$\frac{1}{14}(44,4,12,14,5,-4,-3,-2,-1,6,7)$	$-\frac{299}{140}$
43	$D_2,\ D_3,\ D_4,\ D_{10}$	$\frac{1}{6}(16,1,3,6,5,4,3,2,1,-1,3)$	
45	$D_3,\ D_2,\ D_4,\ D_6$	$\frac{1}{8}(23,5,2,8,3,-2,-1,1,6,4,2)$	$-\frac{383}{168}$
47	$D_3,\ D_2,\ D_4,\ D_{11}$	$\frac{1}{11}(29,6,2,10,12,9,6,3,1,5,-2)$	
49	$D_3,\ D_2,\ D_4,\ D_9$	$\frac{1}{8}(23,5,2,8,9,10,6,2,-2,1,5)$	

Tree No.	D_j, D_k, D_l, D_m	K	$bk(T)$
50	D_2, D_3, D_4, D_9	$\frac{1}{7}(22, 2, 6, 7, 6, 5, 4, 1, -2, 3, 2)$	
50	D_2, D_3, D_4, D_{10}	$\frac{1}{6}(16, 1, 3, 6, 8, 10, 12, 8, 4, -1, 6)$	
55	D_3, D_2, D_4, D_8	$\frac{1}{11}(32, 7, 3, 11, 12, 7, 2, -3, 2, 1, 6)$	
57	D_2, D_3, D_4, D_8	$\frac{1}{3}(10, 1, 3, 3, 2, 1, 0, -1, 2, 1, 0)$	
57	D_2, D_3, D_4, D_{10}	$\frac{1}{4}(12, 1, 3, 4, 4, 4, 4, 2, 1, -1, 2)$	
65	D_4, D_2, D_3, D_5	$\frac{1}{13}(46, 11, 18, 4, -4, -3, -2, -1, 9, 12, 6)$	$-\frac{73}{32}$
66	D_4, D_2, D_3, D_8	$\frac{1}{5}(17, 4, 6, 2, 1, 0, -1, -2, 6, 3, 3)$	
67	D_3, D_2, D_4, D_7	$\frac{1}{14}(41, 9, 4, 14, 8, 2, -4, 3, 2, 1, 7)$	$-\frac{1109}{546}$
72	D_4, D_2, D_3, D_7	$\frac{1}{16}(51, 14, 15, 6, 2, -2, -6, 7, 10, 5, 3)$	$-\frac{157}{70}$
77	D_4, D_2, D_3, D_{11}	$\frac{1}{17}(47, 12, 14, 4, 3, 2, 1, 6, 7, 5, -4)$	$-\frac{59}{26}$
80	D_4, D_2, D_3, D_6	$\frac{1}{3}(10, 3, 3, 1, 0, -1, 2, 1, 2, 1, 0)$	$-\frac{95}{42}$
82	D_2, D_3, D_4, D_9	$\frac{1}{9}(34, 4, 10, 11, 8, 5, 2, -1, -4, -2, 5)$	
82	D_2, D_3, D_4, D_{10}	$\frac{1}{8}(28, 3, 8, 9, 7, 5, 3, 1, -1, 3, 4)$	
84	D_2, D_3, D_4, D_9	$\frac{1}{11}(36, 4, 10, 11, 8, 5, 2, -1, -4, 2, 5)$	
85	D_2, D_3, D_4, D_7	$\frac{1}{11}(38, 4, 12, 11, 6, 1, -4, -2, 9, 6, 3)$	
85	D_2, D_3, D_4, D_{11}	$\frac{1}{10}(32, 3, 9, 10, 8, 6, 4, 2, 5, 1, -3)$	

Tree No.	$D_j,\ D_k,\ D_l,\ D_m$	K	$bk(T)$
86	$D_4,\ D_2,\ D_3,\ D_7$	$\frac{1}{13}(48, 14, 15, 6, 2, -2, -6, -3, 7, 10, 5)$	
87	$D_2,\ D_3,\ D_4,\ D_7$	$\frac{1}{13}(40, 4, 12, 11, 6, 1, -4, 2, 9, 6, 3)$	
87	$D_2,\ D_3,\ D_4,\ D_8$	$\frac{1}{4}(13, 1, 4, 4, 3, 2, 1, -1, 3, 2, 1)$	

In case 6, the only tree for which $(K)^2 = -1$ is tree number (36) with $(D_1)^2 = -1,\ (D_2)^2 = (D_3)^2 = -4,\ (D_4)^2 = -3$ and $(D_j)^2 = -2$ for all $5 \leq j \leq 11$. In this case $K = 2D_1 + D_4 + 2D_5 + \frac{3}{2}D_6 + D_7 + \frac{1}{2}D_8 + \frac{3}{2}D_9 + D_{10} + \frac{1}{2}D_{11}$. But we see that $\sum_{i \neq 2,3} D_i''$ is a fibre – say, F'' – of ϕ''. But then it is easy to see that for either of the horizontal components D_2'' or D_3'' we have $(F''.D_j'') = 2$. This is a contradiction as we need to have $(F''.H'') \geq 6$ for any horizontal component H'' of ϕ''. Hence this case cannot occur.

In case 7, computation of $(K)^2$ eliminates all but the trees listed in the table below with the conventions similar to the above. Column number 1 gives the tree number. Column 2 gives the irreducible component whose self-intersection number is equal to -3. $(D_1)^2 = -1,\ (D_3)^2 = -5,\ (D_4)^2 = -4$ and all the other components of Δ have self-intersection number equal to -2. We need to further study only those trees for which the column 4 is blank.

Tree No.	D_j	K	Remarks
1	D_{10}	$\frac{1}{7}(30, 15, 2, 6, 8, 10, 7, 4, 1, -2, 1, 5)$	
3	D_{11}	$\frac{1}{4}(18, 9, 1, 4, 6, 8, 10, 12, 8, 4, -1, 6)$	$-\frac{20}{9}$
10	D_8	$\frac{1}{9}(42, 21, 3, 9, 12, 7, 2, -3, -2, -1, -8, 4)$	$-\frac{21}{10}$
13	D_{10}	$\frac{1}{6}(28, 14, 2, 6, 8, 10, 7, 4, 1, -2, -15)$	
18	D_{12}	$\frac{1}{2}(6, 3, 0, 1, 2, 3, 4, 3, 2, 1, 2, 0)$	$\sum_4^{11} D_i$

In case 8, computation of $(K)^2$ eliminates all the trees, as for any possible tree we have $(K)^2 \neq -2$.

In case 9, there are exactly two trees for which $(K)^2 = -3$. One is tree number (13) with following weights: $(D_1)^2 = -1$, $(D_3)^2 = (D_{11})^2 = -3$, $(D_4)^2 = -7$ and $(D_j)^2 = -2$ for all $j \neq 1, 3, 4, 11$. For this tree, we have $4K = 22D_1 + 11D_2 + 6D_3 + D_4 + 5D_5 + 9D_6 + 7D_7 + 5D_8 + 3D_9 + D_{10} - D_{11} + 6D_{12} + 3D_{13}$. The other tree is tree number 17 with the following weights: $(D_1)^2 = -1$, $(D_3)^2 = (D_{11})^2 = -3$, $(D_4)^2 = -7$ and $(D_j)^2 = -2$ for all $j \neq 1, 3, 4, 11$. For this tree, we have $2K = 2D_1 + D_2 - D_4 + D_5 + 3D_6 + 5D_7 + 7D_8 + 9D_9 + 5D_{10} + D_{11} + 6D_{12} + 3D_{13}$.

This exhausts all the nine cases listed in Table 3. Upshot of these computations is that we are left with precisely the following 37 trees.

Sl. no.	β_2	Tree no.	Weight set	K
1	10	38	$w_4 = w_6 = -3$	$\frac{1}{9}(2,4,6,-1,9,1,12,6,8,4)$
2	10	39	$w_1 = w_4 = -3$	$\frac{1}{3}(-1,0,0,1,3,6,4,2,4,2)$
3	10	50	$w_8 = w_9 = -3$	$\frac{1}{8}(6,12,18,9,15,10,5,2,-2,1)$
4	10	56	$w_1 = w_6 = -3$	$\frac{1}{7}(-1,4,2,7,10,1,12,6,8,4)$
5	10	72	$w_7 = w_{10} = -3$	$\frac{1}{5}(4,8,12,6,10,8,1,5,2,-1)$
6	10	78	$w_1 = w_3 = -3$	$\frac{1}{3}(-1,0,1,6,3,8,4,6,4,2)$
7	10	79	$w_8 = w_9 = -3$	$\frac{1}{7}(6,12,18,9,15,10,5,2,-2,-1)$
8	10	87	$w_9 = w_{10} = -3$	$\frac{1}{4}(3,6,9,12,15,10,5,8,1,-1)$
9	10	87	$w_{10} = -4,\ w_7 = -3$	$\frac{1}{5}(4,8,12,16,20,11,2,13,6,-1)$
10	10	77	$w_1 = w_6 = w_7 = -3$	$\frac{1}{3}(1,6,3,8,4,0,-1,6,4,2)$
11	10	79	$w_6 = w_7 = w_{10} = -3$	$\frac{1}{9}(8,16,24,12,20,3,-2,13,6,-1)$
12	11	31	$w_2 = -4,\ w_1 = -1$	$\frac{1}{11}(38,4,9,14,8,2,-4,-3,-2,-1,7)$
			$w_3 = w_4 = w_7 = -3$	
13	11	34	$w_2 = -4,\ w_1 = -1$	$\frac{1}{9}(30,3,7,11,12,7,2,-3,-2,-1,6)$
			$w_3 = w_4 = w_8 = -3$	
14	11	36	$w_2 = -4,\ w_1 = -1$	$\frac{1}{10}(28,2,6,10,12,5,-2,-1,9,6,3)$
			$w_3 = w_4 = w_7 = -3$	
15	11	37	$w_2 = -4,\ w_1 = -1$	$\frac{1}{7}(22,2,5,8,9,10,6,2,-2,-1,5)$
			$w_3 = w_4 = w_9 = -3$	
16	11	40	$w_2 = -4,\ w_1 = -1$	$\frac{1}{5}(14,1,3,5,6,7,8,5,2,-1,4)$
			$w_3 = w_4 = w_{10} = -3$	
17	11	43	$w_2 = -4,\ w_1 = -1$	$\frac{1}{6}(16,1,3,6,5,4,3,2,1,-1,3)$
			$w_3 = w_4 = w_{10} = -3$	
18	11	47	$w_3 = -4,\ w_1 = -1$	$\frac{1}{11}(29,6,2,10,12,9,6,3,1,5,-2)$
			$w_2 = w_4 = w_{11} = -3$	
19	11	49	$w_3 = -4,\ w_1 = -1$	$\frac{1}{8}(23,5,2,8,9,10,6,2,-2,1,5)$
			$w_2 = w_4 = w_9 = -3$	
20	11	50	$w_2 = -4,\ w_1 = -1$	$\frac{1}{7}(22,2,6,7,6,5,4,1,-2,3,2)$
			$w_3 = w_4 = w_9 = -3$	

