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A NUMERICAL SCHEME USING ITÔ EXCURSIONS FOR SIMULATING LOCAL TIME RESP. STOCHASTIC DIFFERENTIAL EQUATIONS WITH REFLECTION

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1. Introduction

Simulation of stochastic differential equations or numerical evaluation of their functionals is an important issue in physics, chemical and engineering problems. For diffusion without boundary the problem is investigated very well (see for example the book of Kloeden and Platen [14]). The simplest method is the Euler scheme, where we can go back on many results (see for example the articles of Bally and Talay [2] and Bouleau and Lèpingle [4]). If reflection is concerned, most methods have more or less shortcomings. In this paper we suggest another ansatz of numerical schemes for solving a stochastic differential equation directly including the boundary condition with instantaneous reflection. The idea of this approach is to approximate the underlying Poisson point process arising by cutting the diffusion at the level set of the boundary and parametrizing by the local time these excursions. Additionly, this methods is easily to implement on computers. First, we give a description, second we give a proof of convergence. The last chapter deals with the rate of convergence.

Thus, we are interested in numerical schemes for diffusion with boundary, the so called Skorohod problem. Such results have applications to PDE's with boundary conditions of Neumann or Wentzell type. For ∂D being a hyperplane, i.e. $D = \{x \in \mathbb{R}^n \mid x_0 \geq 0\}$ and $\partial D = \{x \in \mathbb{R}^n \mid x_0 = 0\}$ the problem has been investigated since the early sixties by many authors (just to mention a few: El Karoui [9], Ikeda and Watanabe [11], Lions and Sznitzman [17], McKean [19] and Skorohod [28]).

Now, we are interested in a scheme for approximating the local time which is based on the Euler scheme. The work of Costantini, Pacchierotti and Sartoretto [8] describes an approximation scheme for functionals of reflected diffusion processes by approximating the boundary, which is based on the articles of Smólinski [29], [30] and Costantini [7], respectively Saisho [25]. Furthermore Lépingle has designed in [16] and an algorithm for simulating the reflected Brownian motion. He improved the suggestion of [29] by introducing an exponential distributed random variable. Both algorithm approximate the local time by approximating the boundary. Further, we have to cite

Menaldi [20], where a penalization method is used.

In this paper we suggest an algorithm for simulating the local time, i.e. simulating reflected diffusion. In contrast to well known penalty methods we give a completely new approach, which avoids some of the shortcomings of penalty methods. The proof is given only in one dimension, where we continue with the proof of Blumenthal, who has shown the convergence of algorithm 2. Further, we give the rate of convergence.

1.1. Description of the algorithm: We can assume that the path is approximated the Euler scheme. Now, the question which arises: is there a scheme involving the local time? Here, we confine ourselves to a martingal arising by stochastic integration due to Brownian motion, i.e. a process given by

$$dX_t = \sigma(X_t)\, dB_t, \tag{1}$$

where $\sigma(x) : \mathbb{R} \to \mathbb{R}$ is bounded away from zero, Lipschitz continuous and satisfies the growth condition $|\sigma(x)| < C(1 + |x|)$ for some $0 < C < \infty$. The boundary of the domain $D \subset \mathbb{R}$ is to be taken a singleton, i.e. $\partial D = \{d\}$ and $D = \{x \in \mathbb{R}, x > d\}$. Further, the starting point is given by $X_0 = x$ and X_t^n denotes the process approximated by the Euler scheme with time step size $1/n$. Let $\tau^n = \inf\{t > 0 \mid X_t^n \notin \{x > d\}\}$ the discretized version of the first exit time and $\eta_t^B(x)$ defined by

$$\eta_t^B(A) = \int_A 1/\sqrt{t^3\pi}xe^{-x^2/2t}\, dx.$$

The total mass of η^B at time t is $\|\eta_t^B\| = \eta_t((0,\infty)) = \int_0^\infty 1/\sqrt{t^3\pi}xe^{-x^2/2t}dx = 1/\sqrt{t\pi}$.

Algorithm 1. Fix $n \in \mathbb{N}$ and $\epsilon > 0$. ϵ should not be smaller than $1/n$.

- **Step 1:** $X_0 = x$. Now the process will be simulated by the Euler scheme until it hits the boundary, i.e. if $t = i/n$ for a i, $\hat{X}_t^n$ is computed by

$$\hat{X}_{i/n}^n = X_0 + \sum_{j=1}^{i} \sigma(\hat{X}_{(j-1)/n}^n)\left(B_{j/n} - B_{(j-1)/n}\right) \quad \text{if} \quad t < \tau^n,$$

  where $B_{j/n} - B_{(j-1)/n}$ are identical distributed Gaussian random variables with expectation zero and variance $1/n$. We take as first exit time $\tau^n = \inf\{i/n \mid \exists t \le i/n \text{ with } X_t^n \notin D\}$. Since for arbitrary $(i-1)/n < t < i/n$ it is assumed that the process is generated by the operator at time $(i-1)/n$, the event 'hitting d' happens either if $\{\hat{x}_{i/n}^n < d\}$ or if $\{(i-1)/n < \tau^n \le i/n, \hat{x}_{i/n}^n > d\}$ happens. The second event is simulated by tossing a coin at each time step with probability

$$\mathbb{P}(head) = \exp\left(-n\frac{(x-d)^2}{\|\sigma(d)\|^2}\right)$$

If X_t^n hits the boundary, we continue with Step 2.

- **Step 2:** Let $\sigma = \sigma(d)$. The increment of the local time $\Delta L = t$, where t is exponential distributed with holding parameter $\hat{\beta}^\epsilon = \mathbb{E}[\|\eta_\epsilon^B\|]/\sigma = \sqrt{1/(\epsilon\pi)}1/\sigma$. Hence the original process X_τ itselfs goes on excursion which takes shorter than ϵ, the time t increases as well while $\hat{X}_\tau$ remains at d. The difference is the expectation of the sum over the lengths. Thus, we have to increase the time by $\Delta t = \int_0^\epsilon t\, d\|\mu_t\|\Delta L = \sqrt{\epsilon/\pi}\Delta L$.
- **Step 3:** At the first time step $\hat{X}^n_{\epsilon+\tau^n+\Delta t}$ is approximated by

$$\mathbb{P}\left(\hat{X}_{\tau^n+\epsilon} - d = x\sigma\right) = \frac{\eta_\epsilon^B(x)}{\|\eta_\epsilon^B\|}.$$

- **Step 4:** We continue at Step 1 with starting point $\hat{X}_{\tau^n+\epsilon+\Delta t}$.

REMARK 1.1. Generalization by including a drift term can be easily achieved by a simple modification of the entrance law, i.e. $\eta_\epsilon = \eta_\epsilon^B(x - \epsilon b(d))$.

REMARK 1.2. Similarly smooth boundary instead of a singleton can be involved by analog modifications.

REMARK 1.3. Hence the Excursion point process can be extended to a more general domain (see for example exit systems of Maisonneuve [18] or Motoo theory [21]. Burdzy [6] pursued these ideas and extended it to the multidimensional case).

2. Theoretical background, notation and auxiliaries

The idea for approximating the local time[1] is to approximate the underlying Poisson point process, where the local time corresponds to the parameter and the real time or the time the process X_t passes through corresponds to the sum over the time intervals, where X_t is removed from the boundary. A good introduction to this theory are the books Blumenthal [3], Burdzy [6] and Rogers and Williams [15] and the articles of Motoo [21], Salisburg [26], [27], Watanabe [32] and Roger [24].

Let $\tau = \inf_{t>0}\{X_t \in \partial D\}$. Now, roughly speaking, we can split a path $X_t(\omega)$ into pieces by cutting at the points $\{X_t(\omega) \in \partial D\}$. Each piece is a process starting somewhere at the boundary and being killed upon reaching the boundary again. Such a piece is called excursion. The set of excursion is denoted by $\mathcal{U}$ and the space $\mathcal{U}$ is the set of all continuous functions $e_t : (0,\infty) \mapsto \{x > d\}$, such that $\lim_{t\to 0} e_t 0d$, and

[1] The local time is here defined as the unique continuous additive functional L_t which satisfies $E^x[e^{-\tau}] = E^x[\int_0^\infty e{-}\lambda t\, dL_t]$, $\tau = \inf_{t>0}\{X_t = d\}$.

$e_t = d$ for $t > \tau$. The function e_t is called excursion.

The set of excursions is equipped by an excursion law and the corresponding filtration. The excursion law is strong Makov in regard to the transition probability of X_t. Hence if x is regular, $\mathbb{P}^x(\tau > 0) = 0$, the excursion law has to be seen as 'a limit of properly renormalized distributions of diffusion in $\{x > d\}$ starting at y where $y \to x$' (see Burdzy [5]). The nonrenormalized version coincides with the entrance law. For more details, please see Burdzy [6, p.19] or Blumenthal [3, p.102].

The idea is now using local time to parametrize the excursion. For covering the case X_t spends a positive real time at the boundary, we create an empty path or graveyard δ, defined by $\delta_t = d$ and extend $\mathcal{U}$ by δ. Let X_t be a Markov process. Thus, to every Markov process corresponds a so called excursion process, i.e. a set $\mathcal{U}$ and a point process $e = (e_s, s > 0)$ defined by

$$\begin{aligned} e : [0,\infty) &\to \mathcal{U} \cup \delta \\ e_t(s) \; &= \; \begin{cases} X_{L_{s-}^{-1}+t} & \text{if} \quad L_s^{-1} - L_{s-}^{-1} > 0 \\ \delta & \text{elsewhere.} \end{cases} \end{aligned}$$

It follows $e \mapsto e_t(s) \neq \delta$ for only countable many s. Further, one excursion starts, where the last were stopped, i.e. $e_0(s) = e_\tau(r_0)$, where $r_0 = \sup_{r<s}\{e(r) \neq \delta\}$. The parameter s turns out to be the local time of the boundary. If we sum up the lengths of the excursion,

$$T_s = \sum_{u \leq s} \tau(e(u)),$$

we get the time, the process passed through, i.e. $L_s^{-1} = T_s$ or $L_{T_s} = s$. It follows, that T_t is a Poisson point process. We denote the associated Lévy measure by $\nu([t,\infty)) = \mathbb{P}(\tau > t) = ||\eta_t||$.

Now, we can reverse this construction by starting with a set of excursions and an excursion process and linking together the excursions to get a Markov process. By doing these, our time parameter passes through the local time and the real time of the process is given by the subordinator T_s. But first we want to introduce some definitions. Let be $T_s^- = \lim_{u\to s, u<s} T_s$ the left limit point of T_s, D_{e_ω} these points, where $e(s) \neq \delta$, i.e. $D_{e_\omega} = \{s \,|\, T_s^-(\omega) < T_s(\omega)\}$ and $L_t = \inf_{r\geq 0}\{T_r \geq t\} = \inf_{r\geq 0}\{T_r > t\}$ the inverse of T_s. L_t turns out to be the local time of D at time t. Let $t > 0$ and $s = L_t$. We define X_t by:

$$X_t \; = \; \begin{cases} e_{t-T_s^-}(s) & \text{if} \quad T_s^- < T_s \\ e_{\tau(e(r_0))}(r_0) & \text{elsewhere,} \end{cases}$$

where $r_0 = \sup_{r<s}\{e(r) \neq \delta\} = \sup_{r\leq s}\{e(r) \neq \delta\}$. The last equality holds because $e(s) = \delta$.

Since in almost all examples such as Brownian motion the point d of reflection is assumed to be regular, i.e. $\mathbb{P}^d(\tau = 0) = 1$ or the Lèvy measure ν has infinite mass at zero. This means, the number of points in a finite time interval $[0, s]$, where X_t goes on an excursion with length larger than ϵ tends to infinity as ϵ tends to zero, i.e.

$$\sum_{\substack{\tau(e(r))>\epsilon \\ 0\le r\le s}} 1 \to \infty \quad \text{as} \quad \epsilon \searrow 0.$$

But for example in the case of Brownian motion it holds almost surely (see Karatzas and Shreve [13, p.413])

$$\sqrt{\frac{\pi}{2\epsilon}} \sum_{\substack{\tau(e(r))<\epsilon \\ 0\le r\le s}} \tau(e(r)) \to s \quad \text{as} \quad \epsilon \searrow 0.$$

Further, the time interval between two excursions with length $\tau > \epsilon$ are exponential distributed with holding parameter tending to infinity as $\epsilon \searrow 0$. Therefore, to give an explicit construction, we have to approximate X_t for $\epsilon > 0$ by only considering excursion with length $\tau > \epsilon$ (see Blumenthal [3, p.136]). Let e_t be an excursion with $\tau > \epsilon$. At time ϵ the excursion is removed from the boundary. Roughly speaking, it jumps from the boundary into the interior of D. This fact leads to an entrance law. Then, e_t behaves like X_t up until the hitting time τ where it is killed. Given a regular process X_t, this killed version is called minimal process.

DEFINITION 2.1. The minimal (killed, taboo) process $\bar{X}_t$ of the process X_t is the semigroup, killed upon reaching the first time the boundary.

DEFINITION 2.2. A family $\{\eta_s; s > 0\}$ of measures on the Borel sets $\{x > d\}$ is called an entrance law for the semigroup $\{\bar{X}_t, \bar{P}_t^0\}$ if

$$\eta_s \bar{P}_t^0 = \eta_{t+s}, \ s > 0, \ t > 0 \quad \text{and} \quad \lim_{t\to 0} X_t = d \quad \eta_0 - \text{a.s.}$$

REMARK 2.1. Considering continuous stochastic processes, the entrance law can also be defined by $\eta_t^B(A) = \mathbb{P}(B_t \in A \wedge t < \tau^B)$.

REMARK 2.2. $\{\eta_t\}$ can be described also as last exit decomposition (Roger and Williams [15, p.417]) $P_t(a, \Gamma) = \mathbb{P}(\exists s \in [0, t] : X_s = a) = \mathbb{E}^a[\int_0^t \eta_{t-s}(\Gamma) dL_s]$.

For the sake of clarity, we will write $\bar{\mathbb{E}}$ instead of $\mathbb{E}$, if we compute the expectation value of a function with respect to the minimal process.

