

UNKNOTTING IMMERSED SURFACE-LINKS AND SINGULAR 2-DIMENSIONAL BRAIDS BY 1-HANDLE SURGERIES

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1. Introduction and results

By a *surface-link*, we mean a closed oriented immersed surface F in the 4-space R^4 whose singularities are transverse double points. An embedded surface-link is said to be *unknotted* if it bounds mutually disjoint handlebodies (possibly including 3-balls) in R^4 . We define a surface-link with double points to be *unknotted* if it is obtained from an unknotted embedded surface-link by adding several pieces of trivial positive/negative kink, as illustrated in Fig. 1.

A 3-ball h in R^4 is called a *1-handle attached to F* if $F \cap h$ is a pair of 2-disks on ∂h that is disjoint from the singularities of F ; the surgery results an orientable surface, say F' (cf. [3, 9, 20]). We assign F' the orientation induced from that of F and consider it a surface-link. It is known that every embedded surface-link is transformed into an unknotted one by surgery along 1-handles (cf. [9]). (There are interesting researches on 1-handle surgeries for embedded surface-links in [3, 9, 10, 17, 18, 23].) This result is generalized to surface-links, which are not necessarily embedded.

Theorem 1.1. *Every surface-link is transformed into an unknotted one by surgery along 1-handles.*

We notice that the proof of the above result given in [9] is not applied to our situation, which surface-links may have double points. In order to prove Theorem 1.1, we use 2-dimensional braids and their chart descriptions. (A 2-dimensional braid and its related topics are found in [4, 5, 8, 11, 13, 19, 21, 22, 24].) Viro's theorem [24] states that every embedded surface-link is ambient isotopic to a closed embedded 2-dimensional braid (cf. [13]). This is generalized to surface-links in our sense:

Theorem 1.2 ([16]). *Every surface-link is ambient isotopic to the closure of a singular 2-dimensional braid.*

We introduce the notion of unknottedness of a singular 2-dimensional braid, which

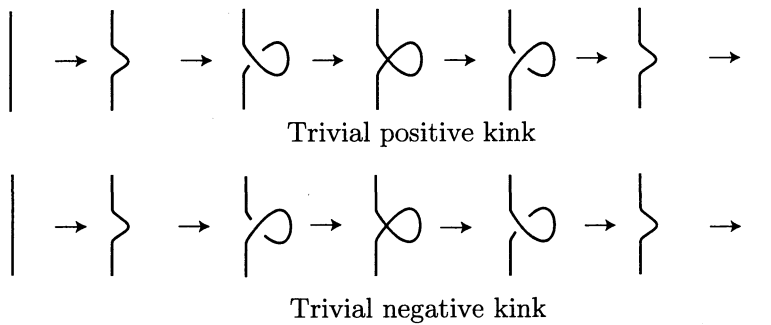


Fig. 1.

is a generalization of an unknotted embedded 2-dimensional braid defined in [11], and prove the following two theorems.

Theorem 1.3. *A surface-link is unknotted if and only if it is ambient isotopic to the closure of an unknotted singular 2-dimensional braid.*

Theorem 1.4. *Every singular 2-dimensional braid is transformed into an unknotted one by surgery along 1-handles.*

Theorem 1.1 is a direct consequence of Theorems 1.2–1.4. Our argument in this paper gives an explicit procedure to find 1-handles along which one can transform a given surface-link into an unknotted one by surgery. (The proof of Theorem 1.2 given in [16] gives a procedure to deform a surface-link into a closed braid. The proofs of Theorems 1.3 and 1.4 given here are very simple algorithm.) These theorems are announced in [16] without proofs. The purpose of this paper is to give the postponed proofs.

For a surface-link F (resp. a singular 2-dimensional braid S), Theorem 1.1 (resp. Theorem 1.4) enables us to define the *unknotting number* $u(F)$ (resp. $u(S)$) by the minimum number of 1-handle surgeries used for transforming F (resp. S) into an unknotted one. If F is ambient isotopic to the closure of a singular 2-dimensional braid S , then by Theorem 1.3 we have

$$u(F) \leq u(S).$$

From the proof of Theorem 1.4, we see the following.