Sl. no.	β_2	Tree no.	Weight set	K
21	11	50	$w_2=-4,\ w_1=-1$ $w_3=w_4=w_{10}=-3$	$\frac{1}{6}(16,1,3,6,8,10,12,8,4,-1,6)$
22	11	55	$w_3=-4,\ w_1=-1$ $w_2=w_4=w_8=-3$	$\frac{1}{11}(32,7,3,11,12,7,2,-3,2,1,6)$
23	11	57	$w_2=-4,\ w_1=-1$ $w_3=w_4=w_8=-3$	$\frac{1}{3}(10,1,3,3,2,1,0,-1,2,1,0)$
24	11	57	$w_2=-4,\ w_1=-1$ $w_3=w_4=w_{10}=-3$	$\frac{1}{4}(12,1,3,4,4,4,4,2,1,-1,2)$
25	11	66	$w_4=-4,\ w_1=-1$ $w_2=w_3=w_8=-3$	$\frac{1}{5}(17,4,6,2,1,0,-1,-2,6,3,3)$
26	11	82	$w_2=-4,\ w_1=-1$ $w_3=w_4=w_9=-3$	$\frac{1}{9}(34,4,10,11,8,5,2,-1,-4,-2,5)$
27	11	82	$w_2=-4,\ w_1=-1$ $w_3=w_4=w_{10}=-3$	$\frac{1}{8}(28,3,8,9,7,5,3,1,-1,-3,4)$
28	11	84	$w_2=-4,\ w_1=-1$ $w_3=w_4=w_9=-3$	$\frac{1}{11}(36,4,10,11,8,5,2,-1,-4,2,5)$
29	11	85	$w_2=-4,\ w_1=-1$ $w_3=w_4=w_7=-3$	$\frac{1}{11}(38,4,12,11,6,1,-4,-2,9,6,3)$
30	11	85	$w_2=-4,\ w_1=-1$ $w_3=w_4=w_{11}=-3$	$\frac{1}{10}(32,3,9,10,8,6,4,2,5,1,-3)$
31	11	86	$w_4=-4,\ w_1=-1$ $w_2=w_3=w_7=-3$	$\frac{1}{13}(48,14,15,6,2,-2,-6,-3,7,10,5)$
32	11	87	$w_2=-4,\ w_1=-1$ $w_3=w_4=w_7=-3$	$\frac{1}{13}(40,4,12,11,6,1,-4,2,9,6,3)$
33	11	87	$w_2=-4,\ w_1=-1$ $w_3=w_4=w_8=-3$	$\frac{1}{4}(13,1,4,4,3,2,1,-1,3,2,1)$
34	12	1	$w_1=-1,\ w_{10}=-3$ $w_4=-4,\ w_3=-5$	$\frac{1}{7}(30,15,2,6,8,10,7,4,1,-2,1,5)$
35	12	13	$w_1=-1,\ w_{10}=-3$ $w_4=-4,\ w_3=-5$	$\frac{1}{6}(28,14,2,6,8,10,7,4,1,-2,-1,5)$
36	13	13	$w_1=-1,\ w_4=-7$ $w_3=w_{11}=-3$	$\frac{1}{4}(22,11,6,1,5,9,7,5,4,1,-1,6,3)$
37	13	17	$w_1=-1,\ w_4=-7$ $w_3=w_{11}=-3$	$\frac{1}{2}(2,1,0,-1,1,3,5,7,9,5,1,6,3)$

5. Completion of the proof of the Theorem

In this section, we complete the proof of proposition 1.1 by eliminating the 37 trees listed at the end of the section 4. We shall study the 37 trees individually.

Tree 1: For this tree, we observe that $D_1+D_2+D_3+D_5+D_7+D_8+D_9+D_{10}$ is free from horizontal components of the elliptic fibration $\phi : X \to \mathbb{P}^1$ and hence it is a part of a (singular) fibre of ϕ. Only way this can happen is that the above components are a part of a fibre F (whose MNC-model) is of the type II*. But then, in F, we see that D_7 is of multiplicity 6. Also we know that there are two multiple fibres $2P_1$ and $3P_2$. Now consider the restriction of ϕ to D_6. Clearly $\phi : D_6 \to \mathbb{P}^1$ is a degree 6 map and the points of intersection of D_6 with the components of fibres are ramification points with appropriate ramification indices. Note that the ramification indices on D_6 due to $3P_2$ are either $(3,3)$ or (6) - i.e., P_2 intersects D_6 at two distinct points transversally so that the ramification indices are $(3,3)$ or P_2 intersects D_6 tangentially exactly at one point so that the ramification index is equal to 6. Similarly ramification indices due to $2P_1$ are either $(2,2,2)$, $(4,2)$ or (6). Further, ramification index due to F is equal to (6). Now, Riemann-Hurewitz formula for $\phi : D_6 \to \mathbb{P}^1$ gives

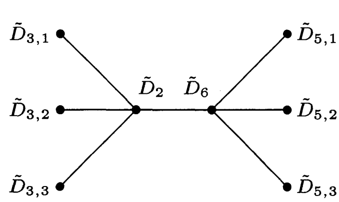
$$2g(D_6) - 2 = 6(2g(\mathbb{P}^1) - 2) + \sum_i (e_i - 1)$$

where e_is are the ramification indices. The l.h.s. in the above equation is equal to -2 whereas r.h.s. is at least equal to 0. This contradiction shows that this tree cannot occur.

Tree 2: Consider the 3-fold ramified cover $f : \overline{X} \to X$ given by

$$3K \sim -D_1 + D_4 + 3D_5 + 6D_6 + 4D_7 + 2D_8 + 4D_9 + 2D_{10}.$$

MNC-model of the inverse image of Δ after resolution of singularities is as follows:

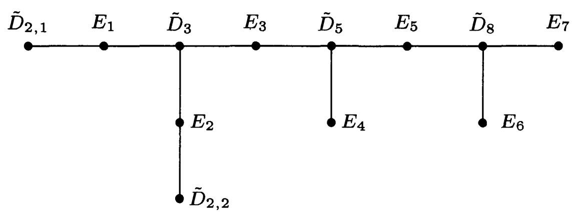


We have $(\tilde{D}_2)^2 = -4$, $(\tilde{D}_6)^2 = -1$ and all the other irreducilbe components are (-2)-curves. We observe that after contracting $\tilde{D}_6$ we have three (-1)-curves passing through a point. But then $\tilde{X}$ is a rational surface. This is a contradiction as $\tilde{X}$ dominates X, which is a non-rational surface. Hence tree 2 does not occur.

Tree 3: Consider the 8-fold ramified cover given by

$$8K \sim 6D_1 + 12D_2 + 18D_3 + 9D_4 + 15D_5 + 10D_6 + 5D_7 + 2D_8 - 2D_9 + D_{10}.$$

MNC-model of the inverse image of Δ after resolution of singularities is as follows:



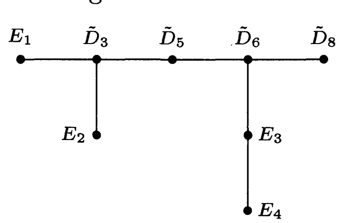
We have, $(E_3)^2 = (E_4)^2 = (E_6)^2 = (E_7)^2 = -3$, $(\tilde{D}_5)^2 = -1$, $(E_5)^2 = -4$ and all the other irreducible components are (-2)-curves. It can be seen that $\tilde{K} = 12\tilde{D}_3 + 11E_3 + 22\tilde{D}_5 + 7E_4 + 6E_5 + 4\tilde{D}_8 + E_6 + E_7 + \cdots$ which implies that $(\tilde{K})^2 = 10$. We have $e(\tilde{X}) = 21$. Thus $\chi(\tilde{X}) = \frac{31}{12}$ which is a contradiction. Hence this tree cannot occur.