Let $\epsilon > 0$. We give here a description of the algorithm getting a process X_t^ϵ which converges to X_t as $\epsilon \searrow 0$ almost surly. Blumenthal [3, p.139] proved that the resolvent operator converges for all λ uniformly in $\mathcal{C}(\mathbb{R}^m)^2$ and further that the limit is a resolvent of a strongly continuous probability semigroup (Blumenthal [3, Theorem 2-8 p.142]).

Algorithm 2.

- **Step 1:** $s = 0$. $X_0 = x_0$. We pick out an minimal process with starting point x_0.
- **Step 2:** Let $d = X_\tau$ be the point, the excursion is killed. The increment of the local time is $t\Delta L$, where t is exponential distributed with holding parameter $\beta_\epsilon = \|\eta_\epsilon\|$. X_t is assumed to remain at d, thus the time increases also by $\Delta t = t(1 + \int_0^\epsilon s d\|\eta_s^X\|)$.
- **Step 3:** We jump into $\{x > d\}$ distributed by the entrance law η_ϵ, i.e. $\mathbb{P}(x \in A) = \eta_\epsilon(x - d)$. Let x be the point we touched.
- **Step 4:** We continue at Step 1 with starting point x, i.e we pick out a minimal process $e(s)$ starting at x.

If we identify an minimal process with entrance law η by an excursion with $\tau > \epsilon$, we formulate the approximated time T_s^ϵ of T_s by

$$T_s^\epsilon = \sum_{\substack{\tau(e(r))>\epsilon \\ r\leq s}} \tau(e(r)) + (m_\epsilon + o(\epsilon))s,$$

where the delay coefficient m_ϵ (see Blumenthal [3, p.144]) is given by

$$m_\epsilon = 1 - \langle \eta_\epsilon, V_1 1\rangle = 1 - \langle \eta_\epsilon, 1 - e^{-\tau}\rangle = 1 - \int_0^\infty \eta_\epsilon(x)\mathbb{E}^x[1 - e^{-\tau}]dx$$

($\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathcal{L}^2$).

REMARK 2.3. If the Lévy measure has infinite mass at zero m is equal to zero. This is the case for the Brownian motion and therefore for all continuous martingal with non zero quadratic variation.

REMARK 2.4. Hence the excursions are approximated by the Euler scheme on a compact interval, almost surely convergence depends on the convergence of the subordinator T_ξ^ϵ.

2.1. Approximation of entrance law and holding parameter For obtaining the algorithm described in (1), first, we approximate the holding parameter in Step 2 and second we approximate the entrance law in Step 3 of Algorithm 2.

[2] We denote by $\mathcal{C}(S)$ the set of real valued continuous functions on S which vanish at ∞.

The approximation of the holding parameter is implicitly given by the approximation of the entrance law, since β_ϵ is the mass of η at time ϵ. In the algorithm the entrance law η^X is approximated by the entrance law η^B of the Brownian motion. The idea is first to map the path $X_.(\omega)$ onto a Brownian path $B_.(\omega)$ by a random time change and then to consider the entrance law of the corresponding Brownian motion. The time change is given by the theorem of Dambis, Dubins-Schwarz (see Karatzas and Shreve [13] or Revuz and Yor [23]).

Corollary 2.1. *Suppose X_t is the stochastic integral defined before, γ the random time change $\gamma_t = \int_0^t \sigma^2(X_s)ds$ and $\rho_t = \int_0^t \sigma^{-2}(X_s)ds$ its inverse. The stopping times of a regular point x is given by $\tau^B = \inf_{s>0}\{B_s = x\}$ and $\tau^X = \inf_{s>0}\{X_s = x\}$ respectively. Then it holds:*

$$\rho_{\tau^B} = \tau^X \quad \textit{and} \quad \gamma_{\tau^X} = \tau^B.$$

Proof. For the stooping time τ^B of the Brownian motion it holds almost surely $X_{\rho_{\tau^B}} = B_{\tau^B} = x$ and for $t < \tau^B$ B_t is smaller than x. Let $t < \rho_{\tau^B}$. Since $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and non decreasing, hence surjectiv, there exists a s, $0 \leq s < \tau^B$ with $B_s = X_t$. Therefore $B_t \neq x$ and it follows that ρ_{τ^B} is the infimum. □

Considering stochastic processes arising by stochastic integration, the entrance law can also be defined by $\eta_t^B(A) = \mathbb{P}(B_t \in A \wedge t < \tau^B)$. In the sequel we denote by η_t^B the entrance law of the Brownian motion and by η_t^X the entrance law of the stochastic integral X_t. In case of Brownian motion, the entrance law is given by the formula (Blumenthal [3, p.110])

$$\eta_t^B(A) = \int_A \frac{\xi}{\sqrt{t^3\pi}} e^{-\xi^2/2t} d\xi$$

with mass

$$\|\eta_t^B\| = \mathbb{P}(t < \tau^B) = \frac{1}{\sqrt{2t}}.$$

What is now the entrance law of X_t, or how can we express η_t^X in terms of the entrance law of the Brownian motion?

$$\begin{aligned}\eta_t^X(A) &= \mathbb{E}\big[1_A(X_t) \wedge t < \tau^X\big] = \mathbb{E}\big[1_A(X_{\gamma_t}) \wedge \gamma_t < \gamma_{\tau^X}\big] \\ &= \mathbb{E}\big[1_A(B_{\gamma_t}) \wedge \gamma_t < \tau^B\big] = \mathbb{E}\big[\eta_{\gamma_t}^B(A)\big].\end{aligned}$$

The mass is given by

$$\|\eta_t^X\| = \mathbb{E}\big[1_{\{t<\tau^X\}}\big] = \mathbb{E}\big[1_{\{\gamma_t<\gamma_{\tau^X}\}}\big] = \mathbb{E}\big[\|\eta_{\gamma_t}^B\|\big].$$

The entrance law: In our algorithm, first the random time $\gamma_t = \int_0^t \sigma^2(X_s)ds$ at time $t = \epsilon$ is approximated by $\hat{\gamma}_\epsilon = \epsilon\sigma^2$ where $\sigma = \sigma(d)$. Now, this approximation is put in the formula for the entrance law. Further we use the scaling property[3] of the Brownian motion or the normalized entrance law, respectively, to get a better expression. The jump of the excursion at time ϵ is distributed like $p_\epsilon^X(A) = \mathbb{P}(X_\epsilon \in A \mid \epsilon < \tau^X)$ which is given by the normalized entrance law, i.e.

$$p_\epsilon^X(A) = \frac{\eta_\epsilon^X}{\|\eta_\epsilon^X\|} = \mathbb{E}\left[\frac{\eta_{\gamma_\epsilon}^B(A)}{\|\eta_{\gamma_\epsilon}^B\|}\right] = \mathbb{E}\left[\frac{\eta_\epsilon^B(A\sqrt{\epsilon/\gamma_\epsilon})}{\|\eta_\epsilon^B\|}\right].$$

Approximating $\gamma_\epsilon = \int_0^\epsilon \sigma^2(x_s)ds$ by $\sigma^2\epsilon$ leads to the following approximation

$$\hat{p}_\epsilon^X(A) = \frac{\eta_\epsilon^X}{\|\eta_\epsilon^X\|} = \mathbb{E}\left[\frac{\eta_{\epsilon\sigma^2}^B(A)}{\|\eta_{\epsilon\sigma^2}^B\|}\right] = \mathbb{E}\left[\frac{\eta_\epsilon^B(A/\sigma)}{\|\eta_\epsilon^B\|}\right].$$

The holding parameter: Further, the approximation of the holding parameter $\beta_\epsilon^X = \mathbb{P}(\epsilon < \tau^X) = \|\eta_\epsilon^X\|$ is given by

$$\hat{\beta}_\epsilon^X = \|\hat{\eta}_\epsilon^X\| = \|\eta_{\epsilon\sigma^2}^B\| = \frac{1}{\sqrt{2\epsilon\sigma}}.$$

Algorithm 3. This algorithm coincides with Algorithm 2 except the approximation of the holding parameter and entrance law.

- **Step 1:** $s = 0$. $X_0 = x_0$. We pick out an minimal process with with starting point x_0.
- **Step 2:** Let $d = X_\tau$ the point, the excursion is killed. The increment of the local time is $\Delta L = t(1 + \langle \eta_\epsilon, V_1 1\rangle) \simeq t$, where t is exponential distributed with with holding parameter $\hat{\beta}_\epsilon = \|\eta_\epsilon^B\|/\sigma$. $\hat{X}_t$ is assumed to remain at d, thus the time increases also by $\Delta t = t(1 + \sqrt{\epsilon/\pi})$.
- **Step 3:** We jump into $\{x{>}d\}$ distributed by the entrance law $\hat{\eta}_\epsilon{=}\eta_\epsilon^B((x-d)/\sigma)$. Let x be the point we touched.
- **Step 4:** We continue at Step 1 with starting point x, i.e we pick out a minimal process $e(s)$ starting at x.

3. Convergence of the algorithm

In this paragraph we show the convergence of the pair $(\hat{X}_T^n, \hat{L}_T^n)$ for a fix $T < \infty$. The convergence is shown in several steps via Algorithm 2 and Algorithm 3. The last step is to approximate the minimal process by the Euler scheme. First, for the sake of clarity we introduce the following notation:

[3] $\eta_{ct}^B(A) = \eta_t^B(A/c)$, $A/c = \{a/c \mid a \in A\}$.

X_t : the process defined by the stochastic integral $dX_t = \sigma(X_t)\, dB_t$

X_t^ϵ : the process arising by only taking into account the excursion longer than ϵ, i.e. Algorithm 2

$\hat{X}_t^\epsilon$: the process arising by X_t^ϵ by approximating the entrance law and the holding parameter, i.e. Algorithm 3

$\hat{X}_t^{\epsilon,n}$: the process arises by X_t^ϵ by approximating the excursion by the Euler scheme, i.e. Algorithm 1 (here, the path between the grid point is a dif-fusion generated by the coefficients of the last grid points.)

$\hat{\hat{X}}_t^{\epsilon,n}$: the mean square Euler scheme is applied

U_λ, U_λ^ϵ, $\hat{U}_\lambda^\epsilon$, $\hat{U}_\lambda^{\epsilon,n}$ denote the corresponding resolvents, i.e.

$$U_\lambda : \mathcal{C}(D) \to \mathcal{C}(D) \quad \text{with} \quad U_\lambda g(x) = \mathbb{E}^x\left[\int_0^\infty e^{-\lambda r} g(X_r) dr\right],$$

where $\mathcal{C}(S)$ denotes the set of real valued continuous functions on S which vanish at ∞. The resolvent of the minimal process, we dente by V_λ, i.e. $V_\lambda g(x) = \bar{\mathbb{E}}^x[\int_0^\tau e^{-\lambda r}\, g(X_r)\, dr]$. Furthermore, we say mean square Euler scheme, if X_t for $i/n \le t < (i+1)/n$ is approximated by $\mathbb{E}[X_t^n]$ where X_t^n satisfy the SDE $X_t^n = X_{i/n} + \int_{i/n}^t \sigma(X_{i/n}^n) dB_t$.

3.1. Convergence of algorithm (2) Blumenthal proved in [3, p.138] the convergence of X_t^ϵ via resolvents by applying the Hille-Yoshida theorem to the limit $\lim_{\epsilon\to 0} U_\lambda^\epsilon$.

Theorem 3.1 (One form of the Hille-Yoshida theorem:). *In order that a family $\{R_\lambda\}$ of endomorphisms of the Banach space $\mathcal{C}(S)$ with the norm $||g||_\infty = \sup_{x\in S} |g(x)|$ be the resolvent of a semigroup $\{Q_t\}$ of contractions it is necessary and sufficient that*

- *R_λ satisfies the resolvent equation*:

$$R_\lambda - R_\mu = (\mu - \lambda) R_\lambda R_\mu$$

- *λR_λ is a contraction*
- *$\lambda R_\lambda \longrightarrow 1$ as $\lambda \to \infty$.*

The theorem is cited from Feller [10, p.461].

A short calculation shows:

$$U_\lambda g(x) = V_\lambda g(x) + \bar{\mathbb{E}}^x[e^{-\lambda\tau}] U_\lambda g(d) \tag{1}$$

$$U_\lambda^\epsilon g(x) = \underbrace{V_\lambda g(x)}_{\textbf{Step1}} + \bar{\mathbb{E}}^x[e^{-\lambda\tau}] U_\lambda^\epsilon g(d). \tag{2}$$

We can get $U_\lambda^\epsilon g(d)$ by the Algorithm 2:

$$\begin{aligned} U_\lambda^\epsilon g(d) &= g(d) \underbrace{\mathbb{E}^d \left[\int_0^J e^{-\lambda t} dt \right]}_{\text{Step3, } J=\Delta} \\ &\quad + \mathbb{E}^d[e^{-\lambda J}] \underbrace{\mathbb{E}^{X_J}[V_\lambda g]}_{\text{Step3 and 4}} + \mathbb{E}^d[e^{-\lambda(J+\tau\circ\theta_J)}] U_\lambda^\epsilon g(d) \\ &= \frac{g(d)}{\lambda+\beta} + \frac{\beta}{\lambda+\beta}\langle\gamma, V_\lambda g\rangle + \frac{\beta}{\lambda+\beta}(1-\lambda\langle\gamma, V_\lambda 1\rangle) U_\lambda^\epsilon g(d), \end{aligned} \tag{3}$$

where J is exponential distributed with holding parameter β, X_J is independent of J and distributed by $\gamma = \eta_\epsilon/\|\eta_\epsilon\|$. Verifying the parameter β we have to take into account, that the time is increased by t/α_ϵ, the local time increases by t. Thus $\beta = \beta_\epsilon\alpha_\epsilon$. Solving this equation we obtain

$$U_\lambda^\epsilon g(d) = \frac{g(d)/\alpha_\epsilon + \beta^\epsilon\langle\eta_\epsilon/\|\eta_\epsilon\|, V_\lambda g\rangle}{\lambda(1/\alpha_\epsilon + \beta^\epsilon\langle\eta_\epsilon/\|\eta_\epsilon\|, V_\lambda 1\rangle)} \tag{4}$$

and $\alpha_\epsilon = 1/\sqrt{\pi\epsilon} + 1 \to 1$ as $\epsilon \to 0$. Further V_λ denotes the λ potential of the corresponding minimal process, i.e.

$$V_\lambda g(x) = \bar{\mathbb{E}}\left[\int_0^\tau e^{-\lambda t} g(X_t) dt\right].$$

Blumenthal [3, Theorem 2-6, p.138] shows first, that for all $\lambda > 0$ fixed

$$\lim_{\epsilon\to 0} U_\lambda^\epsilon g(d)$$

exists uniformly in $\mathcal{C}(D)$ with $0 \le g \le 1$. Hence the first term of the right hand-side of (2) is independent of ϵ, it follows for a fixed λ

$$\lim_{\epsilon\to 0} U_\lambda^\epsilon g(x) =: U_\lambda g(x)$$

exists uniformly in $\mathcal{C}(D)$ with $0 \le g \le 1$ and therefore the limit does it as well. Further, since X_t^ϵ is Markov for each $\epsilon > 0$, each family $\{U_\lambda^\epsilon, \lambda > 0\}$ solves the resolvent equation. Together with the uniform convergence follows that the limit solves the resolvent equation as well. For applying the Hille-Yoshida theorem it remains to show that it holds:

$$\lambda \lim_{\epsilon\to 0} U_\lambda^\epsilon \to 1 \quad \text{as} \quad \lambda \to \infty \tag{5}$$

and that $\lambda \lim_{\epsilon\to 0} U_\lambda^\epsilon$ is a contraction. (5) follows by

$$\lambda V_\lambda g(x) = \int_0^{\lambda\tau} e^{-t} g(X_{t/\lambda}) dt \to g(x) \quad \text{as} \quad \lambda \to \infty$$

and $\lambda U_\lambda^\epsilon g(d) \longrightarrow 0$ as $\lambda \to \infty$, which is proved by Blumenthal [3, Theorem 2-8, p.143]. The fact that $\lambda U_\lambda^\epsilon$ is a contraction, follows by considering the Equation (2). The first part is a contraction since V_λ is the resolvent of the minimal process and therefore is λV_λ a contraction. Thus, it holds by the uniform convergence for the limit. For the second part it follows, since $\mathbb{E}^x[e^{-\lambda\tau}] \leq 1$ and $U_\lambda^\epsilon g(d) \leq g(d)$. Since, the Laplace transform and therefore the resolvent is unique, it follows that the limit coincides with U_λ, the resolvent of X_t.