Proposition 1.5. *Let S be a singular 2-dimensional braid of degree m and Γ a chart description of S . Then*

$$u(S) \leq w(\Gamma) + m - 1,$$

where $w(S)$ is the number of white vertices of Γ .

Although the inequality above may not give an accurate relation between the two invariants, its right hand term is explicitly calculated from the chart description.

This paper is organized as follows. We expound singular 2-dimensional braids and their chart descriptions in §2. In §3, the notion of an unknotted singular 2-dimensional braid is introduced and we prove Theorem 1.3. Theorem 1.4 is proved in §4. In §5, some examples are given.

2. Singular 2-dimensional braids and chart descriptions

Let D_1, D_2 be 2-disks and X_m a fixed set of m interior points of D_1 . We denote by $pr_i : D_1 \times D_2 \rightarrow D_i$ ($i = 1, 2$) the i th factor projection. A *singular 2-dimensional braid* of degree m is a compact oriented surface S in $D_1 \times D_2$ whose singularities are transverse double points such that

- (1) for an immersion $f : S_0 \rightarrow D_1 \times D_2$ associated with S , the composition $pr_2 \circ f$ is an m -fold branched covering map,
- (2) the boundary ∂S is the trivial closed m -braid $X_m \times \partial D_2$ in the solid torus $D_1 \times \partial D_2$, and
- (3) for each $y \in D_2$, $\sharp(S \cap pr_2^{-1}(y)) = m - 1$ or m .

A *singular point* of S means a double point of S or the image by f of a branch point of S_0 of the branched covering map $pr_2 \circ f$. The condition (3) above implies that each fiber $pr_2^{-1}(y) = D_1 \times \{y\}$ contains at most one singular point of S and that the branching index of each branch point is two (cf. [1, 2]). Two 2-dimensional braids are said to be *equivalent* if one is transformed into the other by a fiber-preserving isotopy of $D_1 \times D_2$ (as a D_1 -bundle over D_2) keeping $D_1 \times \partial D_2$ fixed. We often regard equivalent 2-dimensional braids as the same.

Let S be a singular 2-dimensional braid of degree m . Identify D_2 with the product $I_1 \times I_2$ of the unit intervals and put $b_t = S \cap (D_1 \times (I_1 \times \{t\}))$ for $t \in I_2 = [0, 1]$. Then b_t are m -braids in $D_1 \times (I_1 \times \{t\}) \cong D_1 \times I_1$ for all $t \in [0, 1]$ but a finite number of t 's. For each exceptional value t , the braid b_t is a *singular m -braid*, i.e., it has double points in its strands. We call $\{b_t\}$ ($t \in [0, 1]$) a *braid movie* of S . (It depends on the identification between D_2 and $I_1 \times I_2$.) An example is given in Fig. 2. From the second condition of the definition of a singular 2-dimensional braid, we see that b_0 and b_1 are trivial m -braids; for each $t \in [0, 1]$, the boundary ∂b_t is a pair of copies of X_m .

Each singular point of S corresponds to a singular point (double point) of a singular m -braid b_t for some $t \in [0, 1]$ in the braid movie. If the singular point of S is the image of a branch point, then in the braid movie a recombination of two strands occurs as in Fig. 3, which is called an *intercommutation* in [11]. If the singular point of S is a double point singularity, then in the braid movie a crossing change of two strands occurs.