Tree 4: As in the case of tree 1, we see that the degree 6 map $\phi : D_6 \to \mathbb{P}^1$ violates the Riemann-Hurewitz formula. Hence this tree does not occur.

Tree 5: Consider the 5-fold ramified cover $f : \overline{X} \to X$ given by

$$5K \sim 4D_1 + 8D_2 + 12D_3 + 6D_4 + 10D_5 + 8D_6 + D_7 + 5D_8 + 2D_9 - D_{10}.$$

MNC-model of the inverse image of Δ after resolution of singularities is as follows:



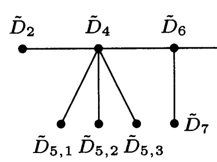
We have $(E_1)^2 = -3$, $(\tilde{D}_3)^2 = (\tilde{D}_6)^2 = -1$, $(\tilde{D}_5)^2 = -10$, $(\tilde{D}_8)^2 = -7$ and all the other irreducible components are (-2)-curves. It can be seen that $\tilde{K} \sim 5E_1 + 16\tilde{D}_3 + 2\tilde{D}_5 + 12\tilde{D}_6 + \tilde{D}_8 + \cdots$ and hence $(\tilde{K})^2 = -2$. In particular, $p_g(\tilde{X}) > 0$. Observe that each $\tilde{D}_j$ and E_i is a rational curve and $\cup\tilde{D}_j \cup E_i$ supports an ample divisor. Hence the image of these irreducible components in the Albanese of $\tilde{X}$ generate the Albanese torus of $\tilde{X}$. Then Albanese of $\tilde{X}$ is trivial and hence $q(\tilde{X}) = 0$. Further, we know that $1 = e(V) = e(X \setminus \Delta)$ where $e(-)$ denotes the topological Euler characteristic. Hence $e(\tilde{X} \setminus \tilde{\Delta}) = 5$. Therefore $e(\tilde{X}) = 5 + e(\tilde{\Delta}) =$

$5+(1+8)=14$. But then $\chi(\tilde{X})=\chi(\tilde{X},\mathcal{O}_{\tilde{X}})=\frac{1}{12}((\tilde{K})^2+e(\tilde{X})=\frac{1}{12}(-2+14)=1$ whence $p_g(\tilde{X})=0$. This contradiction shows that this tree does not occur.

Tree 6: Consider the 3-fold ramified cover $f:\overline{X}\to X$ given by

$$3K\sim -D_1+D_3+6D_4+3D_5+8D_6+4D_7+6D_8+4D_9+2D_{10}.$$

MNC-model of the inverse image of Δ after resolution of singularities is as follows:

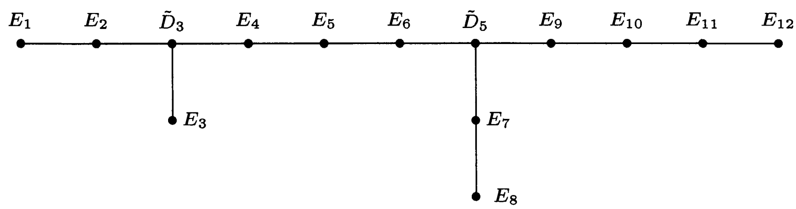


We have $(\tilde{D}_2)^2=(\tilde{D}_8)^2=-4$, $(\tilde{D}_4)^2=-5$, $(\tilde{D}_6)^2=-1$ and all the other irreducible components are (-2)-curves. It can be seen that $\tilde{K}=2\tilde{D}_4+10\tilde{D}_6+2\tilde{D}_8+\cdots$ and hence $(\tilde{K})^2=0$. Also $e(\tilde{X})=12$. But then $\chi(\tilde{X})=1$ which implies that $p_g(\tilde{X})=0$ and this is a contradiction. Thus this tree cannot occur.

Tree 7: Consider the 7-fold ramified cover $f:\overline{X}\to X$ given by

$$7K\sim 6D_1+12D_2+18D_3+9D_4+15D_5+10D_6+5D_7+2D_8-2D_9-D_{10}.$$

MNC-model of the inverse image of Δ after resolution of singularities is as follows:



We have $(E_6)^2=-3$, $(E_{12})^2=-4$ and all the other irreducible components are (-2)-curves. It can be seen that $\tilde{K}=12E_6+E_{12}+\cdots$ and hence $(\tilde{K})^2=14$. Also $e(\tilde{X})=22$ which implies that $\chi(\tilde{X})=3$. In particular, $p_g(\tilde{X})=2$. Now, we apply Lefschetz fixed point formula (1) to the smooth surface $\tilde{X}$ which has a $\mathbb{Z}/7\mathbb{Z}$-action. To compute r.h.s. of (1), we observe that we have at least two fixed curves, $\tilde{D}_3$ and $\tilde{D}_5$. Also, since each E_i is a stable curve, we see that $E_i\cap E_j$, $1\le i\neq j\le 12$ are all isolated fixed points whenever this intersection is non-empty. There are seven such points. In addition to these, there are four more isolated fixed points one each on E_1,E_3,E_8 and E_{12}. We shall denote these four points by q_1,q_3,q_8 and q_{12} (respectively).

We choose $\tilde{D}_3$ as the starting vertex b as in section 1.1.4. Using the definition of L-function given in section 1.1.4, we compute $L(v_i)$ for all the vertices. We have $L(\tilde{D}_3) = 0$, $L(E_2) = L(E_4) = L(E_3) = 1$, $L(E_1) = L(E_5) = 2$, $L(E_6) = 3, L(\tilde{D}_5) = 0$, $L(E_7) = L(E_9) = 4$, $L(E_8) = L(E_{10}) = 1$, $L(E_{11}) = 5$ and $L(E_{12}) = 2$.

Let η denote $e^{2\pi \imath/7}$. By a suitable representation of the cyclic group $\boldsymbol{Z}/7\boldsymbol{Z}$, which is identified with the group of seventh roots unity, we denote the eigen value for the group action on the normal bundle of $\tilde{D}_3$ by ω.

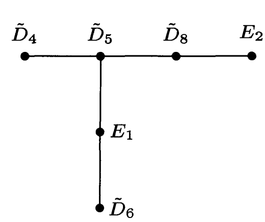
Let $p_1 = E_1 \cap E_2$. By lemma 1.4, $a(p_1, E_1) = L(E_1).a(\tilde{B})$ where we can choose $\tilde{B}$ to be E_2. Let $a := a(\tilde{b})$. The eigen value of the action of g (which corresponds to η) is $\omega = \eta^a$. Then $a(p_1, E_1) = 2a$. Hence $a(q_1, E_1) = -2a \equiv 5a \ mod \ 7$. By lemma 1.1, $n(q_1, E_1) = n(p_1, E_1) - a(p_1, E_1).(E_1)^2 = a(p_1, E_2) - a(p_1, E_1).(E_1)^2 = -a(E_2 \cap \tilde{D}_3, E_2) - a(p_1, E_1).(E_1)^2 = -a - 2a(-2) = 3a$.

Hence the contribution to the r.h.s. of (1) due to q_1 is equal to $\frac{1}{(1-\omega^5)(1-\omega^3)}$. Similarly we compute the eigen values at all the other ten isolated fixed points. In particular, $n(q_3, E_3) = 2a$, $n(q_8, E_8) = 5a$ and $n(q_{12}, E_{12}) = 3a$. Now, we compute the contributions of the fixed curves $\tilde{D}_3$ and $\tilde{D}_5$ to the r.h.s. of (1). Observe that by lemma 1.2, $n(q(E_2, \tilde{D}_3), \tilde{D}_3) = a(q(E_2, \tilde{D}_3), E_2) = a$ and $n(q(E_6, \tilde{D}_5), \tilde{D}_5) = a(q(E_6, \tilde{D}_5), E_6) = 4a$. Thus the contribution of $\tilde{D}_3$ and $\tilde{D}_5$ to the r.h.s. of (1) equals $\frac{1+\omega^{-1}}{(1-\omega^{-1})^2}$ and $\frac{1+\omega^{-4}}{(1-\omega^{-4})^2}$ respectively. Then the r.h.s. of (1) equals $\frac{1+\omega^{-1}}{(1-\omega^{-1})^2} + \frac{1+\omega^{-4}}{(1-\omega^{-4})^2} + \frac{3}{(1-\omega^6)(1-\omega^2)} + \frac{3}{(1-\omega^5)(1-\omega^3)} + \frac{2}{(1-\omega^6)(1-\omega^5)} + \frac{2}{(1-\omega^3)(1-\omega^1)} + \frac{1}{(1-\omega^2)(1-\omega^2)} = 0.7225 - \imath 2.1962$. Since $p_g(\tilde{X}) = 2$, we see that l.h.s. of (1) is equal to $1 + \omega^l + \omega^m$ for $1 \leq l, m \leq 7$. But we see that the imaginary part of this is greater than -2. This contradiction shows that this tree cannot occur.