3.2. Convergence of algorithm (3) For the convergence of Algorithm 3, i.e. the convergence of $\hat{U}_\lambda^\epsilon \longrightarrow U_\lambda$ as $\epsilon \searrow 0$, we have to show that it holds $|\hat{U}_\lambda^\epsilon - U_\lambda^\epsilon| \longrightarrow 0$ as $\epsilon \searrow 0$ uniformly in $C(D)$ for all function $g : \mathbb{R} \to \mathbb{R}$, $0 \leq g \leq 1$. Here we have to add some considerations to Blumenthals proof. Substituting the approximated entrance law and holding parameter in (4), we get

$$\hat{U}_\lambda^\epsilon g(d) = \frac{g(d)/\alpha_\epsilon + \hat{\beta}^\epsilon \langle \hat{\eta}_\epsilon / \|\hat{\eta}_\epsilon\|, V_\lambda g\rangle}{\lambda(1/\alpha_\epsilon + \hat{\beta}^\epsilon \langle \hat{\eta}_\epsilon / \|\hat{\eta}_\epsilon\|, V_\lambda 1\rangle)}.$$

Since for a function g, $g = 1$ it holds $0 \leq g \leq 1$. It remains only considering the case

$$\left|\beta^\epsilon \left\langle \frac{\eta_\epsilon}{\|\eta_\epsilon\|}, V_\lambda g \right\rangle - \hat{\beta}^\epsilon \left\langle \frac{\hat{\eta}_\epsilon}{\|\hat{\eta}_\epsilon\|}, V_\lambda g \right\rangle\right| \longrightarrow 0 \quad \text{as} \quad \epsilon \searrow 0.$$

where $0 \leq g \leq 1$ is arbitrary. First, we write the term in another form:

$$\begin{aligned}
&\left|\beta^\epsilon \left\langle \frac{\eta_\epsilon}{\|\eta_\epsilon\|}, V_\lambda g \right\rangle - \hat{\beta}^\epsilon \left\langle \frac{\hat{\eta}_\epsilon}{\|\hat{\eta}_\epsilon\|}, V_\lambda g \right\rangle\right| \\
&= \left|\bar{\mathbb{E}}^{\eta_\epsilon}[V_\lambda g] - \bar{\mathbb{E}}^{\hat{\eta}_\epsilon}[V_\lambda g]\right| \\
&= \left|\int_0^\infty \eta_\epsilon(x) \bar{\mathbb{E}}^x [V_\lambda g] - \int_0^\infty \eta_\epsilon(x) \bar{\mathbb{E}}^x [V_\lambda g]\right|,
\end{aligned}$$

where $\bar{E}$ denotes the expectation value in regard of the minimal semigroup. Changing the order of integrals, it follows

$$\begin{aligned}
&= \left|\int_0^\infty \mathbb{E}[\eta_{\gamma_\epsilon}^B(x)] \bar{\mathbb{E}}^x [V_\lambda g] - \int_0^\infty \eta_{\sigma^2\epsilon}^B(x) \bar{\mathbb{E}}^x [V_\lambda g]\right| \\
&\leq \mathbb{E}\left[\left|\int_0^\infty \eta_{\gamma_\epsilon}^B(x) \bar{\mathbb{E}}^x [V_\lambda g] - \int_0^\infty \eta_{\sigma^2\epsilon}^B(x) \bar{\mathbb{E}}^x [V_\lambda g]\right|\right] \\
&= \mathbb{E}\Big[\left|\langle \eta_{\gamma_\epsilon}^B, V_\lambda g\rangle - \langle \eta_{\sigma^2\epsilon}^B, V_\lambda g\rangle\right|\Big].
\end{aligned} \tag{6}$$

For a arbitrary minimal semigroup, $\langle \eta_s^B, V_\lambda g\rangle$ is uniformly bounded in s and for $g \in C(D)$, $0 \leq g \leq 1$. Further, we know in the case of the Brownian motion, η_s^B is continuous and differentiable in s. Therefore $\partial/\partial s \langle \eta_s^B, V_\lambda g\rangle = \langle \eta_s^B, V_\lambda \Delta g\rangle$ is differentiable.

Is the derivative also uniformly bounded in s? Hence it holds

$$\begin{aligned}\langle \eta_{s+t}^B, V^\lambda g\rangle &= \bar{\mathbb{E}}^{\eta_s^B}\left[\int_0^\tau e^{-\lambda r} g(X_r)dr; \tau \le t\right] \\ &+\bar{\mathbb{E}}^{\eta_s^B}\left[\int_0^t e^{-\lambda r} g(X_r)dr; \tau > t\right] + e^{-\lambda\tau}\langle \eta_{s+t}^B, V_\lambda g\rangle,\end{aligned}$$

the following inequality can be shown:

$$\begin{aligned}&\langle \eta_s^B, V_\lambda g\rangle - \langle \eta_{s+t}^B, V_\lambda g\rangle \\ &= (e^{\lambda t}-1)\langle \eta_s^B, V_\lambda g\rangle + e^{\lambda t}\langle \eta_s^B, V_\lambda 1\rangle - \langle \eta_{s+t}^B, V_\lambda 1\rangle \\ &= (e^{\lambda t}-1)\langle \eta_s, V_\lambda g\rangle \\ &\quad + e^{\lambda t}\left(\bar{\mathbb{E}}^{\eta_s^B}\left[\int_0^\tau e^{-\lambda r} g(X_r)dr; \tau \le t\right] + \bar{\mathbb{E}}^{\eta_s^B}\left[\int_0^t e^{-\lambda r} g(X_r)dr; \tau > t\right]\right).\end{aligned}$$

Considering the differential quotient we get

$$\begin{aligned}&\frac{\partial}{\partial s}\langle \eta_s^B, V_\lambda g\rangle \\ &= \lim_{t\to 0}\frac{1}{t}\Big[\langle \eta_s^B, V_\lambda g\rangle - \langle \eta_{s+t}^B, V_\lambda g\rangle\Big] \\ &= \lim_{t\to 0}\frac{(e^{\lambda t}-1)}{t}\langle \eta_s^B, V_\lambda g\rangle \\ &\quad + \lim_{t\to 0} e^{\lambda t}\left(\bar{\mathbb{E}}^{\eta_s^B}\left[\int_0^\tau e^{-\lambda r} g(X_r)dr; \tau \le t\right] + \bar{\mathbb{E}}^{\eta_s^B}\left[\int_0^t e^{-\lambda r} g(X_r)dr; \tau > t\right]\right).\end{aligned}$$

By a short calculation we obtain

$$\frac{\partial}{\partial s}\langle \eta_s^B, V_\lambda g\rangle \le \lambda\langle \eta_s^B, V_\lambda g\rangle \le \langle \eta_s^B, 1-e^{-\lambda\tau}\rangle + \underbrace{\|g\|_\infty}_{\le 1},$$

which is bounded in s. Thus, we can give an upper bound independent of g. The next step is applying the Taylor expansion up to the first order to verifying the convergence of (6). Since $\gamma_\epsilon = \int_0^\epsilon \sigma^2(X_t)dt$, it follows $\mathbb{E}[\gamma_\epsilon] = \sigma^2\epsilon + o(\epsilon)$ and we get

$$\begin{aligned}\mathbb{E}\Big[\big|\langle \eta_{\gamma_\epsilon}^B, V_\lambda g\rangle - \langle \eta_{\sigma^2\epsilon}^B, V_\lambda g\rangle\big|\Big] &\le \mathbb{E}\Big[O(|\gamma_\epsilon - \sigma^2\epsilon|)\langle \eta_s^B, 1-e^{-\lambda\sigma}\rangle\Big] \\ &= o(\epsilon)K,\end{aligned}$$

where $K = \langle \eta_s^B, 1-e^{-\lambda\sigma}\rangle$ only depends on λ. Thus, for fix λ,

$$|\hat{U}_\lambda^\epsilon g(d) - U_\lambda^\epsilon g(d)| \longrightarrow 0 \quad \text{as} \quad \epsilon \searrow 0$$

uniformly in $C(D)$ and therefore $\hat{U}_\lambda^\epsilon \longrightarrow U_\lambda$ where U_λ coincides with the resolvent of X_t. Thus, the convergence of algorithm (3) is shown.

3.3. Convergence of algorithm (1) We know from the literature, for example Bouleau and Lèpingle [4], Bally and Talay [2], given the Lipschitz hypothesis and the Hölder property, the Euler scheme converges almost surely and in $\mathcal{L}^p$ on compact sets to the solution of the stochastic differential equation.

The problem which arises is, we have the convergence only on compact sets, but the potential of the minimal process is defined on $[0, \infty)$. The way to manage it is considering a point process being stopped if an excursion takes longer than T. In fact, we simulate the process only until a fixed T, such this restriction is in reality no restriction and we need only the convergence of τ^n on the compact set $[0, T]$.

Thus we must modify the Equations (1), (2) and (3). What happens now introducing the stopping? First, if an excursion takes longer than T, the excursion will be send to a graveyard and we set $\tau = \infty$. Further we introduce the following functions:

$$\begin{aligned}
\Theta(x) &= \mathbb{P}^x(\tau < \infty) \\
&= \mathbb{P}^x(\tau < T) \\
&= \sqrt{\frac{2}{\pi T}} \int_0^d e^{-(x-z)^2/2T} dz \\
\varphi(x) &= \mathbb{P}^x(\tau = \infty) \\
&= 1 - \Theta(x).
\end{aligned}$$

All functions concerning only the stopped version are denoted by a superscript $*$, i.e. $V_\lambda^* g(x) = \mathbb{E}^x[\int_0^{\tau \wedge T} e^{-\lambda r} g(X_r) dr]$ and $\tau^* = \tau \wedge T$. Thus, we have for the resolvents:

$$U_\lambda^{\epsilon,*} g(x) = V_\lambda^* g(x) + \bar{\mathbb{E}}^x[e^{-\lambda\tau} \mid \tau < T] \Theta(x) U_\lambda^{\epsilon,*} g(d).$$

We obtain $U_\lambda^{\epsilon,*} g(d)$ by

$$\begin{aligned}
U_\lambda^{\epsilon,*} g(d) &= \underbrace{g(d)\, \mathbb{E}^d\left[\int_0^J e^{-\lambda t} dt\right]}_{=(\lambda+\beta)^{-1}} + \underbrace{\mathbb{E}^d[e^{-\lambda J}]}_{=\beta/(\lambda+\beta)}\ \underbrace{\mathbb{E}^{X_J}[V_\lambda^* g]}_{\langle \eta_\epsilon/\|\eta_\epsilon\|, V_\lambda^* g\rangle} \\
&\quad + \mathbb{E}^d[e^{-\lambda(J + \tau\circ\theta_J)} \wedge \tau < T] U_\lambda^{\epsilon,*} g(d).
\end{aligned}$$

The third summand is $\mathbb{E}^d[e^{-\lambda J}] U_\lambda^{\epsilon,*} g(d)$ times

$$\begin{aligned}
\mathbb{E}^{X_J}[e^{-\lambda\tau} \wedge \tau < T] &= \mathbb{E}^{X_J}[e^{-\lambda(\tau\wedge T)} - \varphi(x) e^{-\lambda T}] \\
&= \left\langle \frac{\eta_\epsilon}{\|\eta_\epsilon\|}, e^{-\lambda(\tau\wedge T)} \right\rangle - \left\langle \frac{\eta_\epsilon}{\|\eta_\epsilon\|}\varphi, 1 \right\rangle e^{-\lambda T} \\
&= 1 - \lambda \left\langle \frac{\eta_\epsilon}{\|\eta_\epsilon\|}, V_\lambda^* 1 \right\rangle - \left\langle \frac{\eta_\epsilon}{\|\eta_\epsilon\|}\varphi, 1 \right\rangle e^{-\lambda T},
\end{aligned}$$

since it holds $1 - e^{-\lambda(\tau\wedge T)} = \lambda \int_0^{\tau\wedge T} e^{\lambda r} dr$. Solving the equation we obtain for the resolvent at d:

$$U_\lambda^{\epsilon,*} g(d) = \frac{g(d)/\alpha_\epsilon + \langle \eta_\epsilon, V_\lambda^* g\rangle}{\lambda/\alpha_\epsilon + \lambda\langle \eta_\epsilon, V_\lambda^* 1\rangle - \langle \eta_\epsilon \varphi, 1\rangle e^{-\lambda T}}.$$