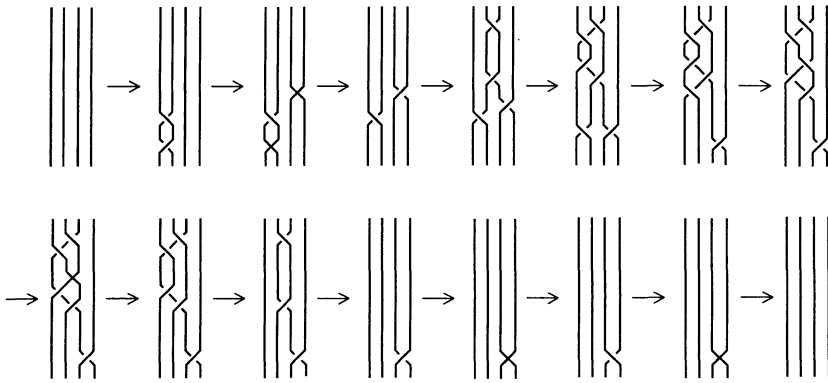


Fig. 2.

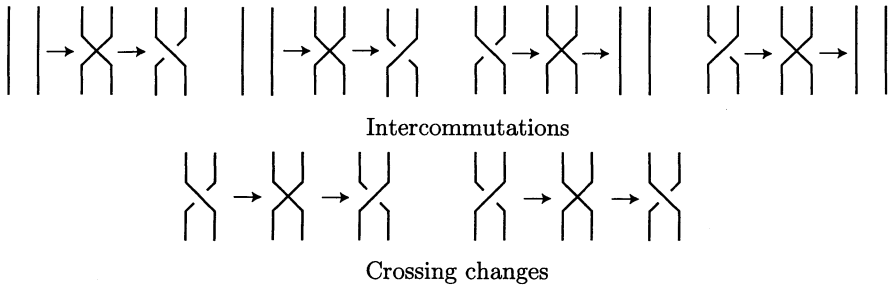


Fig. 3.

An m -chart is a (possibly empty) finite graph in the interior of D_2 , which may have *hoops* (that are closed edges without vertices), satisfying the following conditions:

- (1) Every vertex has degree one, two, four or six.
- (2) Every edge is directed, and labeled an integer in $\{1, 2, \dots, m - 1\}$.
- (3) For each vertex of degree six, three consecutive edges are directed inward and the other three are directed opposite, or outward; these six edges are labeled i and $i + 1$ alternately for some i .
- (4) For each vertex of degree four, edges in a diagonal position have the same label and are directed coherently; their labels i and j of the diagonals satisfy $|i - j| > 1$.
- (5) For each vertex of degree two, the two edges are labeled the same integer and directed noncoherently.

A vertex of degree one, two, four or six is called a *black* vertex, a *node*, a *crossing*, and a *white* vertex respectively. We often denote a white vertex by a fat vertex colored white and consider a crossing vertex to be a crossing point of two edges intersecting transversely, see Fig. 4. An edge attached to a white vertex is called a *middle edge* if it is the middle of the three consecutive edges directed either inward or

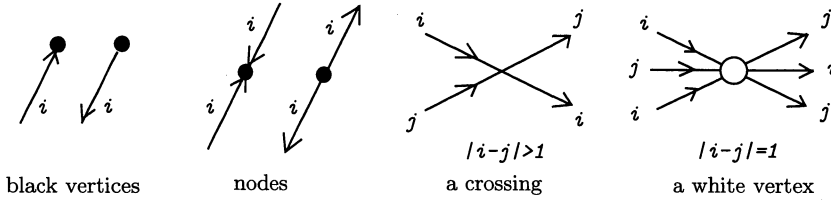


Fig. 4.

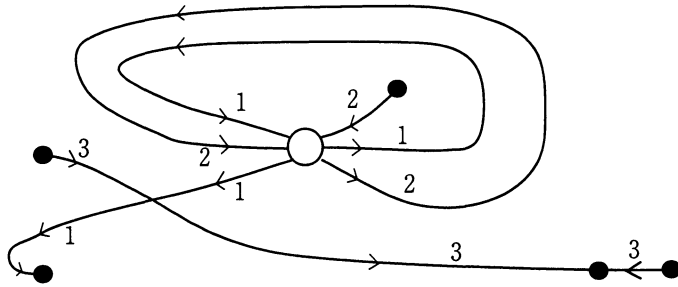


Fig. 5.

outward; otherwise a *non-middle edge*.