Tree 8: Consider the 4-fold ramified cover $f : \overline{X} \to X$ given by

$$4K \sim 3D_1 + 6D_2 + 9D_3 + 12D_4 + 15D_5 + 10D_6 + 5D_7 + 8D_8 + D_9 - D_{10}.$$

MNC-model of the inverse image of Δ after resolution of singularities is as follows:

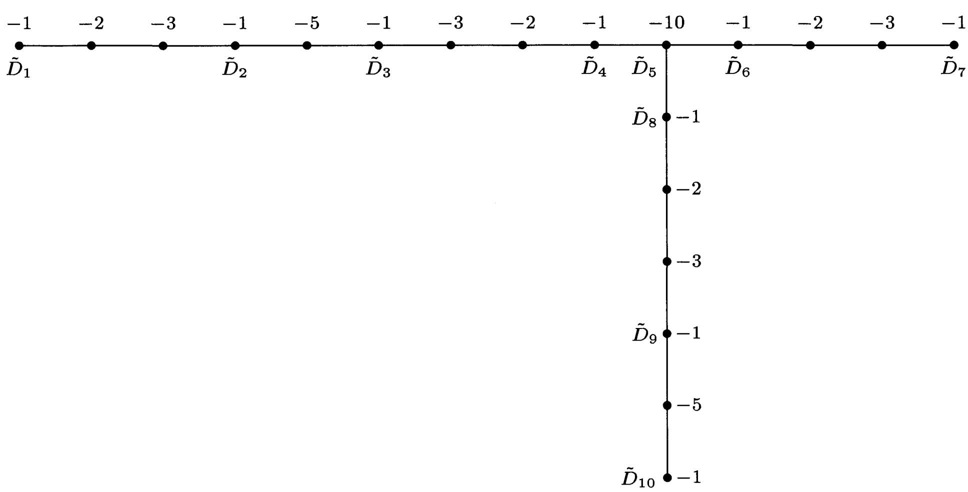


We have $(\tilde{D}_4)^2 = -5$, $(\tilde{D}_5)^2 = -1$, $(\tilde{D}_8)^2 = -7$ and all the other irreducible components are (-2)-curves. It can be seen that $\tilde{K} = 3\tilde{D}_4 + 18\tilde{D}_5 + 2\tilde{D}_8 + \cdots$ and hence $(\tilde{K})^2 = 1$. Also $e(\tilde{X}) = 11$ which implies that $\chi(\tilde{X}) = 1$ whence $p_g(\tilde{X}) = 0$ and this is a contradiction. Hence this tree does not occur.

Tree 9: Consider the 5-fold ramified cover $f : \overline{X} \to X$ given by

$$5K \sim 4D_1 + 8D_2 + 12D_3 + 16D_4 + 20D_5 + 11D_6 + 2D_7 + 13D_8 + 6D_9 - D_{10}.$$

The inverse image of Δ after resolution of singularities is as follows:

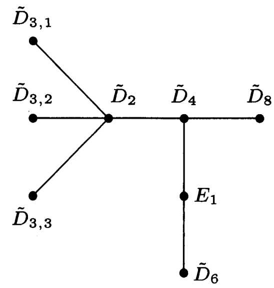


where $\tilde{D}$ denotes the proper transform of D_j. Blowing down all the (-1)-curves successively we see that there are two adjacent (-1)-curves and this contradiction shows that this tree cannot occur.

Tree 10: Consider the 3-fold ramified cover $f : \overline{X} \to X$ given by

$$3K \sim D_1 + 6D_2 + 3D_3 + 8D_4 + 4D_5 - D_7 + 6D_8 + 4D_9 + 2D_{10}.$$

MNC-model of the inverse image of Δ after resolution of singularities is as follows:

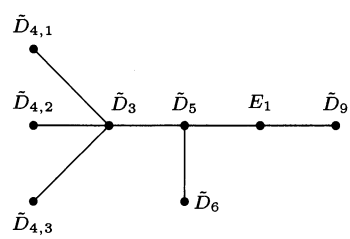


We see that $(\tilde{D}_2)^2 = -5$, $(\tilde{D}_4)^2 = -1$, $(\tilde{D}_6)^2 = -7$, $(\tilde{D}_8)^2 = -4$ and all the other irreducible components are (-2)-curves. It can be seen that $\tilde{K} \sim 2\tilde{D}_2 + 10\tilde{D}_4 + 2\tilde{D}_8 + \cdots$ and hence $(\tilde{K})^2 = 0$. Also $e(\tilde{X}) = 12$. But then $\chi(\tilde{X}) = 1$ whence $p_g(\tilde{X}) = 0$ and this is a contradiction. This shows that this tree does not occur.

Tree 11: Consider the 3-fold ramified cover $f : \overline{X} \to X$ given by

$$3(3K) \sim 8D_1 + 16D_2 + 24D_3 + 12D_4 + 20D_5 + 3D_6 - 2D_7 + 13D_8 + 6D_9 - D_{10}.$$

MNC-model of the inverse image of Δ after resolution of singularities is as follows:

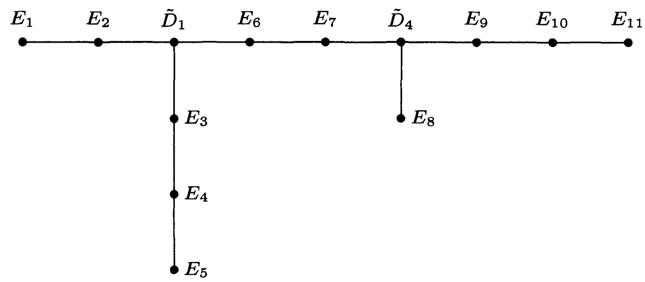


We see that $(\tilde{D}_3)^2 = (\tilde{D}_9)^2 = -4$, $(\tilde{D}_5)^2 = -1$, $(\tilde{D}_6)^2 = -8$ and all the other irreducible components are (-2)-curves. It can be seen that $\tilde{K} \sim \frac{12}{9}(\tilde{D}_{4,1}\tilde{D}_{4,2}\tilde{D}_{4,3}) + \frac{24}{9}\tilde{D}_3 + (\frac{60}{9} + 2)\tilde{D}_5 + \frac{3}{9}\tilde{D}_6 + \frac{6}{9}\tilde{D}_9 + \cdots$ and hence $(\tilde{K})^2 = 0$. Also $e(\tilde{X}) = 12$. But then $\chi(\tilde{X}) = 1$. We see that Lefschetz's fixed point formula gives a contradiction for this tree and hence it cannot occur.

Tree 12: Consider the 11-fold ramified cover $f : \overline{X} \to X$ given by

$$11K \sim 38D_1 + 4D_2 + 9D_3 + 14D_4 + 8D_5 + 2D_6 - 4D_7 - 3D_8 - 2D_9 - D_{10} + 7D_{11}.$$

MNC-model of the inverse image of Δ after resolution of singularities is as follows:

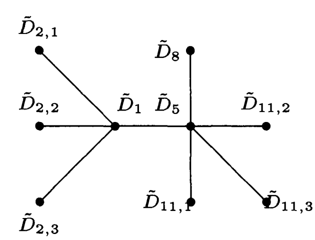


We see that $(E_7)^2 = -6$, $(\tilde{D}_4)^2 = -1$, $(E_9)^2 = -7$ and all the other components are (-2)-curves. It can be seen that $\tilde{K} \sim 48\tilde{D}_1 + 28E_6 + 8E_7 + 24\tilde{D}_4 + 3E_9 + 2E_{11} + E_{11} + \cdots$ and hence $(\tilde{K})^2 = 23$. Also $e(\tilde{X}) = 25$ and hence $\chi(\tilde{X}) = 4$. We see that Lefschetz's fixed point formula gives a contradiction for this tree and hence it cannot occur.

Tree 13: Consider the 3-fold ramified cover $f : \overline{X} \to X$ given by

$$3(3K) \sim 30D_1 + 3D_2 + 7D_3 + 11D_4 + 12D_5 + 7D_6 + 2D_7 - 3D_8 - 2D_9 - D_{10} + 6D_{11}.$$

MNC-model of the inverse image of Δ after resolution of singularities is as follows:



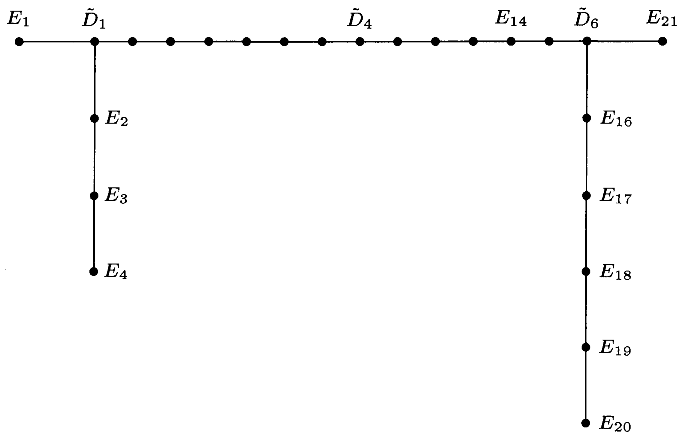
We see that $(\tilde{D}_{2,1})^2 = (\tilde{D}_{2,2})^2 = (\tilde{D}_{2,3})^2 = -4$, $(\tilde{D}_1)^2 = -1$, $(\tilde{D}_8)^2 = -5$, $(\tilde{D}_5)^2 = -3$ and all the other irreducible components are (-2)-curves. It can be seen that $\tilde{K} \sim \frac{1}{3}(\tilde{D}_{4,1} + \tilde{D}_{4,2} + \tilde{D}_{4,3}) + \frac{10}{3}\tilde{D}_1 - \frac{1}{3}\tilde{D}_8 + \frac{4}{3}\tilde{D}_5 + \frac{2}{3}(\tilde{D}_{11,1} + \tilde{D}_{11,2} + \tilde{D}_{11,3})$ and hence $(\tilde{K})^2 = -1$. Also $e(\tilde{X}) = 13$ and hence $\chi(\tilde{X}) = 1$. We see that Lefschetz's fixed point formula gives a contradiction for this tree and hence it cannot occur.