Now, we can approximate the excursions by the Euler scheme.

$$\begin{aligned} |V_\lambda^{*,n} g(x) - V_\lambda^* g(x)| &= \Bigg| \mathbb{E}^x \Bigg[\int_0^{T\wedge\tau} e^{-\lambda s}[g(X_s) - g(X_s^n)]ds \\ &\qquad + e^{-\lambda(T\wedge\tau)} \int_{T\wedge\tau}^{T\wedge\tau^n} e^{-\lambda s} g(X_s^n) ds \Bigg] \Bigg| \\ &\le \Bigg| \mathbb{E}^x \Bigg[\int_0^{T\wedge\tau} e^{-\lambda s}[g(X_s) - g(X_s^n)]ds \Bigg] \Bigg| \\ &\qquad + \mathbb{E}^x[|T\wedge\tau - T\wedge\tau^n|]. \end{aligned}$$

Since we have convergence in distribution of X_t^n to X_t uniformly on $[0,T]$, $g(X_t^n)$ converges to $g(X_t)$ uniformly on $[0,T]$, which implies the convergence of the Laplace transform, i.e. the resolvent. Hence, the first summand of the term of the right hand side tends to zero. The second term tends to zero, since we have almost surely convergence in $[0,T]$. Thus we have

$$V_\lambda^{*,n} g(x) \longrightarrow V_\lambda^* g(x)$$

as $n \to \infty$, where g is a uniformly continuous real valued function. It follows $U_\lambda^{\epsilon,n,*} g(d) \to U_\lambda^{\epsilon,*} g(d)$ as $n \to \infty$ and therefore $U_\lambda^{\epsilon,n,*} g(x) \to U_\lambda^{\epsilon,*} g(x) \to U_\lambda g(x)$ as $n \to \infty$ and $\epsilon \searrow 0$. Going back, we have convergence in distribution of the pair $(\hat{X}_T^{\epsilon,n}, \hat{L}_T^\epsilon)$ to (X_T, L_T). □

4. Rate of Convergence

Assuming we have simulated the process X_t described by the stochastic integral in (1) until a fixed time T, we are interested in the difference

$$\mathbb{E}^x[f(X_T)] - \mathbb{E}^x[f(\hat{X}_T^{\epsilon,n})] \quad \text{and} \quad \mathbb{E}^x[L_T] - \mathbb{E}^x[\hat{L}_T^{\epsilon,n}],$$

where f is twice differentiable. The diffusion coefficient is assumed to satisfy the linear grow condition, further to be differentiable and bounded away from zero.

The algorithm arises by first cutting the path at the time points X_t hits the boundary and then sticking the pieces together. Thus, analyzing the convergence, we have to go the same way by dividing the estimation into two steps. First, we consider the corresponding Poisson point process, parametrized by the local time and being killed if an excursion takes longer than T.

That is, we investigate in the difference

$$T_{s\wedge\xi} - \hat{T}^{\epsilon,n}_{s\wedge\xi},$$

where $\xi = \inf_{t>0}\{e_t(s) \geq T\}$ is a $\mathcal{F}_{T_s}$ stopping time. Further, hence the process is killed at least at time T, we restrict the set of excursion onto $\mathcal{U}^T = \{e \in \mathcal{U} \mid \tau(e) \leq T\}$. Therefore we consider the convergence of the Euler schema on a compact set $[0, T]$.

In the second part, the error of the displacement of time is lifted up to the error of X_T by letting start the process ones at time 0 and once at time ΔT, where ΔT is the error. The last chapter deals with the accuracy of the local time.

4.1. Convergence of the Subordinator T: Hence, if s increases the error of T_s increases as well, we can confine ourselves to the case $s = \xi$. ξ^n denotes the stopping time of the modified Euler scheme. Now, the error decays into four independent parts in a natural way:

$$\begin{aligned}
\text{error}(T) &\leq \mathbb{E}\big[|T_\xi - \hat{T}^{\epsilon,n}_\xi|\\
&\leq \underbrace{|T_\xi - T^\epsilon_\xi|}_{\mathrm{I}} + \underbrace{|T^\epsilon_\xi - \hat{T}^\epsilon_\xi|}_{\mathrm{II}} + \underbrace{|\hat{T}^\epsilon_\xi - \hat{T}^{\epsilon,n}_\xi|}_{\mathrm{III}} + \underbrace{|\hat{T}^{\epsilon,n}_\xi - \hat{T}^{\epsilon,n}_{\xi^n}|}_{\mathrm{IV}}\big].
\end{aligned}$$

The first part I is just the sum over all excursion not taking longer than ϵ, i.e. the excursion which we cancelled out. We add the expectational time the cancelled excursions takes, but we have to take into account, that this expectation value is an approximation. The second part is the error arising by approximating the entrance law. The two summands III and IV arise by applying the Euler scheme to simulate the excursion.

Hence, we have a Poisson point process, the distribution of ξ is independent of T_t and exponentially distributed with holding parameter $\beta_T = \|\eta^X_T\|$. Therefore, if we can give an upper bound of $1/\beta_T$, it is enough to consider the error of one excursion.

4.1.1. The upper bound of $1/\beta_T$: The upper bound is given by

$$\beta_T = \mathbb{E}\left[\sqrt{\frac{1}{\pi\gamma_T}}\right].$$

i.e.

$$\begin{aligned}
\mathbb{E}[\gamma_T] = \mathbb{E}[X_T^2] &= \mathbb{E}\left[\int_0^T \sigma^2(X_s)ds\right]\\
\text{(linear growth condition)} &\leq \mathbb{E}\left[K^2\int_0^T (1+X_s^2)ds\right]
\end{aligned}$$

$$\leq K^2 \left(T + \int_0^T \mathbb{E}[\gamma_s] ds \right).$$

Applying the Gronwall inequality, we can give an estimate for $\mathbb{E}[\gamma_T]$ by

$$\mathbb{E}[\gamma_T] \leq K^2 \left(T + K^2 \int_0^T s e^{K^2(T-s)} ds \right) = e^{K^2 T} - 1.$$

Hence the function $1/\sqrt{x}$ is convex, we have by the Jensen inequality

$$\beta_T = \mathbb{E}\left[\sqrt{\frac{1}{\pi \gamma_T}} \right] \geq \sqrt{\frac{1}{\pi \mathbb{E}[\gamma_T]}}.$$

Therefore $1/\beta_T$ is bounded.

4.1.2. The Error of Part I: While the local time increases by ΔL, the real time increases by

$$\Delta t = \sum_{\substack{\tau(e(s)) < \epsilon \\ 0 < s \leq \Delta L}} \tau(e(s)),$$

where the expectation value is equal to $\int_0^\epsilon s \mathbb{P}^d(\tau^X = s) ds$. In our algorithm we approximate the expectation by the expectation of a Brownian motion with variance σ^2. The difference is given by

$$\int_0^\epsilon s \mathbb{P}^d({}_{\tau^X = s}) ds - \int_0^\epsilon s \mathbb{P}^d({}_{\tau^B = \sigma^2 s}) ds.$$

Replacing $\mathbb{P}^d({}_{\tau^X = s})$ by $\mathbb{P}^d({}_{\rho \tau^B = s})$ resp. by $\mathbb{P}^d({}_{\tau^B = \gamma_s})$ and substituting γ_s by $\sigma^2 s$, we get

$$\int_0^\epsilon s \mathbb{P}^d({}_{\tau^X = s}) ds - \mathbb{E}\left[\int_0^{\gamma_\epsilon / \sigma^2} \rho_{\sigma^2 s} \frac{\sigma^2(B_s)}{\sigma^2} \mathbb{P}^d({}_{\tau^B = \sigma^2 s}) ds \right].$$

The next step is to split the difference in the following sum of differences:

$$= \mathbb{E}\left[\int_0^\epsilon \cdots ds - \int_0^{\gamma_\epsilon/\sigma^2} \cdots ds \right] - \mathbb{E}\left[\int (s - \rho_{\sigma^2 s}) \cdots ds \right]$$
$$- \mathbb{E}\left[\int \left(1 - \frac{\sigma^2(B_s)}{\sigma^2} \right) \cdots ds \right].$$

Taking into account that it holds $\mathbb{P}^d({}_{\tau^B = \sigma^2 s}) = O(\epsilon^{-3/2})$, we can give the following estimates for each difference:

- $\mathbb{E}[\epsilon - \gamma_\epsilon/\sigma^2] = O(\epsilon^2)$
  $\Rightarrow\ \mathbb{E}[\int_0^\epsilon \cdots ds - \int_0^{\gamma_\epsilon/\sigma^2} \cdots ds] = \epsilon O(\epsilon^{-3/2})O(\epsilon^2)$.
- $\mathbb{E}[s - \rho_{\sigma^2 s}] = O(s^2)$
  $\Rightarrow\ \mathbb{E}[\int(s - \rho_{\sigma^2 s})\cdots ds] = \int_0^\epsilon O(s^2)O(s^{-3/2})ds = O(\epsilon^{3/2})$.
- $\mathbb{E}[\sigma^2 - \sigma^2(B_s)] = O(s)$
  $\Rightarrow\ \mathbb{E}[\int(\sigma^2 - \sigma^2(B_s))\cdots ds] = \int_0^\epsilon O(s^{-3/2})O(s)sds = O(\epsilon^{3/2})$.

It follows that the difference is of order $O(\epsilon^{3/2})$.

4.1.3. The Error of Part II: We can put the error arising by the approximation of the holding parameter into the error of the Lévy measure arising by the approximation the entrance law, i.e. the error of $\tau^* = \tau \wedge T$ arising by starting at a wrong place. The approximation can be formulated by an approximation of the underlying minimal process:

$$\hat{X}_0^\epsilon = 0$$
$$\hat{X}_t^\epsilon = \begin{cases} \displaystyle\int_0^t \sigma_\epsilon(s, \hat{\hat{X}}_s^\epsilon)dB_s & t \le \tau \wedge T \\ d & t > \tau \wedge T \end{cases}$$

where

$$\sigma_\epsilon(s,x) = \begin{cases} \sigma^2 & \text{for} \quad t \le \epsilon \\ \sigma(t,x) & \text{otherwise} \end{cases}$$

$\hat{X}_t^\epsilon$ can be seen as a process, equal to the Brownian motion with variance σ^2 until time ϵ and then agreeing with the original process X_t. Now, we denote by $\hat{\tau}$ the stopping time of $\hat{X}_t^\epsilon$ upon reaching the boundary. The error can be formulated as

$$|\mathbb{E}[\hat{\tau}^* \circ \theta_\epsilon] - \mathbb{E}[\tau^* \circ \theta_\epsilon]|,$$

which is equivalent to $|\langle \hat{\eta}_\epsilon, \hat{\tau}^* \rangle - \langle \eta_\epsilon, \tau^* \rangle|$. Hence, the stochastic differential equations agree on $[\epsilon, T]$, we can remove the hat at τ and get

$$|\langle \hat{\eta}_\epsilon, \tau^* \rangle - \langle \eta_\epsilon, \tau^* \rangle| = |\langle \hat{\eta}_\epsilon - \eta_\epsilon, \tau^* \rangle|.$$

As we pointed out in chapter (2.1.), we obtained $\hat{\eta}_\epsilon$ by approximating γ_ϵ by $\sigma^2\epsilon$. The inverse transformation, we denote by ρ_t, and it holds $\rho_t = \int_0^t \sigma^{-2}(B_s)ds$. It follows for the expectation $\mathbb{E}[\rho_\epsilon] = \sigma^2\epsilon + o(\epsilon)$ and for the second moment $\mathbb{E}[\rho_\epsilon^2] = o(\epsilon)$. Let $s, t > 0$ and $s+t$ smaller than T. Now, we want to look up at the function $\langle \eta_s^B, V_\lambda 1 \rangle$, where the entrance law is the law of the Brownian motion and the second term is determined by the process X_t. For clarity, we denote the shift operator by θ_s^B, if the process follows in the time interval $[0, s)$ along a Brownian motion. Now, we can write

$$\langle \eta_{s+t}^B, V_\lambda 1 \rangle - \langle \eta_s^B, V_\lambda 1 \rangle$$

$$= \bar{\mathbb{E}}\left[\bar{\mathbb{E}}\left[\int_0^{\tau^{*X}\circ\theta^B_{s+t}} e^{-\lambda r}dr\right] - \int_0^{\tau^{*X}\circ\theta^B_s} e^{-\lambda r}dr\right]$$

$$= \bar{\mathbb{E}}\left[\int_0^{\tau^{*X}\circ\theta^B_{s+t}} e^{-\lambda r}dr - \int_0^{\tau^{*X}\circ\theta^B_s} e^{-\lambda r}dr, \underbrace{\tau^X\circ\theta^B_s \geq \rho_t}_{=\tau^B\circ\theta^B_s\geq t}\right]$$

$$-\bar{\mathbb{E}}\left[\int_0^{\tau^{*X}\circ\theta^B_s} e^{-\lambda r}dr, \tau^X\circ\theta^B_s < \rho_t\right]$$