Let Γ be an m -chart. For a smooth path $a : [0, 1] \rightarrow D_2$ intersecting Γ transversely (missing all the vertices), assign an intersection a letter σ_i (resp. σ_i^{-1}) if its intersecting edge of Γ is labeled i and directed from left to right (resp. right to left) with respect to a . Reading off all these letters along a , we have a word on standard generators $\sigma_1, \dots, \sigma_{m-1}$ of the m -braid group B_m , and we denote this word by $w_\Gamma(a)$. There exists a unique, up to equivalence, singular 2-dimensional braid $\lambda(\Gamma)$ of degree m having the following properties (cf. [4, 5, 11, 12, 15, 16]):

- (1) The restriction of $\lambda(\Gamma)$ to $D_1 \times (D_2 \setminus N(\Gamma))$ is the product $X_m \times (D_2 \setminus N(\Gamma))$ for a regular neighborhood $N(\Gamma)$ of Γ .
- (2) For any smooth path $a : [0, 1] \rightarrow D_2$ such that $a(0), a(1) \in D_2 \setminus N(\Gamma)$ and it intersects Γ transversely, the geometric m -braid $b \subset D_1 \times [0, 1]$ determined by $pr_1(b \cap (D_1 \times \{t\})) = pr_1(\lambda(\Gamma) \cap (D_1 \times \{a(t)\}))$ for $t \in [0, 1]$ is presented by the word $w_\Gamma(a)$, where pr_1 are projections $D_1 \times [0, 1] \rightarrow D_1$ and $D_1 \times D_2 \rightarrow D_1$.

Conversely any singular 2-dimensional braid S of degree m is equivalent to $\lambda(\Gamma)$ for some Γ . Then we say that Γ *presents* S . For example, the singular 2-dimensional braid of degree 4 whose braid movie is illustrated in Fig. 2 is presented by a 4-chart depicted in Fig. 5.

Operations listed below (and their inverses) are called a C_{I^-} , C_{II^-} , C_{III^-} , C_{IV^-} and C_V -move respectively. Two m -charts are *C-move equivalent* if they are related by a finite sequence of such C-moves and ambient isotopies.