Tree 14: As in the case of tree 1, we see that the degree 6 map $\phi : D_2 \to \mathbb{P}^1$ violates the Riemann-Hurewitz formula. Hence this tree does not occur.

Tree 15: Consider the 7-fold ramified cover $f : \overline{X} \to X$ given by

$$7K \sim 22D_1 + 2D_2 + 5D_3 + 8D_4 + 9D_5 + 10D_6 + 6D_7 + 2D_8 - 2D_9 - D_{10} + 5D_{11}.$$

MNC-model of the inverse image of Δ after resolution of singularities is as follows:

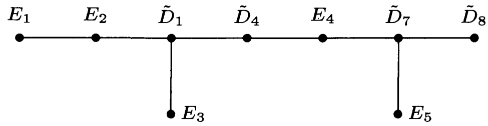


We see that $(E_1)^2 = (E_{14})^2 = (E_{20})^2 = -3$ and all the other irreducible components are (-2)-curves. It can be seen that $\tilde{K} = 9E_1 + 6E_{14} + E_{20} + \cdots$ which implies that $(\tilde{K})^2 = 16$. Also $e(\tilde{X}) = 32$. Thus $\chi(\tilde{X}) = 4$. We see that Lefschetz's fixed point formula gives a contradiction for this tree and hence it cannot occur.

Tree 16: Consider the 5-fold ramified cover $f : \overline{X} \to X$ given by

$$5K \sim 14D_1 + D_2 + 3D_3 + 5D_4 + 6D_5 + 7D_6 + 8D_7 + 5D_8 + 2D_9 - D_{10} + 4D_{11}.$$

MNC-model of the inverse image of Δ after resolution of singularities is as follows:

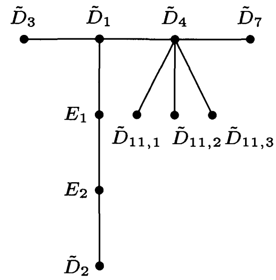


We see that $(\tilde{D}_1)^2 = (\tilde{D}_7)^2 = -1$, $(E_3)^2 = -4$, $(\tilde{D}_4)^2 = -12$, $(E_4)^2 = -3$, $(\tilde{D}_8)^2 = -7$ and all the other components are (-2)-curves. It can be seen that $\tilde{K} \sim 18\tilde{D}_1 + 4E_3 + \tilde{D}_4 + 4E_4 + 12\tilde{D}_7 + \tilde{D}_8 + \cdots$ and hence $(\tilde{K})^2 = -3$. Also $e(\tilde{X}) = 15$. Thus $\chi(\tilde{X}) = 1$ whence $p_g(\tilde{X}) = 0$ which is a contradiction. Hence this tree does not occur.

Tree 17: In this case we consider the 3-fold cover given by

$$3(2K) \sim 16D_1 + D_2 + 3D_3 + 6D_4 + 5D_5 + 4D_6 + 3D_7 + 2D_8 + D_9 - D_{10} + 3D_{11}$$

MNC-model of the inverse image of Δ after resolution of singularities is as follows:



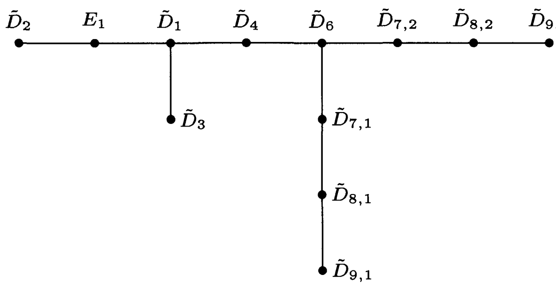
We see that $(\tilde{D}_1)^2 = -1$, $(\tilde{D}_3)^2 = -8$, $(\tilde{D}_4)^2 = -7$ and all the other components are (-2)-curves. It can be seen that $\tilde{K} \sim 10\tilde{D}_1 + \frac{1}{2}\tilde{D}_3 + \tilde{D}_4 + \cdots$ which implies that $(\tilde{K})^2 = -2$. Also $e(\tilde{X}) = 14$. But then $\chi(\tilde{X}) = 1$. We see that Lefschetz's fixed point formula gives a contradiction for this tree and hence it cannot occur.

Tree 18: As in the case of tree 1, we see that the degree 6 map $\phi : D_3 \to \mathbb{P}^1$ violates the Riemann-Hurewitz formula. Hence this tree does not occur.

Tree 19: Consider the 2-fold ramified cover $f : \overline{X} \to X$ given by

$$2(4K) \sim 23D_1 + 5D_2 + 2D_3 + 8D_4 + 9D_5 + 10D_6 + 6D_7 + 2D_8 - 2D_9 + D_{10} + 5D_{11}.$$

MNC-model of the inverse image of Δ after resolution of singularities is as follows:

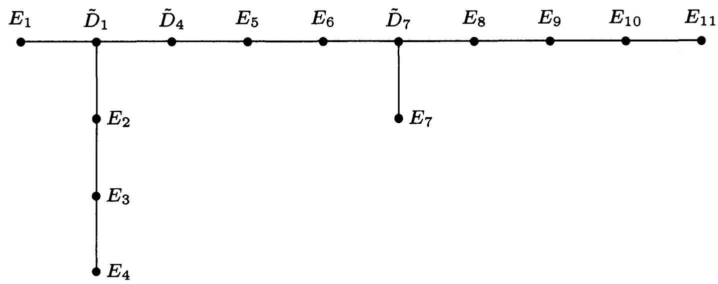


We see that $(\tilde{D}_1)^2 = -1$, $(\tilde{D}_3)^2 = -7$, $(\tilde{D}_{9,1})^2 = (\tilde{D}_{9,2})^2 = -3$, $(\tilde{D}_4)^2 = -5$ and all the other components are (-2)-curves. It can be seen that $\tilde{K} = (\frac{23}{8}.2 + 1)\tilde{D}_1 + \frac{2}{8}\tilde{D}_3 + \tilde{D}_4 - \frac{2}{8}(\tilde{D}_{9,1} + \tilde{D}_{9,2}) + \cdots$ which implies that $(\tilde{K})^2 = -3$. Also $e(\tilde{X}) = 15$. Thus $\chi(\tilde{X}) = 1$ We see that Lefschetz's fixed point formula gives a contradiction for this tree and hence it cannot occur.

Tree 20: Consider the 7-fold ramified cover $f : \overline{X} \to X$ given by

$$7K \sim 22D_1 + 2D_2 + 6D_3 + 7D_4 + 6D_5 + 5D_6 + 4D_7 + D_8 - 2D_9 + 3D_{10} + 2D_{11}.$$

MNC-model of the inverse image of Δ after resolution of singularities is as follows:

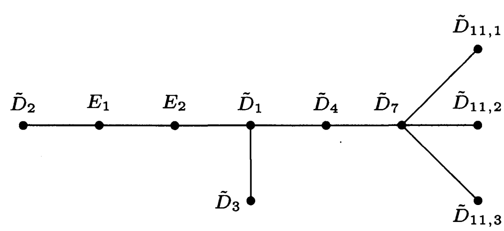


We see that $(E_1)^2 = -5$, $(\tilde{D}_1)^2 = -1$, $(\tilde{D}_4)^2 = -17$ and all the other components are (-2)-curves. It can be seen that $\tilde{K} = 5E_1 + 28\tilde{D}_1 + \tilde{D}_4 + \cdots$ which implies that $(\tilde{K})^2 = 2$. Also $e(\tilde{X}) = 22$. Thus $\chi(\tilde{X}) = 2$. We see that Lefschetz's fixed point formula gives a contradiction for this tree and hence it cannot occur.

Tree 21: Consider the 3-fold ramified cover $f : \overline{X} \to X$ given by

$$3(2K) \sim 16D_1 + D_2 + 3D_3 + 6D_4 + 8D_5 + 10D_6 + 12D_7 + 8D_8 + 4D_9 - D_{10} + 6D_{11}.$$

MNC-model of the inverse image of Δ after resolution of singularities is as follows:

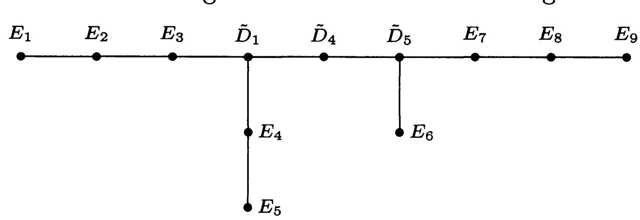


We see that $(\tilde{D}_1)^2 = -1$, $(\tilde{D}_3)^2 = -8$, $(\tilde{D}_4)^2 = -7$ and all the other components are (-2)-curves. It can be seen that $\tilde{K} = (\frac{16}{6}.3 + 2)\tilde{D}_1 + \frac{3}{6}\tilde{D}_3 + \tilde{D}_4 + \cdots$ which implies that $(\tilde{K})^2 = -2$. Also $e(\tilde{X}) = 14$. Thus $\chi(\tilde{X}) = 1$. We see that Lefschetz's fixed point formula gives a contradiction for this tree and hence it cannot occur.