Hence $\tau^{*X}\circ\theta^B_t + \rho_t = \tau^{*X}$ if $\tau^{*X} \geq \rho_t$, we can combine the two integrals of the first term in one:

$$= \bar{\mathbb{E}}\left[e^{-\lambda\rho_t}\int_{\rho_t\circ\theta^B_s}^{\tau^{*X}\circ\theta^B_{s+t}+\rho_t\circ\theta^B_s} e^{-\lambda r}dr - \int_{\rho_t\circ\theta^B_s}^{\tau^{*X}\circ\theta^B_s} e^{-\lambda r}dr, \tau^B\circ\theta^B_s \geq t\right]$$

$$-\bar{\mathbb{E}}\left[\int_0^{\tau^{*X}\circ\theta^B_s\wedge\rho_t\circ\theta^B_s} e^{-\lambda r}dr\right]$$

$$= \bar{\mathbb{E}}\left[(e^{-\lambda\rho_t\circ\theta^B_s}-1)\int_{\rho_t\circ\theta^B_s}^{\tau^{*X}\circ\theta^B_s} e^{-\lambda r}dr, \tau^B\circ\theta^B_s \geq t\right]$$

$$-\bar{\mathbb{E}}\left[\int_0^{\tau^{*X}\circ\theta^B_s\wedge\rho_t\circ\theta^B_s} e^{-\lambda r}dr\right],$$

where $\wedge$ denotes the minimum. Taking the differential quotient

$$\begin{aligned}\frac{\partial}{\partial s}\langle\eta^B_s, V_\lambda 1\rangle &= \lim_{t\to 0}\frac{\langle\eta^B_{s+t}, V_\lambda 1\rangle - \langle\eta^B_s, V_\lambda 1\rangle}{t}\\ &= \lambda\lim_{t\to 0}\bar{\mathbb{E}}\left[\frac{\rho_t\circ\theta^B_s}{t}\int_{\rho_t\circ\theta^B_s}^{\Upsilon_{\tau^B_*}\circ\theta^B_s} e^{-\lambda r}dr, \tau^B\circ\theta^B_s\geq t\right]\\ &\quad -\lim_{t\to 0}\frac{1}{t}\bar{\mathbb{E}}\left[(\tau^{*X}\circ\theta^B_s)\wedge(\rho_t\circ\theta^B_s)\right],\end{aligned}$$

we obtain for the first term $\lambda\bar{\mathbb{E}}\left[1/\sigma^2(B_s)\right]\langle\eta^B_s, V^*_\lambda 1\rangle$. For the second term we can give an upper and lower threshold

$$\frac{1}{t}\mathbb{P}(\tau^B\circ\theta^B_s\geq t)\bar{\mathbb{E}}\left[\rho_t\circ\theta^B_s, t\leq\tau^B\circ\theta^B_s\right] \leq \frac{1}{t}\bar{\mathbb{E}}\left[(\tau^{*X}\circ\theta^B_s)\wedge(\rho_t\circ\theta^B_s)\right] \leq \frac{1}{t}\bar{\mathbb{E}}\left[\rho_t\circ\theta^B_s\right]$$

$$\mathbb{P}(\tau^B\circ\theta^B_s\geq t)\bar{\mathbb{E}}\left[(1/\sigma^2(B_s))+(o(t)/t)\right] \leq \bar{\mathbb{E}}\left[(\tau^{*X}\circ\theta^B_s)\wedge(\rho_t\circ\theta^B_s)\right] \leq \bar{\mathbb{E}}\left[(1/\sigma^2(B_s))+(o(t)/t)\right].$$

Since $\mathbb{P}(\tau^B\circ\theta^B_s\geq t)\nearrow 1$ as $t\downarrow 0$, we get for the limit

$$\lim_{t\to 0}\frac{1}{t}\bar{\mathbb{E}}\left[(\tau^{*X}\circ\theta^B_s)\wedge(\rho_t\circ\theta^B_s)\right] = \bar{\mathbb{E}}\left[1/\sigma^2(B_s)\right] = \langle\eta^B_s, 1/\sigma^2\rangle.$$

Now, we have found the derivative:

$$\frac{\partial}{\partial s}\langle \eta_s^B, V_\lambda^* 1\rangle = \bar{\mathbb{E}}\left[1/\sigma^2(B_s)\right]\left(\lambda\langle \eta_s^B, V_\lambda 1\rangle - 1\right).$$

Next, we know for the Laplace transform

$$\frac{\partial^m}{\partial \lambda^m}\langle \eta_s^B, V_\lambda 1\rangle = (-1)^m \left\langle \eta_s^B, \int_0^{\tau^{*B}} r^m e^{-r\lambda} dr \right\rangle.$$

Partial integration on the right side leads to

$$\frac{\partial}{\partial \lambda}\langle \eta_s^B, V_\lambda^* 1\rangle = \frac{1}{\lambda}\left(\langle \eta_s^B, \tau^{*B} e^{-\tau_*^B \lambda}\rangle - \langle \eta_s^B, V_\lambda^* 1\rangle\right).$$

To give an upper bound, we apply the Tayler approximation of the first order and change the order of differentiation. The difference $\sigma^2\epsilon - \gamma_\epsilon = \int_0^\epsilon \sigma^2 - \sigma^2(X_s) ds$ is of order ϵ. Further, we have assumed that the diffusion coefficient $\sigma : \mathbb{R} \to \mathbb{R}^+$ is bounded away from zero. Thus we can say

$$\begin{aligned}
\left|\bar{\mathbb{E}}\left[\langle \eta_{\sigma^2\epsilon} - \eta_{\gamma_\epsilon}, \tau^{*B}\rangle\right]\right| &= \left|\bar{\mathbb{E}}\left[\left(\lambda(\partial/\partial\lambda)+1\right)\langle \eta^B_{\sigma^2\epsilon} - \eta^B_{\gamma_\epsilon}, V_\lambda^* 1\rangle\right]\Big|_{\lambda=0}\right| \\
&= \left|\bar{\mathbb{E}}\left[\left(\langle \eta^B_{\sigma^2\epsilon}, V_\lambda^* 1\rangle + 1\right) 1/\sigma^2(B_{\sigma^2\epsilon})|\sigma^2\epsilon - \gamma_\epsilon|\right]\right| \\
&= \left(\langle \eta^B_{\sigma^2\epsilon}, V_\lambda^* 1\rangle + 1\right) K O(\epsilon^2).
\end{aligned}$$

4.1.4. The Error of Part III: The error can be seen as the expectation of the difference $\mathbb{E}^x[\tau - \bar{\tau}^n]$ due to the entrance law. Thus, in the sequel, we consider a diffusion starting at x. The term above we can split in two independent parts: $\mathbb{E}^x[\tau - \bar{\tau}^n] = \mathbb{E}^x[\tau - \tau^n] + \mathbb{E}^x[\tau^n - \bar{\tau}^n]$. The second term is always smaller than $1/n$, hence the grid is of that size. One can write for the first term

$$\begin{aligned}
|\mathbb{E}^x[\tau - \tau^n]| &= \left|\int_0^T \left(1 - P(\tau \le t)\right) dt - \int_0^T \left(1 - P(\tau^n \le t)\right) dt\right| \\
&= \left|\int_0^T \left(P(\tau^n \le t) - P(\tau \le t)\right) dt\right| \\
&\le \int_0^T |P(\tau^n \le t) - P(\tau \le t)| dt
\end{aligned}$$

We can give an upper bound for the difference $P(\tau^n \le t) - P(\tau \le t)$ by applying corollary (5-1):

$$|\mathbb{P}^x(\tau \le t) - \mathbb{P}^x(\tau^n \le t)| \le \left(\frac{1}{n}\right)^{1-u} K(T)(1 + ||x||^q).$$

Now, we have to evaluate the estimate due to the entrance law, i.e.

$$\begin{aligned}\mathbb{E}^{\eta}[\mathbb{E}^{x}[\tau-\tau^{n}]] &\leq \mathbb{E}^{\eta}\left[\int_0^T |\mathbb{P}^x(\tau\leq t)-\mathbb{P}^x(\tau^n\leq t)|dt\right]\\ &\leq \left(\frac{1}{n}\right)^{1-u}\mathbb{E}^{\eta}\left[\int_0^T |1+|x|^q|dt\right]K'(T)\\ &= O\left(\frac{1+\Gamma(1+q/2)\epsilon^{q/2}}{n^{1/2-u}}\right)\end{aligned}$$

Thus we know: The difference $\tau^n-\bar{\tau}^n$ is smaller than $1/n$, thus the expectation is of order T/n^{1-u} and the error of Part III of the same order.

4.1.5. The Error of Part IV: The error is induced by simulating the scheme until a time ξ^n instead of ξ. Since the holding constant for an excursion longer than T is $1/\sqrt{\gamma_T}$, the expectation of $\xi-\xi^n$ is equal to

$$\left|\frac{1}{\mathbb{E}^x[1/\sqrt{\pi\gamma_T}\,]}-\frac{1}{\mathbb{E}^x[1/\sqrt{\pi\gamma_T^n}]}\right|\leq\frac{\mathbb{E}\left[|\sqrt{\gamma_T}-\sqrt{\gamma_T^n}|/\sqrt{\gamma_T\gamma_T^n}\right]}{\mathbb{E}^x[1/\sqrt{\gamma_T^n}]\mathbb{E}^x[1/\sqrt{\gamma_T}\,]}.$$

Hence it holds $\mathbb{E}[|f(X_T^n)-f(X_T)|]\sim 1/n$ for f two times differentiable, we get

$$\begin{aligned}\mathbb{E}^{\eta_\epsilon}\left[\gamma_T-\gamma_T^n\right] &= \mathbb{E}^{\eta_\epsilon}\left[\int_0^T\sigma^2(X_s)-\sigma^2(X_s^n)ds\right]\\ &\leq \mathbb{E}^{\eta_\epsilon}\left[\int_0^T\frac{1}{n}(1+|x|^2)ds\right]\\ &\leq \frac{1}{\sqrt{\pi}}(1+2\epsilon)T\frac{1}{n}\end{aligned}$$

Now, knowing γ_T is bounded away from zero and applying the Taylor formula, we get

$$\mathbb{E}[\xi-\xi^n]\leq O\left((1+2\epsilon)T\frac{1}{n}\right).$$

4.2. Convergence rate of $\mathbb{E}[f(X_T)]$: In the chapter below, we have shown the the error can be estimated by the following function:

$$\begin{aligned}\text{error}(T) = {}& K_I O(\epsilon^{3/2})+K_{II}O(\epsilon^2)+K_{III}\frac{1+\epsilon^{q/2}}{n^{1-u}}K(T)(1+T)+\\ & K_{IV}(1+2\epsilon)T\frac{1}{n},\end{aligned}$$

where $u > 0$ arbitrary. We want to give an estimate of the amount $\mathbb{E}^x[f(X_T)] - \mathbb{E}[f(\widehat{\widehat{X}}_T^{\epsilon,n})]$, where f is twice differentiable. Let Υ and $\hat{\Upsilon}^{\epsilon,n}$, respectively be the last time, X_T respectively $\hat{X}_T^{\epsilon,n}$ hits the boundary. Assume, we have approximated the process until time $\hat{T}^{\epsilon,n}$, the parameter is denoted by s and it holds $X_\Upsilon = X_{\hat{\Upsilon}^{\epsilon,n}}$. Thus, the error of the approximated process to the original process arises the displacement of time, i.e. $\Upsilon - \hat{\Upsilon}^{\epsilon,n}$ and applying the Euler scheme to the last Excursion, i.e. $X_{T-\Upsilon} - \hat{X}^n_{T-\Upsilon}$. Let $u(x,t)$ defined by $\mathbb{E}^x[f(X_t)]$.

$$\begin{aligned}
&\mathbb{E}^x[f(X_T)] - \mathbb{E}[f(\hat{X}_T^{\epsilon,n})] \\
&= \mathbb{E}^x\left[u(X_{\Upsilon_{s_T}}, \Upsilon_{s_T}) - u(\widehat{\widehat{X}}^{\epsilon,n}_{\Upsilon^{\epsilon,n}_{s_T}}, \Upsilon^{\epsilon,n}_{s_T}) + u(\widehat{\widehat{X}}^{\epsilon,n}_{\Upsilon^{\epsilon,n}_{s_T}}, \Upsilon^{\epsilon,n}_{s_T}) - u(\widehat{\widehat{X}}_T^{\epsilon,n}, d)\right] \\
&= \mathbb{E}^x\left[u(d, \Upsilon_{s_T}) - u(d, \Upsilon^{\epsilon,n}_{s_T}) + u(d, \Upsilon^{\epsilon,n}_{s_T}) - u(\widehat{\widehat{X}}_T^{\epsilon,n}, d)\right] \\
&\leq \mathbb{E}^x\left[\frac{\partial}{\partial t}u(d,t)\bigg|_{t=\Upsilon_{s_T}} \underbrace{|\Upsilon - \hat{\Upsilon}^{\epsilon,n}|}_{\leq |T_\xi - \hat{T}^{\epsilon,n}_\xi|} + C(T)\frac{1}{n}\right].
\end{aligned}$$

where $C(T)$ is a constant. Since f is two times differentiable, the derivative $(\partial/\partial t)u(d,t)$ is bounded and we have

$$\mathbb{E}^x[f(X_T)] - \mathbb{E}[f(\hat{X}_T^{\epsilon,n})] \leq C(T)(1 + |x|^{k_\alpha(T)})\mathrm{error}(T) + C(T)\frac{1}{n}.$$

4.3. Convergence rate of the local time: The excursions are parametrized by the local time. The waiting time between two excursions are exponential distributed with holding parameter β. Thus, the error of the local time between two excursions is induced by the approximation of the holding parameter. Therefore we have for one excursions

$$\frac{1}{\beta} - \frac{1}{\hat{\beta}} = \frac{\mathbb{E}\left[\sqrt{\gamma_\epsilon} - \sigma\sqrt{\epsilon}/\sqrt{\gamma_\epsilon}\right]}{\mathbb{E}\left[1/\sqrt{\gamma_\epsilon}\right]} \leq \frac{\mathbb{E}\left[O(\epsilon^{3/2})/\sqrt{\gamma_\epsilon}\right]}{\mathbb{E}\left[1/\sqrt{\gamma_\epsilon}\right]} = O(\epsilon^{3/2}).$$

Thus, the error of $\hat{L}^\epsilon$ and L^ϵ at one time hitting the bundary is of order $O(\epsilon^{3/2})$. Hence the error of the local time is the difference between the parameters of the point process which arises by approximation and the point process, only the approximation of the holding parameter influences the error, not the lenght of the excursions. Taking into account the number of excursions which takes place, we can conclude that the error is smaller than

$$\underbrace{e^{-\beta_\epsilon \xi}\sum_{n=0}^{\infty}\frac{(\beta_\epsilon \xi)^n}{n!}n}_{=\text{number of expected excursions}} O\left(\epsilon^{3/2}\right) = \beta_\epsilon \underbrace{\mathbb{E}[\xi]}_{=1/\beta_T} O\left(\epsilon^{3/2}\right)$$

where $1/\beta_T$ is bounded (please see Chapter 4.1..1). Thus, this is the order of convergence for the local time.