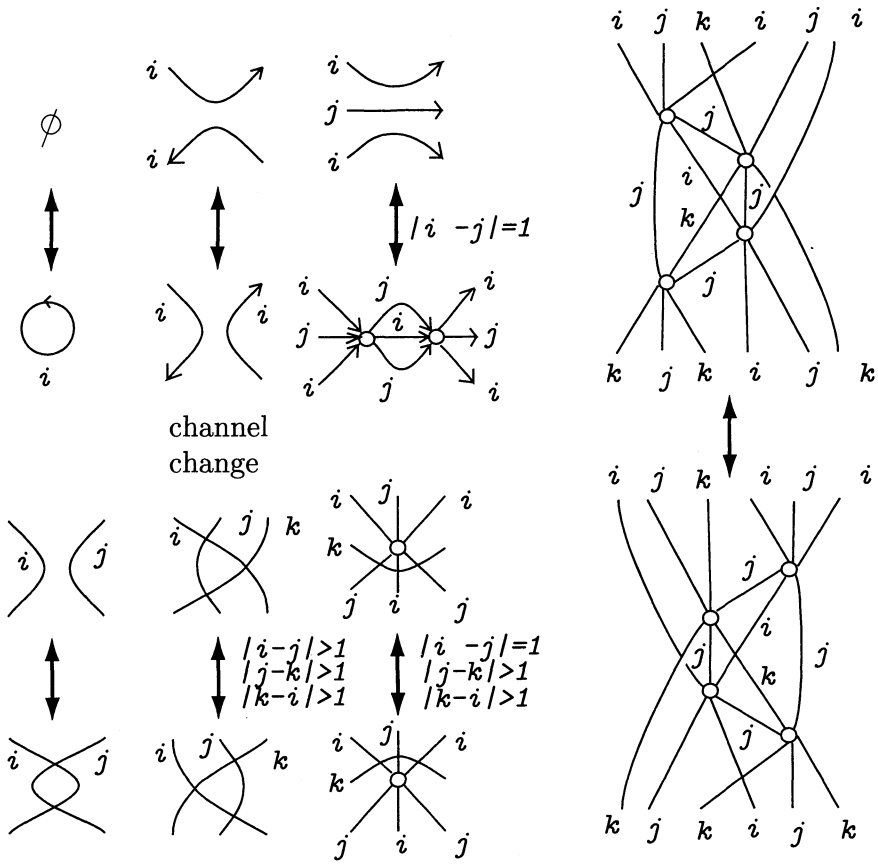


Fig. 6.

- (C_I) For a 2-disk E on D_2 such that $\Gamma \cap E$ has neither black vertices nor nodes, replace $\Gamma \cap E$ with arbitrary chart that has neither black vertices nor nodes.
- (C_{II}) Suppose that an edge e connects a crossing vertex v^4 and a black vertex v^1 . Remove e and v^4 , attach v^1 to the edge of v^4 opposite to e , and connect the other two edges naturally.
- (C_{III}) Let a black vertex v^1 and a white vertex v^6 be connected by a non-middle edge e of v^6 . Remove e and v^6 , attach v^1 to the edge of v^6 opposite to e , and connect other four edges in a natural way.
- (C_{IV}) Let a vertex v^4 of degree four and a node v^2 be connected by an edge. Transfer v^2 to its opposite side across v^4 .
- (C_V) Let a node v^2 and a white vertex v^6 be connected by a non-middle edge of v^6 . Transfer v^2 to its opposite side across v^6 .

We illustrate examples of C_I -moves in Fig. 6 and C_{II} – C_V -moves in Fig. 7. The second move in Fig. 6 is called a *channel change*. J. S. Carter and M. Saito [6] proved that any C_I -move is a consequence of the C_I -moves depicted in Fig. 6.

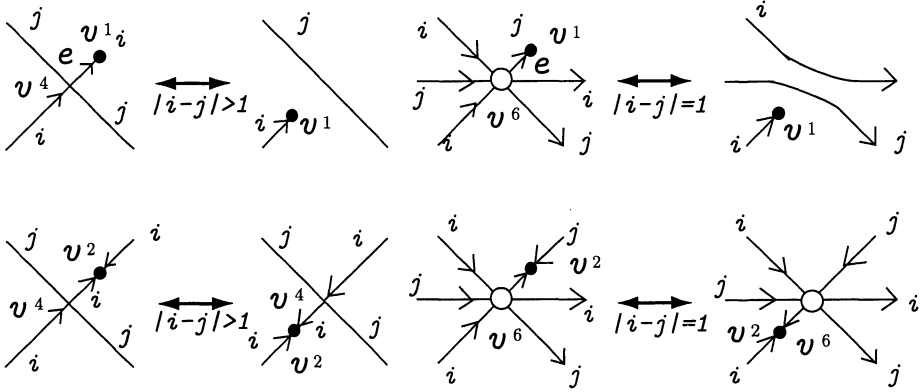


Fig. 7.

Lemma 2.1 ([15, 16]). *Two m -charts are C-move equivalent if and only if their presenting singular 2-dimensional braids are equivalent.*

Let S be a singular 2-dimensional braid of degree m . Embed $D_1 \times D_2$ into R^4 and assume that S is in R^4 . Then the boundary of S bounds naturally m trivial disks in $R^4 - \text{int}(D_1 \times D_2)$. The union of S and the m trivial disks form a surface-link. We call it the *closure* of S .