Tree 22: Consider the 11-fold ramified cover $f : \overline{X} \to X$ given by

$$11K \sim 32D_1 + 7D_2 + 3D_3 + 11D_4 + 12D_5 + 7D_6 + 2D_7 - 3D_8 + 2D_9 + D_{10} + 6D_{11}.$$

MNC-model of the inverse image of Δ after resolution of singularities is as follows:

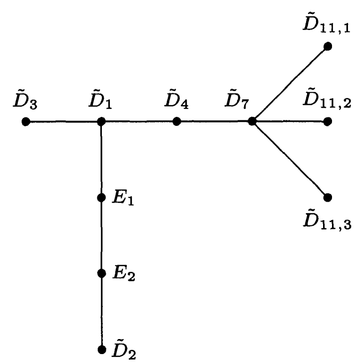


We see that $(E_3)^2 = -4$, $(\tilde{D}_1)^2 = (\tilde{D}_5)^2 = -1$, $(\tilde{D}_4)^2 = -33$, $(E_7)^2 = -3$ and all the other components are (-2)-curves. It can be seen that $\tilde{K} = 12E_3 + 42\tilde{D}_1 + \tilde{D}_4 + 22\tilde{D}_5 + 9E_7 + \cdots$ which implies that $(\tilde{K})^2 = 0$. Also $e(\tilde{X}) = 24$. Thus $\chi(\tilde{X}) = 2$. We see that Lefschetz's fixed point formula gives a contradiction for this tree and hence it cannot occur.

Tree 23: Consider the 3-fold ramified cover $f : \overline{X} \to X$ given by

$$3K \sim 10D_1 + D_2 + 3D_3 + 3D_4 + 2D_5 + D_6 - D_8 + 2D_9 + D_{10}.$$

MNC-model of the inverse image of Δ after resolution of singularities is as follows:

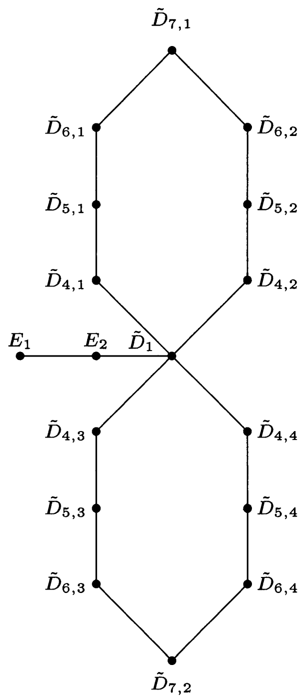


We see that $(\tilde{D}_3)^2 = (\tilde{D}_4)^2 = -7$, $(\tilde{D}_1)^2 = -1$ and $(\tilde{D}_7)^2 = -3$. It can be seen that $\tilde{K} = \tilde{D}_3 + 12\tilde{D}_1 + \tilde{D}_4 + \cdots$ which implies that $(\tilde{K})^2 = -2$. Also $e(\tilde{X}) = 14$. Thus $\chi(\tilde{X}) = 1$ whence $p_g(\tilde{X}) = 0$ which is a contradiction and hence this tree does not occur.

Tree 24: Consider the 4-fold ramified cover $f : \overline{X} \to X$ given by

$$4K \sim 12D_1 + D_2 + 3D_3 + 4D_4 + 4D_5 + 4D_6 + 4D_7 + 2D_8 + D_9 - D_{10} + 2D_{11}.$$

MNC-model of the inverse image of Δ after resolution of singularities is as follows:

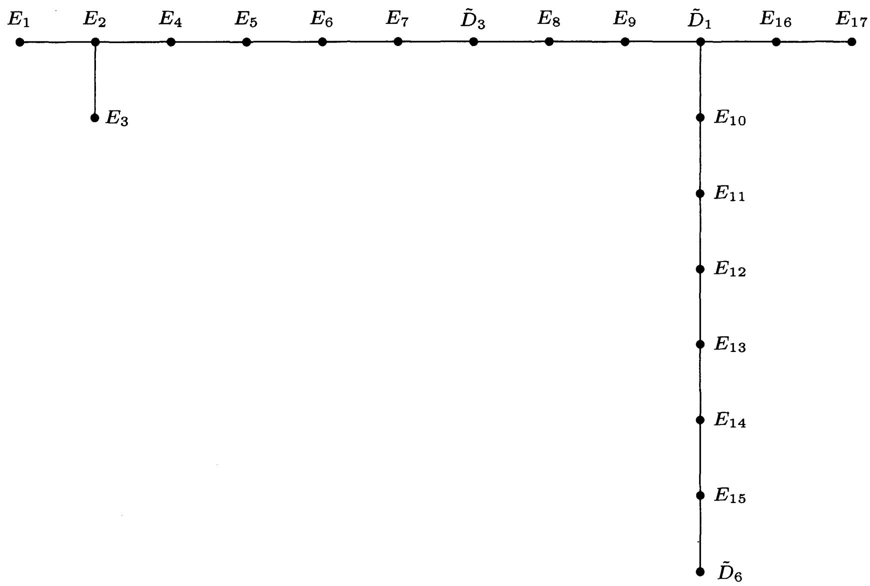


We see that $(\tilde{D}_{4,1})^2 = (\tilde{D}_{4,2})^2 = (\tilde{D}_{4,3})^2 = (\tilde{D}_{4,4})^2 = -3$ and all the other components are (-2)-curves. It can be seen that $\tilde{K} \sim \tilde{D}_{4,1}+\tilde{D}_{4,2}+\tilde{D}_{4,3}+\tilde{D}_{4,4}+\cdots$ which implies that $(\tilde{K})^2 = 4$. Also $e(\tilde{X}) = 20$. Thus $\chi(\tilde{X}) = 2$. We see that Lefschetz's fixed point formula gives a contradiction for this tree and hence it cannot occur.

Tree 25: Consider the 5-fold ramified cover $f : \overline{X} \to X$ given by

$$5K \sim 17D_1 + 4D_2 + 6D_3 + 2D_4 + D_5 - D_7 - 2D_8 + 6D_9 + 3D_{10} + 3D_{11}.$$

MNC-model of the inverse image of Δ after resolution of singularities is as follows:

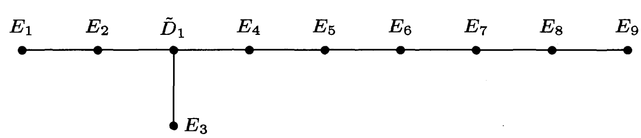


We see that $(E_9)^2 = -3$, $(\tilde{D}_6)^2 = -5$ and all the other components are (-2)-curves. It can be seen that $\tilde{K} = 21\tilde{D}_1 + 3\tilde{D}_6 + 10E_9 + \cdots$ and hence $(\tilde{K})^2 = 10$. Also $e(\tilde{X}) = 26$. This implies that $\chi(\tilde{X}) = 3$. We see that Lefschetz's fixed point formula gives a contradiction for this tree and hence it cannot occur.

Tree 26: Consider the 9-fold ramified cover $f : \overline{X} \to X$ given by

$$9K \sim 34D_1 + 4D_2 + 10D_3 + 11D_4 + 8D_5 + 5D_6 + 2D_7 - D_8 - 4D_9 - 2D_{10} + 5D_{11}.$$

MNC-model of the inverse image of Δ after resolution of singularities is as follows:

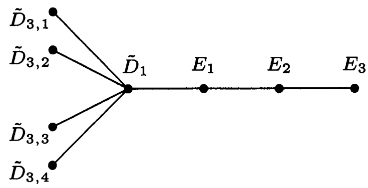


$(E_1)^2 = -3$, $(\tilde{D}_1)^2 = -1$, $(E_3)^2 = -4$, $(E_4)^2 = -7$ and all the other components are (-2)-curves. It can be seen that $\tilde{K} \sim 8E_1 + 42\tilde{D}_1 + 10E_3 + 6E_4 + \cdots$ which implies that $(\tilde{K})^2 = 16$. Also $e(\tilde{X}) = 20$ and hence $\chi(\tilde{X}) = 3$. We see that Lefschetz's fixed point formula gives a contradiction for this tree and hence it cannot occur.

Tree 27: Consider the 8-fold ramified cover $f : \overline{X} \to X$ given by

$$8K \sim 28D_1 + 3D_2 + 8D_3 + 9D_4 + 7D_5 + 5D_6 + 3D_7 + D_8 - D_9 - 3D_{10} + 4D_{11}.$$

MNC-model of the inverse image of Δ after resolution of singularities is as follows:

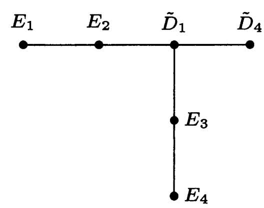


We see that $(\tilde{D}_{3,j})^2 = -5$ for $j = 1,2,3,4$, $(\tilde{D}_1)^2 = -1$, $(E_1)^2 = -3$ and all the other components are (-2)-curves. It can be seen that $\tilde{K} = \tilde{D}_{3,1} + \tilde{D}_{3,2} + \tilde{D}_{3,3} + \tilde{D}_{3,4} + 8\tilde{D}_1 + 3E_1 + \cdots$ which implies that $(\tilde{K})^2 = 7$. Also $e(\tilde{X}) = 17$. Thus $\chi(\tilde{X}) = 2$. We see that Lefschetz's fixed point formula gives a contradiction for this tree and hence it cannot occur.