5. Appendix

We know from Theorem 3.1 of Bally and Talay [2] that under

- (UH) $C_L; + \inf_{x \in \mathbb{R}^d} V_L(x) > 0$ for some integer L
- (C) The derivative of any order of the function b, σ are bounded i.e. b, $\sigma \in \mathcal{C}^\infty$.

where $b : \mathbb{R}^r \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^r \to \mathbb{R}^d \times \mathbb{R}^r$. It holds

$$\mathbb{E}[f(X_T(x)) - f(X_T^n(x))] \le K(T)\|f\|_\infty \frac{1 + \|x\|^Q}{T^q} \frac{1}{n} \tag{1}$$

for some Q, q and nondecreasing function K. Our problem is no, that the right side tends to infinity if T tends to zero. Thus, what happens if T is rather small? For small T we the order of convergence is $1/n^{2/2-u}$ where $u > 0$ can be arbitrary small.

To investigate what happens, we follow the idea of Bally and Talay in [2] and use some localization argument. If t is small, i.e. $t < 1/n$, the process X_t^n arising by the Euler scheme can be seen a Taylor expansion of $\sigma(X)$ up to the first order. For increasing t the coefficients of the Taylor expansion are updated at each grid point k/n, $k = 1, \cdots, [Tn]$. The exact definition will be given below. For applying a localization argument one need a quantity to measure the difference between X_t and X_t^n. A good choice is the difference $\sigma^2(X_s) - \sigma^{n2}(X_s^n)$[4] which is dominated by the Malliavin matrix.

For simplicity we define the function $[\cdot]_n : \mathbb{R} \to \mathbb{N} : s \mapsto [sn]/n$ and denote by θ the shift operator.

To classify the speed of a stochastic process starts at $t = 0$ and removes from the boundary, we introduce the notion of flat functions:

DEFINITION 5.1. A function $f : [0, \infty) \to \mathbb{R}$ is called flat, if $\lim_{\epsilon \to 0} \epsilon^{-p} f(\epsilon) = 0$ for all $p \in \mathbb{N}$.

Clearly if f and g are flat, $f + g$ is flat. In addition if f is flat, $\int_0^\epsilon f(s)ds$ is flat as well. Now one can classify the speed a process X_t starts at X_0 by considering the functions $\epsilon \mapsto \mathbb{P}(X_\epsilon^* \ge t^{k-\delta})$ respective $\epsilon \mapsto \mathbb{P}(X_\epsilon^* \le t^{k+\delta})$ where $X_T^* = \sup_{t \le T} |X_t - X_0|$ and taking this exponent, the function is flat.

- Let $u > 0$ be arbitrary, but small enough. Let Ω_0 be the set of events where

$$|\sigma^2(X_{s-[s]_n} \circ \theta_{[s]_n}) - \sigma^{n2}(X^n_{s-[s]_n} \circ \theta_{[s]_n})|^* \le \gamma_s^* \, (s - [s]_n)^{-u-1/2}$$

[4] σ^n denotes the diffusion matrix of X^n, where the diffusion coefficient is constant on every intervall $[k/n, (k+1)/n)$ and equal to $\sigma(X_{k/n})$, i.e. $\sigma^n(X_s) = \sigma(X_{k/n})$ for all $s \in [k/n, (k+1)/n)$.

for all $s \in [0,T]$. Here we use the fact, $\mathbb{P}(\Omega_0)$ is small, for example $\mathbb{P}(|\sigma^2(X_s) - \sigma^{n2}(X_s)|^* \leq \gamma_s^* s^{-u})$ for $s < 1/n$ is flat.

- on the complementary set of Ω_0, $\sigma^2(X_{s-[s]_n} \circ \theta_{[s]_n}) - \sigma^{n2}(X_{s-[s]_n} \circ \theta_{[s]_n})$ is small, i.e. X_t behaves like X_t^n and the Monte Carlo error is small.

We are operating on the space $(\Omega, \mathcal{F}, P)$, $(\mathcal{F})_{t \geq 0}$ on which the Brownian motion is defined. Further one considers the following classes of stochastic processes:

$$\mathcal{C}_0 = \{X : [0,\infty) \times \Omega \to \mathbb{R} : X \text{ is a continuous, adapted process which is constant at } t = 0 \text{ and } \mathbb{E}[(X_T^*)^p] \leq \infty \quad \forall T < \infty \forall p > 1\},$$

where $X_T^* = \sup_{t \leq T} |X_t - X_0|$ and

$$\mathcal{C}_{k+1} = \left\{ X_t = X_0 + \sum_{j=0}^{1} \int_0^t X_s^j dB_s^j ; X^j \in \mathcal{C}_k \quad \text{for} \quad j = 0,1 \right\},$$

where B_t^1 is a 1-dimensional Brownian motion and $B_t^0 = t$. Further we define

$$\mathcal{C}_\infty = \cap_{k=0}^{\infty} \mathcal{C}_k.$$

Next we consider a projection of $\mathcal{C}_\infty$ onto $\mathcal{C}_k$. Fix $j = 0,1$. First we define

$$\begin{aligned} Pr_j : \mathcal{C}_1 &\to \mathcal{C}_0 \\ X &\mapsto X^j. \end{aligned}$$

Now, for a multiindex $\kappa = (\kappa_1, \kappa_2, \ldots \kappa_m)$, $\kappa_i \in \{0,1\}$, $i = 1, \ldots m$, the projection is defined by

$$\begin{aligned} Pr_\kappa : \mathcal{C}_\infty &\to \mathcal{C}_\infty \\ X &\mapsto Pr_{\kappa_m} \circ \cdots \circ Pr_{\kappa_1}(X) \end{aligned}$$

and

$$\begin{aligned} pr_\kappa : \mathcal{C}_\infty &\to \mathbb{R} \\ X &\mapsto (Pr_\kappa X)_0. \end{aligned}$$

To get unitary representation we define the projection for the void index ϕ by

$$Pr_\phi X = X, \quad pr_\phi X = X_0.$$

In addition, to get an ordering of the multiindecis we define $p(\kappa) = p((\kappa_1, \kappa_2, \ldots \kappa_m))$ $= \#\{k \mid \kappa_k \neq 0\} + 2\#\{k \mid \kappa_k = 0\}$. In addition we write $|\kappa|$ for the number of components of a multiindex κ and $\bar{\kappa}$ for the multiindex where the components are cancelled which are equal to zero and κ^- for the multiindex obtaining by deleting the last component of κ. Further we define the symbol $o(X) = \min\{p(\kappa) \,|\, pr_\kappa(X) \neq 0\}$. Throughout this chapter we denote by κ_m the last component of a multiindex.

Now one can represent a process $X \in \mathcal{C}_\infty$ as a stochastic Taylor expansion in terms of iterated integrals driven by Brownian motion B_t or the time t, i.e.

$$X_t = \sum_{m=0}^{\infty} \sum_{p(\kappa)=m} pr_\kappa(X) B_t^{(\kappa)} = \sum_{m=0}^{\infty} t^{m/2} \sum_{p(\kappa)=m} pr_\kappa(X) B_1^{(\kappa)}, \tag{2}$$

where $B_t^{(\kappa)} = \int_0^t B^{(\kappa^-)} dB_s^{\kappa_m}$. The convergence in $\mathcal{L}^2$ is given since first, X can be represented as a sequence of iterated integrals due to functions over $[0,T]^m$, i.e. $X = \sum_{m=0}^{\infty} I^m(f_m)$[5], where $f_m(\cdot) = \mathbb{E}[D^m X_T \,|\, \mathcal{F}.] \in \mathcal{L}^2[0,T]^{|\kappa|}$ and $I^m(f_m) = \int_0^{t_1} \cdots \int_0^{t_m} f_m(t_1, \dots t_m) dB_{t_1} \dots dB_{t_m}$. Since σ is an infinitely differentiable function in x with bounded derivatives of all order greater or equal to one, $D^k X_t$ exists and belongs to $\mathcal{L}^p(\Omega, [0,T])$ for all $p \geq 1$ and $k \leq 0$. (please see Nualart [22, Theorem 2.2.2]). The convergence of this sequence is given in $\mathcal{L}^2(\Omega, \mathcal{F}, P)$. Second, each function f can be approximated by polynominals in $\mathcal{L}^\infty[0,T]^m$. The last equality holds because of the scaling property of the Brownian motion, i.e. $B_t^{(\kappa)} = t^{p(\kappa)/2} B_1^{(\kappa)}$. The coefficients $pr_\kappa(X)$ can be founded by iterating the Itô-formula:

$$\begin{aligned}
X_t &= X_0 + \int_0^t \sigma(X_{t_1}) dB_{t_1} \\
&= X_0 + \int_0^t \sigma(X_0) + \int_0^{t_1} \sigma(X_0)\sigma'(X_0) + \int_0^{t_2} (\sigma''(X_{t_3})\sigma(X_{t_3}) + \sigma'(X_{t_3})^2) dB_{t_3} \\
&\quad + \frac{1}{2} \int_0^{t_2} (\sigma'''(X_{t_3})\sigma(X_{t_3}) + 3\sigma'(X_{t_3})\sigma''(X_{t_3})) dt_3 dB_{t_2} \\
&\quad + \frac{1}{2} \int_0^{t_1} \sigma''(X_{t_2})\sigma(X_{t_2})^2 dB_{t_1} dt_2 + \cdots \\
&= X_0 + \sigma(X_0) B_t^{(1)} + \sigma(X_0)\sigma'(X_0)) B_t^{(11)} + (\sigma''(X_0)\sigma(X_0) + \sigma'(X_0)^2) B_t^{(111)} \\
&\quad + \frac{1}{2}(\sigma'''(X_0)\sigma(X_0) + 3\sigma'(X_0)\sigma''(X_{t_3})) B_t^{(110)} + \frac{1}{2}\sigma''(X_0)\sigma(X_0)^2 B_t^{(10)} + \cdots \\
\Rightarrow \quad & pr_\Phi(X) = X_0, pr_{(1)}(X) = \sigma(X_0), pr_{(11)}(X) = \sigma(X_0)\sigma'(X_0) \cdots.
\end{aligned}$$

For clarity we omit in the sequel (X) and write pr_κ instead of $pr_\kappa(X)$. Let L be the infinitesimal generator of the Ornstein-Uhlenbeck semigroup. We summerize some properties in the following proposition:

Proposition 1. *Let $X \in \mathcal{C}_\infty$ with stochastic Taylor expansion given in* (2) *and let Γ the inverse of γ (For definition of γ please see Ikeda and Watanabe* [12] *or Nualart* [22]*). The norm $||\%||_p$ is defined by $||F||_p = \mathbb{E}[|F|^p]^{1/p}$. Then it follows*:

[5] for definition of I and D, please see for example the book of Nualart [22, chapter 1.1]

1. $\|LX_t\|_p = \sqrt{t}\, C_p^{LX_t}$ *where* $C_p^{LX_t} < \infty$.
2. $\|\gamma_t\|_p = tC_p^{\gamma_t}$ *where* $C_p^{\gamma_t} < \infty$.
3. $\|\Gamma_t\|_p = (1/t)C_p^{\Gamma_t}$ *where* $C_p^{\Gamma_t} < \infty$.

Proof. First, it is easy to verify that the following holds:

$$LB_1^{(\kappa)} = \begin{cases} B_1^{(\kappa)} B_1^{(\kappa^-,0)} + \frac{1}{2} LB_1^{(\kappa^-)} & : \ \kappa_m = 0 \\ -|\bar{\kappa}| LB_1^{(\kappa)} & : \ \kappa_m = 1 \end{cases}$$

$$DB_1^{(\kappa)} = \begin{cases} B_1^{(\kappa^-)} DB_1^{(\kappa^-)} & : \ \kappa_m = 0 \\ |\bar{\kappa}| s^{p(\kappa^-)/2} B_1^{(\kappa)^-} & : \ \kappa_m = 1. \end{cases}$$

where $(\kappa^-, 0) = (\kappa_1 \ldots, \kappa_{m-1}, 0)$ for $\kappa = (\kappa_1, \ldots, \kappa_m)$.

1. 1. holds just by computation:

$$\begin{aligned} LX_t &= -t^{1/2} pr_{(1)} B_1^{(1)} - 2t pr_{(11)} B_1^{(11)} \\ &\quad -t^{3/2}(-3pr_{(111)} B_1^{(111)} + pr_{(01)} B_1^{(01)}) - \cdots \\ &= \sqrt{t}\, \underbrace{\left(pr_{(1)} B_1^{(1)} - 2\sqrt{t} pr_{(11)} B_1^{(11)} - \cdots\right)}_{\in \mathcal{L}^p([0,T],\Omega)}. \end{aligned}$$

2. To show 2. first we have to compute DX_t:

$$D_s X_t = \sum_{m=1}^{\infty} t^{m/2} \underbrace{\sum_{p(\kappa)=m} |\bar{\kappa}| pr_\kappa D_s B_1^{(\kappa)}}_{=C_s^m}.$$

Then we have

$$\begin{aligned} \gamma_t &= \langle DX_t, DX_t \rangle_{HS} = \sum_{m=1}^{\infty} t^m \sum_{r=0}^{m} \langle C_s^r, C_s^{m-r} \rangle_{HS} \\ &= tpr_{(1)}^2 + \frac{8}{3} t^{3/2} pr_{(1)} \times pr_{(11)} B_1^{(1)} \\ &\quad + t^2 (3pr_{(1)} pr_{(111)} B_1^{(11)} + 2pr_{(1)} pr_{(10)} B_1^{(1)} + pr_{(1)} pr_{(01)} + 2pr_{(1)}^2) + \cdots. \end{aligned}$$

3. Since $\Gamma_t \gamma_t = 1$ we see at once that

$$\Gamma_t = \frac{1}{t} \frac{1}{pr_{(1)}^2} - \frac{1}{\sqrt{t}} \frac{4pr_{(11)}}{pr_{(1)}^3} B_1^{(10)} + \cdots$$

and, since σ is bounded away from zero, $pr_{(1)}$ also and therefore all coefficients belong to $\mathcal{L}^p(\Omega, [0,T])$ for $p \geq 1$. Thus we can verify 3.). □

REMARK 5.1. In analogy we can show that it holds $\|\langle D^2X_t, DX_t \otimes DX_t\rangle_{HS}\|_p = t^2C_p$ resepctively $\|\langle DLX_t, X_t\rangle\|_p = \|tpr^2_{(1)} + \ldots\|_p = tC'_p$ for some constant C_p, $C'_p < \infty$.