Theorem 1.2 ([16]). *Every surface-link is ambient isotopic to the closure of a singular 2-dimensional braid.*

If a surface-link F is ambient isotopic to the closure of a singular 2-dimensional braid presented by a chart Γ , then we say simply that Γ *presents* F .

Let Γ be an m -chart. Consider an m -chart which is obtained from Γ by adding some hoops parallel to ∂D_2 surrounding Γ . We say that the chart is obtained from Γ by a *conjugation*. The inverse operation is also called a conjugation.

Let Γ be an m -chart and Γ' an $(m + 1)$ -chart which is the union of Γ (regarded as an $(m + 1)$ -chart naturally) and a single edge labeled m whose endpoints are black vertices. Then we say that Γ' is obtained from Γ by a *stabilization*.

A singular 2-dimensional braid S' is said to be obtained from another S by a *conjugation* (resp. a *stabilization*) if they have chart descriptions Γ' and Γ such that Γ' is obtained from Γ by a conjugation (resp. stabilization). The notion of a conjugation and a stabilization defined here is equivalent to that in [14]. Thus we have the following lemma, which is also verified directly by considering braid movies.

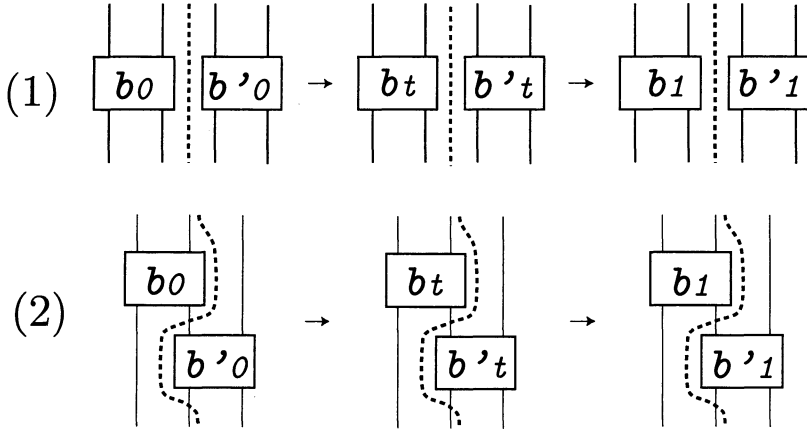


Fig. 8.

Lemma 2.2. *Suppose that a singular 2-dimensional braid S' is obtained from another S by a conjugation or a stabilization. Then their closures are ambient isotopic in R^4 .*

Let \mathcal{C}_m denote the set of m -charts in D_2 . For non-negative integers a and b , we define a natural injection

$$\iota_a^b : \mathcal{C}_m \rightarrow \mathcal{C}_{m+a+b}$$

such that $\iota_a^b(\Gamma)$ is an $(m+a+b)$ -chart obtained from Γ by adding the integer a to each label in Γ and regarding it as an $(m+a+b)$ -chart naturally.

Lemma 2.3 (Split sum and Knot sum). *Let F_1 and F_2 be surfaces-links presented by an m -chart Γ_1 and an n -chart Γ_2 . Then the $(m+n)$ -chart*

$$\iota_0^n(\Gamma_1) \amalg \iota_m^0(\Gamma_2)$$

presents a split sum of F_1 and F_2 , and an $(m+n-1)$ -chart

$$\iota_0^{n-1}(\Gamma_1) \amalg \iota_{m-1}^0(\Gamma_2)$$

presents a knot sum (connected sum) of them.