Tree 28: Consider the 11-fold ramified cover $f : \overline{X} \to X$ given by

$$11K \sim 36D_1 + 4D_2 + 10D_3 + 11D_4 + 8D_5 + 5D_6 + 2D_7 - D_8 - 4D_9 + 2D_{10} + 5D_{11}.$$

MNC-model of the inverse image of Δ after resolution of singularities is as follows:

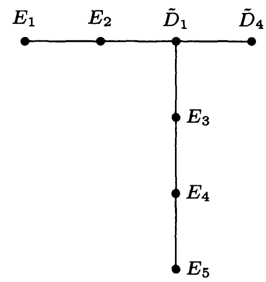


We see that $(E_1)^2 = -4$, $(\tilde{D}_1)^2 = -1$, $(E_3)^2 = -3$, $(\tilde{D}_4)^2 = -24$ and all the other components are (-2)-curves. It can be seen that $\tilde{K} = 6E_1 + 26E_2 + 46\tilde{D}_1 + 18E_3 + 9E_4 + \tilde{D}_4 + \cdots$ which implies that $(\tilde{K})^2 = 6$. Also $e(\tilde{X}) = 18$. Thus $\chi(\tilde{X}) = 2$. We see that Lefschetz's fixed point formula gives a contradiction for this tree and hence it cannot occur.

Tree 29: Consider the 11-fold ramified cover $f : \overline{X} \to X$ given by

$$11K \sim 38D_1 + 4D_2 + 12D_3 + 11D_4 + 6D_5 + D_6 - 4D_7 - 2D_8 + 9D_9 + 6D_{10} + 3D_{11}.$$

MNC-model of the inverse image of Δ after resolution of singularities is as follows:

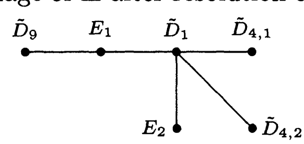


We see that $(E_2)^2 = -5$, $(\tilde{D}_1)^2 = -1$, $(\tilde{D}_4)^2 = -25$ and all the other components are (-2)-curves. It can be seen that $\tilde{K} = 10E_2 + 48\tilde{D}_1 + \tilde{D}_4 + \cdots$ which implies that $(\tilde{K})^2 = 5$. Also $e(\tilde{X}) = 19$. Thus $\chi(\tilde{X}) = 2$. We see that Lefschetz's fixed point formula gives a contradiction for this tree and hence it cannot occur.

Tree 30: Consider the 10-fold ramified cover $f : \overline{X} \to X$ given by

$$10K \sim 32D_1 + 3D_2 + 9D_3 + 10D_4 + 8D_5 + 6D_6 + 4D_7 + 2D_8 + 5D_9 + D_{10} - 3D_{11}.$$

MNC-model of the inverse image of Δ after resolution of singularities is as follows:

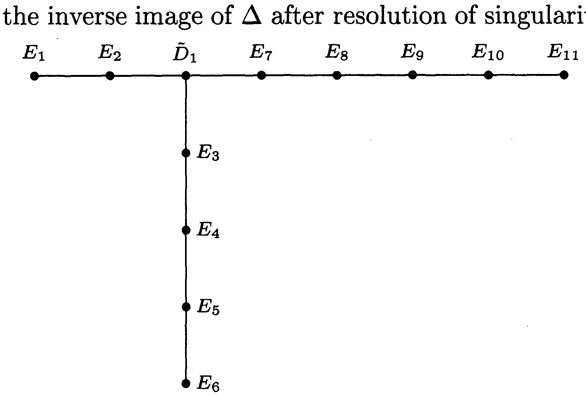


We see that $(\tilde{D}_9)^2 = -3$, $(\tilde{D}_1)^2 = -1$, $(\tilde{D}_{4,1})^2 = (\tilde{D}_{4,2})^2 \doteq -11$, $(E_1)^2 = -3$ and $(E_2)^2 = -2$. It can be seen that $\tilde{K} = 20\tilde{D}_1 + (\tilde{D}_{4,1} + \tilde{D}_{4,2}) + 2\tilde{D}_9 + 7E_1 + 10E_2$ which implies that $(\tilde{K})^2 = 7$. Also $e(\tilde{X}) = 17$. Thus $\chi(\tilde{X}) = 2$. We see that Lefschetz's fixed point formula gives a contradiction for this tree and hence it cannot occur.

Tree 31: Consider the 13-fold ramified cover $f : \overline{X} \to X$ given by

$$13K \sim 48D_1 + 14D_2 + 15D_3 + 6D_4 + 2D_5 - 2D_6 - 6D_7 - 3D_8 + 7D_9 + 10D_{10} + 5D_{11}.$$

MNC-model of the inverse image of Δ after resolution of singularities is as follows:

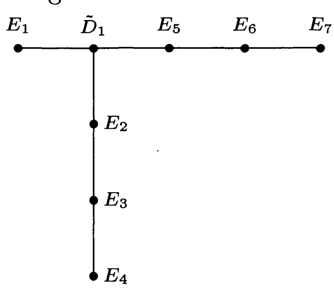


We see that $(E_1)^2 = -4$, $(E_8)^2 = -3$, $(E_{11})^2 = -4$ and all the other components are (-2)-curves. It can be seen that $\tilde{K} = 8E_1 + 60\tilde{D}_1 + 16E_8 + E_{11} + \cdots$ which implies that $(\tilde{K})^2 = 34$. Also $e(\tilde{X}) = 26$. Thus $\chi(\tilde{X}) = 5$. We see that Lefschetz's fixed point formula gives a contradiction for this tree and hence it cannot occur.

Tree 32: Consider the 13-fold ramified cover $f : \overline{X} \to X$ given by

$$13K \sim 40D_1 + 4D_2 + 12D_3 + 11D_4 + 6D_5 + D_6 - 4D_7 + 2D_8 + 9D_9 + 6D_{10} + 3D_{11}.$$

MNC-model of the inverse image of Δ after resolution of singularities is as follows:

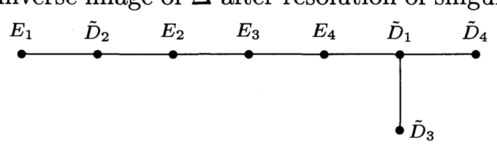


We see that $(E_1)^2 = -9$, $(\tilde{D}_1)^2 = -1$, $(E_4)^2 = -3$, $(E_5)^2 = -6$ and all the other components are (-2)-curves. It can be seen that $\tilde{K} = 5E_1 + 52\tilde{D}_1 + 7E_4 + 9E_5 + \cdots$ which implies that $(\tilde{K})^2 = 26$. Also $e(\tilde{X}) = 22$. Thus $\chi(\tilde{X}) = 4$. We see that Lefschetz's fixed point formula gives a contradiction for this tree and hence it cannot occur.

Tree 33: Consider the 4-fold ramified cover $f : \overline{X} \to X$ given by

$$4K \sim 13D_1 + D_2 + 4D_3 + 4D_4 + 3D_5 + 2D_6 + D_7 - D_8 + 3D_9 + 2D_{10} + D_{11}.$$

MNC-model of the inverse image of Δ after resolution of singularities is as follows:

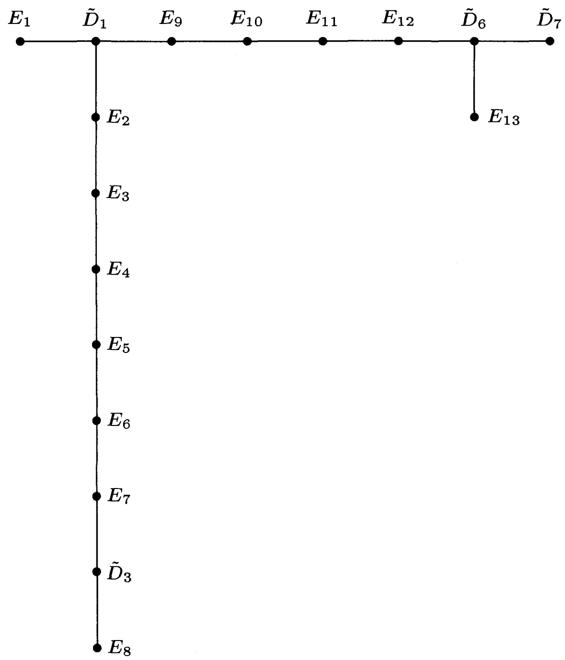


We see that $(\tilde{D}_2)^2 = -2$, $(E_1)^2 = -3$, $(\tilde{D}_1)^2 = -1$, $(\tilde{D}_3)^2 = (\tilde{D}_4)^2 = -9$ and all the other components are (-2)-curves. It can be seen that $\tilde{K} = 16\tilde{D}_1 + \tilde{D}_3 + \tilde{D}_4 + E_1 \cdots$ and hence $(\tilde{K})^2 = -1$. Also $e(\tilde{X}) = 13$. This implies that $\chi(\tilde{X}) = 1$ whence $p_g(\tilde{X}) = 0$ and this is a contradiction. Thus this tree does not occur.