Next, we can give an exact definition of a numerical scheme of g'th order:

DEFINITION 5.2. Now, the approximation of gth order can be seen as approximation the stochastic processes $b(X)$ and $\sigma(X)$ by a stochastic Taylor expansion up to order g

$$\sum_{p(\kappa)\leq g} pr_\kappa(b(X))B^{(\kappa)} \quad \text{and} \quad \sum_{p(\kappa)\leq g} pr_\kappa(\sigma(X))B^{(\kappa)}.$$

Further, the coefficients $pr_\kappa(b(X))$ and $pr_\kappa(\sigma(X))$ are updated at each grid point k/n.

Now we can give an answer to the question: How is the asymptotic behavior of X_t for small t? Therefore we introduce the following classification due to the speed, a process grows at zero. Let $X \in \mathcal{C}_\infty$ and $k \in \mathbb{N}$ arbitrary:

$$\mathcal{S}_{k/2} = \{Y \in \mathcal{C}_\infty \,|\, \text{the function} \epsilon \mapsto \mathbb{P}(\sup_{0\leq t\leq \epsilon} Y_t \leq \epsilon^{k/2+u}) \text{ is flat}\}$$

and

$$\mathcal{L}_{k/2} = \{Y \in \mathcal{C}_\infty \,|\, \text{the function} \epsilon \mapsto \mathbb{P}(\sup_{0\leq t\leq \epsilon} Y_t \geq \epsilon^{k/2-u}) \text{ is flat}\}.$$

We list some elementary properities of the classes $\mathcal{L}_q$ and $\mathcal{S}_q$. The proof can be found in the paper of Bally [1, Proposition 1.4.]:

Proposition 2.
1. $\mathcal{C} = \mathcal{L}_0 \supseteq \mathcal{L}_{1/2} \supseteq \mathcal{L}_1 \supseteq \mathcal{L}_{3/2} \supseteq \cdots$
2. $\mathcal{S}_{1/2} \subseteq \mathcal{S}_1 \subseteq \mathcal{S}_{3/2} \subseteq \cdots$
3. $X, Y \in \mathcal{L}_q \Rightarrow X + Y \in \mathcal{L}_q$
4. $X \in \mathcal{S}_q$, $Y \in \mathcal{L}_p$ *for some* $p \geq q \Rightarrow X + Y \in \mathcal{S}_q$
5. $B^{(\iota)} \in \mathcal{L}_{p(\iota)/2} \cap \mathcal{S}_{p(\iota)/2} \Rightarrow \sum_{m=k}^{\infty} \sum_{p(\iota)=m} c_\iota B^{(\iota)} \in \mathcal{L}_{k/2} \cap \mathcal{S}_{k/2}$
6. $X \in \mathcal{L}_{k/2}$, $Y \in \mathcal{L}_{l/2}$ *and* $X_0 = Y_0 = 0 \Rightarrow XY \in \mathcal{L}_{(k+l)/2}$

Proof. Our goal is to get an asymptotic behavior of $\sigma^2(X_s) - \sigma^{n2}(X_s)$ in terms of γ_s. Let $s < 1/n$. The difference can be written by the Ito formula as an integral with respect to the Brownian motion and time, i.e.

$$\sigma^s(X_s) - \sigma^2(X_0) = 2\int_0^s \sigma'(X_r)\sigma^2(X_r)dB_r + \int_0^s ((\sigma'(X_r))^2 + \sigma''(X_r))\sigma^2(X_r)dr.$$

Thus $pr_\Phi(\sigma^2(X)) = \sigma^2(X_0)$ and $pr_{(1)}(\sigma^2(X)) = 2\sigma'(X_0)\sigma^2(X_0)$. Therefore it holds $o(\sigma^2(X) - \sigma^2(X_0)) \geq 1$ and it yields (please see [1, Theorem 3.3])

$$\sigma^2(X) - \sigma^{2n}(X) = \sigma^2(X) - \sigma^2(X_0) \in \mathcal{L}_{1/2}.$$

To classify γ_s one can go an analog way and write first γ_s as an integral[6]:

$$\gamma_s = \int_0^s (D_r X_s)^2 dr = \int_0^s \sigma^2(X_r) \exp\Big(2\int_r^s \sigma'(X_t)dB_t - \int_r^s (\sigma^2)'(X_t)dt\Big)dr.$$

Therefore it holds $o(\gamma) = 1$ and by [1, Theorem 3.3] we have $\gamma \in \mathcal{S}_1$.

Since the scheme can be seen as expanding the Taylor formula of first order at the grid points k/n, $k = 1, \cdots, [T]_n$ and updating the coefficient, one has to apply the consideration above to each interval $[k/n, (k+1)/n)$ for classifying $\sigma^2(X_{s-[s]_n}) \circ \theta_{[s]_n} - \sigma^{n2}(X_{s-[s]_n}) \circ \theta_{[s]_n}$ and $\gamma_{s-[s]_n} \circ \theta_{[s]_n}$ for arbitrary $s \in [0, T]$. Therefore it holds

$$\sigma^2(X_{s-[s]_n}) \circ \theta_{[s]_n} - \sigma^{n2}(X_{s-[s]_n}) \circ \theta_{[s]_n} \in \mathcal{L}_{1/2} \quad \text{and} \quad \gamma \circ \theta_{s-[s]_n} \in \mathcal{S}_1$$

Let $\phi \in \mathcal{L}_b^\infty(\mathbb{R})$ such that $\phi(x) = 1$ for $\|x\| \leq 1$, $\phi(x) = 0$ for $\|x\| \geq 5/4$ and $0 \leq \phi(x) \leq 1$ for $\|x\| \in (1, 5/4)$.

Set

$$r_k := \sup_{k/n \leq s < (k+1)/n} \frac{|\sigma^2(X_{s-[s]_n}) \circ \theta_{[s]_n} - \sigma^{n2}(X_{s-[s]_n}) \circ \theta_{[s]_n}|^*}{|\gamma^*_{s-[s]_n} \circ \theta_{[s]_n}|^*(s - [s]_n)^{-u}}$$

for $k = 1, \cdots, [Tn] - 1$ and for $k = [Tn]$:

$$r_k := \sup_{k/n \leq s < T} \frac{|\sigma^2(X_{s-[s]_n}) \circ \theta_{[s]_n} - \sigma^{n2}(X_{s-[s]_n}) \circ \theta_{[s]_n}|^*}{|\gamma^*_{s-[s]_n} \circ \theta_{[s]_n}|^*(s - [s]_n)^{-u}}.$$

We have

$$\begin{aligned}
\mathbb{E}[f(X_T(x)) - f(X_T^n(x))] &= \mathbb{E}[(f(X_T(x)) - f(X_T^n(x)))\,(1 - \Pi_k \phi(r_k))] \\
&\quad + \mathbb{E}[(f(X_T(x)) - f(X_T^n(x)))\,\Pi_k \phi(r_k)] \\
&=: I + II.
\end{aligned}$$

To upper bound $|I|$ we use the estimate in (1):

$$|I| \leq \|f\|_\infty K(T) \frac{1 + |x|^q}{T^q} \frac{1}{n} \mathbb{E}^x[(1 - \Pi_k \phi(r_k))]. \tag{3}$$

[6] please see [22, Excercise 2.2.1].

It holds $1 - \Pi_k \phi(r_k) = 0$ iff there exists a k, such that $|r_k| \leq 1$. Thus we have to consider the case that for all k it holds $|r_k| \geq 1$ and we have

$$\begin{aligned}
&\mathbb{E}[1 - \Pi_k \phi(r_k)] \leq \Pi_k \mathbb{P}(|r_k| \geq 1) \\
&= \sum_k \int_0^T \mathbb{P}\Big(|\sigma^2(X_{s-[s]_n}) \circ \theta_{[s]_n} - \sigma^{n2}(X_{s-[s]_n}) \circ \theta_{[s]_n}|^* \geq \\
&\qquad\qquad |\gamma_{s-[s]_n} \circ \theta_{[s]_n}|^2 (s - [s]_n)^{-u}\Big) ds \\
&\leq \sum_k \Big(\int_0^T \mathbb{P}\Big(\underbrace{\gamma_{s-[s]_n} \circ \theta_{[s]_n}}_{=:I} \leq (s - [s]_n)^{1-u}\Big) ds \\
&\quad + \int_0^T \mathbb{P}\Big(\underbrace{|\sigma^2(X_{s-[s]_n}) \circ \theta_{[s]_n} - \sigma^{n2}(X_{s-[s]_n}) \circ \theta_{[s]_n}|^*}_{=:II} \geq (s - [s]_n)^{2(1/2+u)}\Big) ds \Big).
\end{aligned}$$

From the consideration above we know, that for all k the functions $s \mapsto \mathbb{P}(I \leq s^{1+u})$ respective $s \mapsto \mathbb{P}(II \geq s^{1/2-u})$ are flat for $u > 0$ arbitrary but small enough. Therefore the sum is flat and since the integral over a flat function is also flat, it follows that $\mathbb{E}[1 - \Pi_k \phi(r_k)]$ is flat. Since $\mathbb{E}^x[(1 - \Pi_k \phi(r_k))]$ in (3) is bounded by any polynom in $min(1/n, T)$ and tends to zero for $T \to 0$, we can neglect the term.

To give an upper bound of the second part we choose the same way as Bally in [2] and define $u(t,x) = \mathbb{E}^x[f(X^n_{T-t})]$. Thus we have

$$\begin{aligned}
(4) \quad & \mathbb{E}^x[f(X_T)] - \mathbb{E}^x[f(X^n_T)] = \mathbb{E}[u(T,0) - u(0,X_T)] \\
&= \sum_{k=1}^{[Tn]-1} \mathbb{E}[u((k+1)/n, X_{(k+1)/n}) - u(k/n, X_{k/n})] \\
&\qquad + \mathbb{E}[u(T,X_T) - u([T]_n, X_{[T]_n})] \\
(5) \quad &= \sum_{k=1}^{[Tn]} \int_{k/n}^{(k+1)/n \wedge T} \int_{k/n}^t \mathbb{E}[\bar{L}\bar{L}u(s,X_s) - \bar{L}\bar{L}^n u(s,X_s) - \bar{L}^n \bar{L} u(s,X_s) \\
&\qquad + \bar{L}^n \bar{L}^n u(s,X_s)] ds dt,
\end{aligned}$$

where we denote the infinitesimal generator of the process X_t by $\bar{L}$ and the generater of the process X^n_t by $\bar{L}^n$. Hence the coefficients are constant in each interval $[k/n, (k+1)/n)$ and updated at each grid point, we have $\bar{L}^n(u(t,X_s)) = \bar{L}(u(t, X_{[s]_n}))$. Thus we can evaluate the expectation in regard to the grid points. Further in order to avoid writing clumsy terms we assume that it holds $[s]_n = 0$. Now we get

$$= \frac{1}{4}\left[\frac{\partial^2}{x^2}\left(\left(\frac{\partial^2}{x^2}u(s,X_s)\right)\left(\sigma^2(X_s) - \sigma^{n2}(X_s)\right)\right)\right]\left(\sigma^2(X_s) - \sigma^{n2}(X_s)\right)$$

$$\leq \frac{1}{4}\left[\frac{\partial^2}{x^2}\left(\left(\frac{\partial^2}{x^2}u(s,X_s)\right)\gamma_s\right)\right]\gamma_s s^{-2u}.$$

Further we denote by Γ the inverse of γ and by ϕ the function:

$$\phi(s,X_s) = \left(\frac{\partial^2}{x^2}u(s,X_s)\right)\gamma_s s^{-u}.$$

Applying Theorem 2-1 of Watanabe [31] or rule (4.5) of Ikeda and Watanabe [12] we get

$$\begin{aligned}\mathbb{E}\left[\left(\frac{\partial^2}{x^2}\phi(s,X_s)\right)\gamma_s\right] &= -\mathbb{E}\left[\left(\frac{\partial}{x}\phi(s,X_s)\right)\delta\big(\gamma_s\Gamma_s DX_s\big)\right] \\ &= \mathbb{E}\left[\left(\frac{\partial}{x}\phi(s,X_s)\right)LX_s\right],\end{aligned}$$

where δ the adjoint operator to D (for the exact defintion please see the articles mentioned above). Further for shortness we write just $\langle .,.\rangle$ instead of $\langle .,.\rangle_{HS}$. Applying the theorem twice and the rules described in Theorem 3-4 and (4.4) in Ikeda and Watanabe [12] we get

$$\begin{aligned}&= \mathbb{E}\big[\phi(s,X_s)\delta(\Gamma_s LX_s DX_s)\big] \\ &= \mathbb{E}\big[\phi(s,X_s)(\Gamma_s\langle DLX_s,DX_s\rangle + LX_s\langle D\Gamma_s,DX_s\rangle + \Gamma_s LX_s LX_s\big] \\ &= \mathbb{E}\big[\phi(s,X_s)(\Gamma_s\langle DLX_s,DX_s\rangle - LX_s\Gamma_s^2\langle D\gamma_s,DX_s\rangle + \Gamma_s LX_s LX_s)\big] \\ &= \mathbb{E}\big[\phi(s,X_s)\underbrace{(\Gamma_s\langle DLX_s,DX_s\rangle - LX_s\Gamma_s^2 2\langle D^2X_s,DX_s\otimes DX_s\rangle + \Gamma_s LX_s LX_s)}_{=H_s}\big].\end{aligned}$$

Substituting the Taylor expansion of Proposition 1, we can see that the constant terms of $\langle DLX_s,DX_s\rangle$ and LX_sLX_s cancelled each other. That it holds $H_s = \sqrt{s}\hat{H}_s$ for some random variable $\hat{H}_s$ belonging to $\mathcal{L}^p(\Omega,[0,T])$ for all $p \geq 1$.

$$\begin{aligned}&= \mathbb{E}\left[\sqrt{s}\left(\frac{\partial^2}{x^2}u(s,X_s)\right)\gamma_s\hat{H}_s\right] \\ &= \mathbb{E}\left[\sqrt{s}\left(\frac{\partial}{x}u(s,X_s)\right)\delta(\Gamma_s\gamma_s\hat{H}_s DX_s)\right] \\ &= \mathbb{E}\left[\sqrt{s}u(s,X_s)\underbrace{(\langle D\hat{H}_s,DX_s\rangle + \hat{H}_s LX_s)}_{=\bar{H}_s}\right]\end{aligned}$$

$$\begin{aligned}(6)\quad &= \mathbb{E}\big[\sqrt{s}u(s,X_s) \\ &\quad \underbrace{\times\,(\Gamma_s\langle D\bar{H}_s,DX_s\rangle \;+ 2\bar{H}_s\Gamma_s^2\langle D^2X_s,DX_s\otimes DX_s\rangle_{HS} + \Gamma_s\bar{H}_s LX_s)\big]}_{=\widehat{\hat{H}}_s}\end{aligned}$$