Tree 34: Consider the 7-fold ramified cover $f : \overline{X} \to X$ given by

$$7K \sim 30D_1 + 15D_2 + 2D_3 + 6D_4 + 8D_5 + 10D_6 + 7D_7 + 4D_8 + D_9 - 2D_{10} + D_{11} + 5D_{12}.$$

MNC-model of the inverse image of Δ after resolution of singularities is as follows:

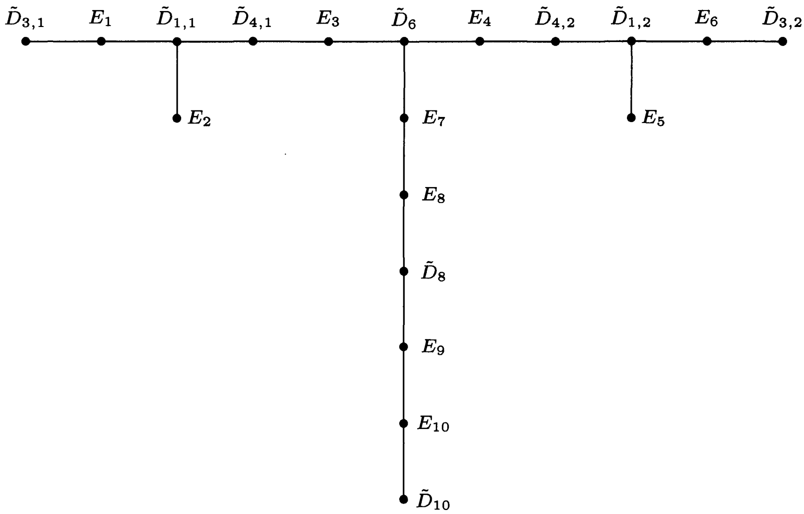


We see that $(E_{10})^2 = (E_{12})^2 = -3$, $(E_{11})^2 = -4$, $(\tilde{D}_6)^2 = -1$, $(\tilde{D}_7)^2 = -9$ and all the other components are (-2)-curves. It can be seen that $\tilde{K} = 36\tilde{D}_1 + 8E_{10} + 3E_{11} + 6E_{12} + 16\tilde{D}_6 + \tilde{D}_7 \cdots$ and hence $(\tilde{K})^2 = 11$. Also $e(\tilde{X}) = 25$. This implies that $\chi(\tilde{X}) = 3$. We see that Lefschetz's fixed point formula gives a contradiction for this tree and hence it cannot occur.

Tree 35: Consider the 6-fold ramified cover $f : \overline{X} \to X$ given by

$$6K \sim 28D_1 + 14D_2 + 2D_3 + 6D_4 + 8D_5 + 10D_6 + 7D_7 + 4D_8 + D_9 - 2D_{10} - D_{11} + 5D_{12}.$$

MNC-model of the inverse image of Δ after resolution of singularities is as follows:

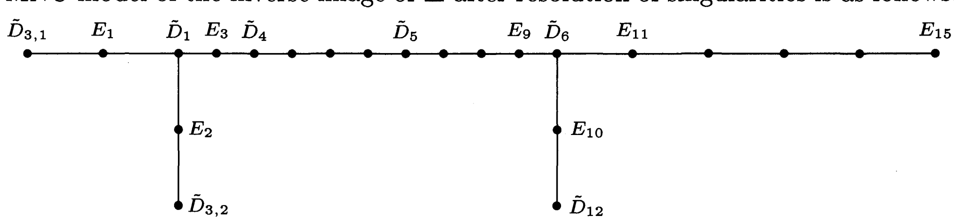


We see that $(\tilde{D}_{1,1})^2 = (\tilde{D}_{1,2})^2 = -1$, $(\tilde{D}_{4,1})^2 = (\tilde{D}_{4,2})^2 = -11$, $(E_1)^2 = (E_6)^2 = -3$ and all the other irreducible components are (-2)-curves. It can be seen that $\tilde{K} = 16(\tilde{D}_{1,1}+\tilde{D}_{1,2})+(\tilde{D}_{4,1}+\tilde{D}_{4,2})+3(\tilde{D}_{3,1}+\tilde{D}_{3,2})+6(E_1+E_6)+\cdots$ which implies that $(\tilde{K})^2 = -2$. Also $e(\tilde{X}) = 26$. But then $\chi(\tilde{X}) = 2$. We see that Lefschetz's fixed point formula gives a contradiction for this tree and hence it cannot occur.

Tree 36: Consider the 4-fold ramified cover $f : \overline{X} \to X$ given by

$4K \sim 22D_1+11D_2+6D_3+D_4+5D_5+9D_6+7D_7+5D_8+3D_9+D_{10}-D_{11}+6D_{12}+3D_{13}$.

MNC-model of the inverse image of Δ after resolution of singularities is as follows:

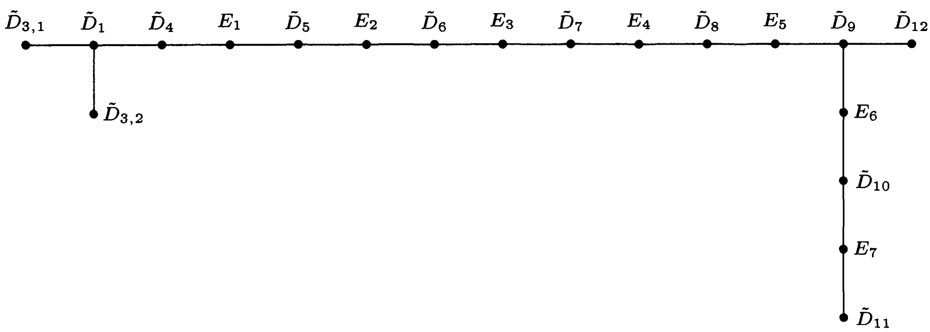


We see that $(\tilde{D}_4)^2 = (E_{11})^2 = -3$ and all the other components are (-2)-curves. It can be seen that $\tilde{K} = 4\tilde{D}_4 + 5E_{11} + \cdots$ which implies that $(\tilde{K})^2 = 9$. Also $e(\tilde{X}) = 27$. But then $\chi(\tilde{X}) = 3$. We see that Lefschetz's fixed point formula gives a contradiction for this tree and hence it cannot occur.

Tree 37: Consider the 2-fold ramified cover $f : \overline{X} \to X$ given by

$2K \sim 2D_1 + D_2 - D_4 + D_5 + 3D_6 + 5D_7 + 7D_8 + 9D_9 + 5D_{10} + D_{11} + 6D_{12} + 3D_{13}$.

MNC-model of the inverse image of Δ after resolution of singularities is as follows:



Here we see that $(\tilde{D}_1)^2 = -1$, $(\tilde{D}_{3,1})^2 = (\tilde{D}_{3,2})^2 = (\tilde{D}_{12})^2 = -3$, $(\tilde{D}_4)^2 = -4$ and all the other components are (-2)-curves. It can be seen that $\tilde{K} \sim \tilde{D}_1 + 3\tilde{D}_{12} + \cdots$ and hence $(\tilde{K})^2 = 2$. Also $e(\tilde{X}) = 22$. But then $\chi(\tilde{X}) = 2$. We see that Lefschetz's fixed point formula gives a contradiction for this tree and hence it cannot occur.

This completes the proof of our main Theorem.

6. Proofs of the Corollaries

Proof of Corollary 1 (cf. Introduction). Let V be a $\mathbb{Q}$-homology plane. By our main result V is rational. M. P. Murthy has shown that on a smooth affine rational surface any algebraic vector bundle splits as a direct sum of a trivial bundle and a line bundle (cf. [10]).

Proof of Corollary 2. Let W be a smooth affine 3-fold with trivial reduced rational homology and with a non-trivial action of $\mathbb{C}^\star$. Let $V := W//\mathbb{C}^\star$ be the normal affine variety corresponding to the ring of invariants $\Gamma(W)^{\mathbb{C}^\star}$ and let $\pi : W \to V$ be the quotient morphism. The rationality of W is proved by first proving the rationality of V and then analysing the fibers of π.
Let $S := W^{\mathbb{C}^\star}$ be the fixed point set. By III.10 of [3], $S \neq \phi$.
Case 1. dim $V = 0$.
In this case Luna has proved that $W \cong \mathbb{C}^3$ and $\mathbb{C}^\star$ acts linearly on W (cf. section 3, corollary 2, [9]).
Case 2. dim $V = 1$.
Any fiber F of π has dimension 2. It is well-known that F contains a unique closed orbit, say O_F, and any other orbit contained in F has exactly one point in its closure which lies in O_F. It follows that O_F is a single point. Thus any closed orbit in W is a point. In this situation Kraft and G.Schwarz have proved that

$W \cong \mathbb{C} \times V$ (cf. [8]). Hence V also has trivial reduced rational homology. By our main result V is rational and hence so is W.

Case 3. dim $V = 2$.

Theorem B of [7] proves that V is a (possibly singular) $\mathbb{Q}$-homology plane. By the main result in part I and the main result in parts II,III, we know that V is rational.

If a general fiber of π is a closed orbit, then for a Zariski open subset $U \subset V$, $\pi^{-1}(U) \cong U \times \mathbb{C}^\star$. Hence W is rational.

Suppose that a general fiber of π is not a closed orbit. Then a general fiber is isomorphic to $\mathbb{C}$. The fixed point set S is a section of π over a Zariski open subset of V. Again, $W - S$ has a Zariski open subset which is a trivial $\mathbb{C}^\star$-bundle over a Zariski open subset of V. Hence W is rational.

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