Hence the terms DX_s and LX_s appears in the definition of $\bar{H}_s$, we can conclude that it holds $||\bar{H}_s||_p = \sqrt{s}C_p''$ for some constant $C_p'' < \infty$. Further, the term of the worst order is $\Gamma_s \bar{H}_s LX_s$. Applying [12, Theorem 3-3] to $\hat{\bar{H}}_s$ we get

$$||\hat{\bar{H}}_s||_2 = ||\Gamma||_6 ||LX_s||_6 ||\bar{H}_s||_6 \le \frac{1}{s} C_6^{\Gamma_s} \sqrt{s} C_6^{LX_s} \sqrt{s} C_6'' < C < \infty.$$

A second and third application leads to the following estimate of $\mathbb{E}\big[(\partial^2/x^2)\phi(s,X_s)\gamma_s\big]$:

$$\begin{aligned}\mathbb{E}\left[\frac{\partial^2}{x^2}\phi(s,X_s)\gamma_s\right] &\le \sqrt{s}\mathbb{E}[u(s,X_s)\hat{\bar{H}}_s] \le \sqrt{s}||u(s,X_s)||_2 ||\hat{\bar{H}}_s||_2 \\ &\le \sqrt{s}||f||_\infty C\end{aligned}$$

for some constant $C < \infty$. Going back to Equation (4) and applying the consideration above to the differentiation we can give an upper bound of the second part

$$\begin{aligned}&\mathbb{E}^x[f(X_T)] - \mathbb{E}^x[f(X_T^n)] = \mathbb{E}[u(T,0) - u(0,X_T)] \\ &= \sum_{k=1}^{[Tn]-1} \mathbb{E}\left[u\left(\frac{k+1}{n}, X_{(k+1)/n}\right) - u\left(\frac{k}{n}, X_{k/n}\right)\right] + \mathbb{E}[u(T,X_T) - u([T]_n, X_{[T]_n})] \\ &\le K \sum_{k=1}^{[Tn]} \int_{k/n}^{(k+1)/n \wedge T} \int_{k/n}^{t} (s - [s]_n)^{-1/2-u} ds dt = K \frac{1}{(1/2-u)(3/2-u)} \left(\frac{1}{n}\right)^{3/2-u},\end{aligned}$$

where $u > 0$ arbitrary but small enough. □

REMARK 5.2. Let $f \in \mathcal{H}^{-(m,\infty)}$, where $\mathcal{H}^{(m,p)}(\mathbb{R}^d)$ denotes the space of all functions f, whose derivative of order m is in $\mathcal{L}^P(\mathbb{R}^d)$ and $\mathcal{H}^{-(m,p)} = \mathcal{H}^{(-m,p\prime)}$, $p\prime = (p-1)/p$ denotes the dual space of $\mathcal{H}^{(m,p)}(\mathbb{R}^d)$. Then the Monte-Carlo error is given by $1/n^{3/2-u}(T \vee (1/n))^{-m/2}$ where the norm of $||f||$ is the norm of $\mathcal{H}^{-(m,\infty)}$, i.e. the sup-norm of the m-th integral. This fact arises by applying the [31, Theorem 2-1] m times to function $f(X_{T-s})$ in Equation (6). Each time a factor $\Gamma_{T-s} LX_{T-s}$ arises which is of order $O(\sqrt{T-s})$. Integration of the last step of the theorem gives the result.

Corollary 5.1. *Let $0 \in \mathbb{R}$ regular and X_t be a real valued process and solution to $dX_t = \sigma(X_t)dB_t$. Further, σ fulfills the conditions mentioned above. Let $\tau = \inf_{t>0}\{X_t \in D\}$ and $\tau^n = \inf_{t>0}\{X_t^n \in D\}$, where X_t^n denotes the by the Euler scheme approximated process. Then we can see that it holds for $x > 0$*

$$|\mathbb{P}^x(\tau \le T) - \mathbb{P}^x(\tau^n \le T)| \le \frac{1}{n^{1-u}}(1 + |\mathbb{P}^x(\tau^n = t)|_\infty)$$

Proof.

$$\mathbb{P}^x(\tau \le T) = \mathbb{E}^x[1_{\mathbb{R}^-}(X_T)] - \mathbb{E}^x\left[\int_0^T \delta_t(\tau)1_{\mathbb{R}^+}(X_T)dt\right]$$
$$\mathbb{P}^x(\tau^n \le T) = \mathbb{E}^x[1_{\mathbb{R}^-}(X_T^n)] - \mathbb{E}^x\left[\int_0^T \delta_t(\tau^n)1_{\mathbb{R}^+}(X_T^n)dt\right]$$

Thus it follows

$$\begin{aligned}
&\mathbb{P}^x(\tau \le T) - \mathbb{P}^x(\tau^n \le T) = \mathbb{E}^x[1_{\mathbb{R}^-}(X_T)] - \mathbb{E}^x[1_{\mathbb{R}^-}(X_T)] \\
&\quad + \sum_{k=1}^{[Tn]} \mathbb{E}^x\left[\mathbb{E}^{X_{k/n}}\left[\int_0^{1/n} \delta_t(\tau)1_{\mathbb{R}^+}(X_{(T-k)/n})dt\right]\right. \\
&\qquad\qquad \left. - \mathbb{E}^{X^n_{k/n}}\left[\int_0^{1/n} \delta_t(\tau)1_{\mathbb{R}^+}(X_{(T-k)/n})dt\right]\right] \\
&\quad + \sum_{k=0}^{[Tn]-1} \int_0^{1/n} \mathbb{E}^x\left[\mathbb{E}^{X^n_{k/n}}\left[\mathbb{E}^0[(\delta_{X^n_{k/n}}(X_{\tau\vee t}) - \delta_{X^n_{k/n}}(X^n_{\tau^n\vee t}))]1_{\mathbb{R}^+}(X_{T-(k/n)-t})\right]\right]dt \\
&\quad + \mathbb{E}^x\left[\mathbb{E}^{X^n_{k/n}}\left[\int_0^{1/n} \delta_t(\tau^n)\big(\mathbb{E}^{X^n_t}[1_{\mathbb{R}^+}(X_{T-(k/n)-t}) \mid \mathcal{F}_{0,t}]\right.\right. \\
&\qquad\qquad\qquad \left.\left. - \mathbb{E}^{X^n_t}[1_{\mathbb{R}^+}(X^n_{T-(k/n)-t}) \mid \mathcal{F}_{0,t}]\big)dt\right]\right]
\end{aligned}$$

The third difference arises by interpreting the entrance law as last exit time. Since $\delta_y(x) \in \mathcal{H}^{-(1,\infty)}$ (for the definition please see remark 5.3). Furthermore, the Doob s-tooping theorem should be added before applying Watanabe [31, Theorem 2-1]. Since $\delta_x(X_\tau) = 0$ for $x \neq 0$, the error is of order $(1/n)^{(2/2)-u}t^{-1/2}$, $u > 0$ arbitrary.

$$\begin{aligned}
&|\mathbb{P}^x(\tau \le T) - \mathbb{P}^x(\tau^n \le T)|q \\
&\le (1/n \vee T)^{(2/2)-u} + \sum_{k=1}^{[Tn]}(1/n \vee T)^{(2/2)-u}\left|\mathbb{E}^x\left[\int_0^{1/n} \delta_{t-k/n}(\tau)1_{\mathbb{R}^+}(X_{T-k/n})dt\right]\right|_\infty \\
&\quad + \sum_{k=0}^{[Tn]-1} \mathbb{E}^x\left[\int_0^{1/n} (1/n \vee T)^{(2/2)-u}t^{-1/2}dt\right] \\
&\quad + \sum_{k=0}^{[Tn]-1} \mathbb{E}^x\left[\int_0^{1/n} (1/n \vee T - (k/n) - t)^{(2/2)-u}\delta_t(\tau^n)|1_{\mathbb{R}^+}(x)|_\infty dt\right] \\
&\quad + \mathbb{E}^x\left[\int_0^{T-[Tn]/n} (1/n \vee T)^{(2/2)-u}t^{-1/2}\right. \\
&\qquad\qquad \left. + (1/n \vee T - (k/n) - t)^{(2/2)-u}\delta_t(\tau^n)|1_{\mathbb{R}^+}(x)|_\infty dt\right]
\end{aligned}$$

$$\leq (1/n \vee T)^{(2/2)-u}(1 + (1/n \vee T)^{-1/2} + |\mathbb{P}^x(\tau^n = t)|_\infty) \sim \left(\frac{1}{n}\right)^{1/2-u}. \qquad \square$$

References

[1] V. Bally: *On the connection between the malliavin covariance matrix and hörmander's condition*, J. Functional Anal. **96** (1991), 219–255.

[2] V. Bally and D. Talay: *The law of the Euler schema for stochastic differential equations: I convergence rate of the distribution function*, Probability Theory and Related Fields, **104** (1996), 43–60.

[3] R. Blumenthal: Excursions of Markov Processes, Birkhäuser Berlin, 1992.

[4] N. Bouleau and D. Lepingle: Numerical Methods for Stochastic Processes, Wiley Series in Probility and Mathematical Statistics, John Wiley and Sons, 1993.

[5] K. Burdzy: *Brownian excursions from hyperplans and smooth surfaces*, Trans. Amr. Math. Soc. **295** (1987), 35–57.

[6] K. Burdzy: *Multidimensional Brownian excursions and potential theory*, Pitman research notes in mathamtics series, **164**, Longman Scientific and Technical, 1987.

[7] C. Costantini: *The Skorohod oblique reflection problem in domains with corners and applications to stochastic differential equations*, Probability Theory and Related Fields, **91** (1992), 43–70.

[8] C. Costantini, B. Pacchierotti and F. Sartoretto: *Numerical Approximation for functionals of reflecting diffusion processes*, (submitted), May 1996.

[9] N.E. Karoui: *Processus de réflexion sur* $\mathbb{R}^n$, In Séminar de Probabilité IX, **465**, Lectures Notes in Mathematics, 194–203. Springer Verlag, 1975.

[10] W. Feller: An Introduction to Probility Theory and Its Applications, Wiley Series in Probility and Mathematical Statistics in Probility and Mathematical statistics, John Wiley and Sons, 2 edition, 1971.

[11] N. Ikeda and S. Watanabe: Stochastic differetial equations and diffusion processes, North-Holland, Amsterdam, 1981.

[12] N. Ikeda and S. Watanabe: *An introduction to Malliavin calculus*, Stochastic Analysis, Proceeding of the Taniguchi International Symposium on Stochastic Analysis 1982, Ito, K., (1983), 1–53.

[13] I. Karatzas and S.E. Shreve: Browninan Motion and Stochastic Calculus, Graduate Texts in Mathematics, **113**, Springer Verlag, 1 edition, 1987.

[14] P.E. Kloeden and E. Platen: Numerical Solution of Stochastic Differential Equations, Appli. Math. **23**, Springer Verlag, 1992.

[15] L.C.G. and D. Williams.: *Diffusions, Markov processes, and martingales*, Ito Calculus of Wiley Series in Probility and Mathematical Statistics, **2**, Springer Verlag, 1987.

[16] D. Lépingle: *Euler schema for reflected stochastic differential equations*, Math. and Computers in Simulation, **38** (1995), 119–126.

[17] P.L. Lions and A.S. Sznitzman: *Stochastic differential equations with reflecting boundary conditions*, Comm. Pure and Appl. Math. **37** (1984), 511–537.

[18] B. Maisonneuve: *Exit systems*, Annals of Probability, **3**(3) (1975), 399–411.

[19] H.P. McKean: *A skorohod integral equation for reflecting barrier diffusion*, J. Math. Kyoto Univ. **3** (1963), 86–88.

[20] J.L. Menaldi: *Stochastic variational inequality for reflected diffusion*, Indiana Univ. J. Math. **32**(5) (1983), 733–743.

[21] M. Motoo: *Application of additive functionals to the boundary problems of Markov processes (Levy's systems of u-processes)*, In Proc. 5th Berkley Symp. Math. Stat. Prob. **2**, 75–110. U. of California Press, 1965.

[22] D. Nualart: The Malliavin Calculus and related Topics, Springer Verlag, 1995.

[23] D. Revuz and M. Yor: *Continous Martingale and Brownian Motion*, Grundlehren der Mathematischen Wissenschaften, **293**, Springer Verlag, 1 edition, 1991.

[24] L.C.G. Rogers: *Ito excursion theory via resolvents*, Zur Wahrscheinlichkeit verwandte Gebiete, **63** (1983), 237–255.

[25] Y. Saisho: *Stochastic differentail equations for multi-dimensional domain with reflecting boundary*, Probability Theory and Related Fields, **74** (1987), 455–477.

[26] T.S. Salisburg: *On the Ito excursion processes*, Probability Theory and Related Fields, **73** (1986), 319–350.

[27] T.S. Salisburg: *Right processes from excursions*, Probability Theory and Related Fields, **73** (1986), 351–367.

[28] A.V. Skorohood: *Stochastic equation for diffusion processes in a bounded region*, Theory of Probability and its Appli. **6**(2) (1975), 264–274.

[29] L. Smolinski: *On existence, uniqueness and stability of solutions of multidimensional sde's with reflecting boundary conditions*, Annales de l'Institut Henri Poincaré, Probabilités et Statistiques, **29**(2) (1993), 163–198.

[30] L. Smolinski: *On approximation of solutions of multidimensional sde's with reflecting boundary conditions*, Stochastic processes and their Applications, **50** (1994), 197–219.

[31] S. Watanabe: *Analysis of wiener functionals (Malliavin calculus) and its applications to heat kernels*, Annals of Probability, **15** (1987), 1–39.

[32] S. Watanabe: *Construction of semimartingales from pieces by the method of excursion point porocesses*, Annales de l'Institut Henri Poincaré, Probabilités et Statistiques, **23**(2) (1987), 297–320.

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