Proof. Let $\{b_t\}$ and $\{b'_t\}$ be braid movies of the 2-dimensional braids S_1 and S_2 presented by Γ_1 and Γ_2 . The braid movies depicted as (1) and (2) in Fig. 8 correspond to $\iota_0^n(\Gamma_1) \amalg \iota_m^0(\Gamma_2)$ and $\iota_0^{n-1}(\Gamma_1) \amalg \iota_{m-1}^0(\Gamma_2)$ respectively. Let D be a 3-ball in $D_1 \times D_2 = D_1 \times (I_1 \times I_2)$ such that for each $t \in [0, 1]$, the t -level section

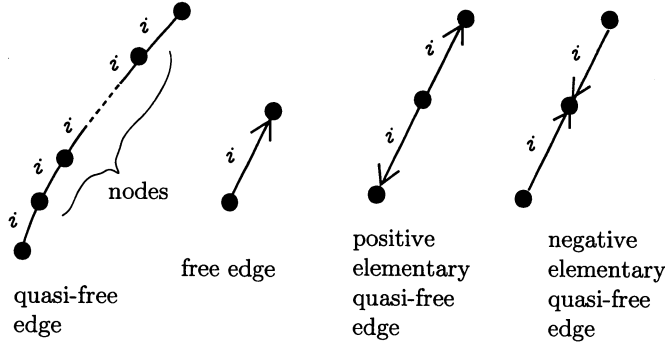


Fig. 9.

$D \cap (D_1 \times (I_1 \times \{t\}))$ is a 2-disk indicated by the dotted line in Fig. 8(1). In R^4 , the 3-ball D is extended to a 3-sphere which separates the closures of S_1 and S_2 . Thus the closure of the braid movie depicted in Fig. 8(1) is a split sum of F_1 and F_2 . Similarly we see that the closure of the braid movie depicted in Fig. 8(2) is a knot sum of F_1 and F_2 . \square

3. Unknotted singular 2-dimensional braids

Let Γ be a chart. A *quasi-free edge* means a smooth arc lying on Γ whose endpoints are black vertices and the other vertices on it are nodes (see Fig. 9). A *free edge* is a single edge whose endpoints are black vertices. A *positive* (resp. *negative*) *elementary quasi-free edge* is a quasi-free edge with a single node such that the two edges of it are directed outward (resp. inward). A *quasi-hoop* is a simple loop on Γ such that every vertex on it is a node.

DEFINITION. A singular 2-dimensional braid is *unknotted* if it is presented by a chart that is empty or the union of some quasi-free edges.

Let U_0, T_0, U_+ and U_- denote a standard 2-sphere, a standard torus, a Whitney 2-sphere of positive type, and of negative type in R^4 respectively (Fig. 10).

EXAMPLE 3.1.

- (1) The empty m -chart presents a trivial 2-dimensional braid of degree m whose closure is an unknotted surface, in R^4 , that is m parallel copies of the standard 2-sphere U_0 .
- (2) Let Λ_n ($n = 1, 2, \dots$) be a 2-chart consisting of n free edges. It presents an unknotted embedded surface in R^4 , which is a connected surface of genus $n - 1$ (cf. [11]). In particular, the standard torus T_0 is presented by Λ_2 .
- (3) Let Λ_+ (resp. Λ_-) be a 2-chart consisting of a single positive (resp. negative) el-

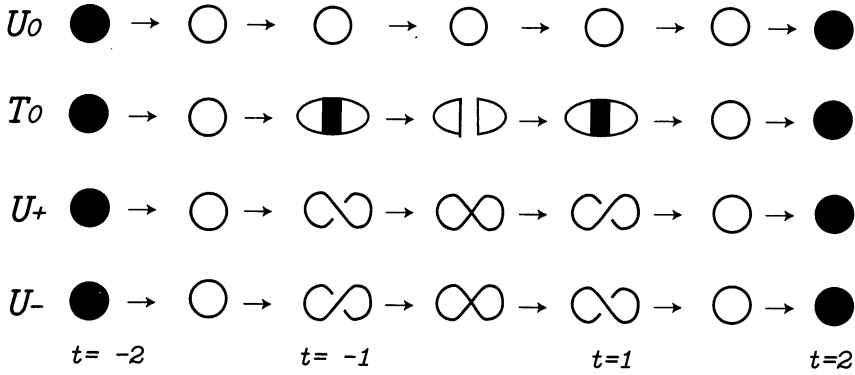


Fig. 10.

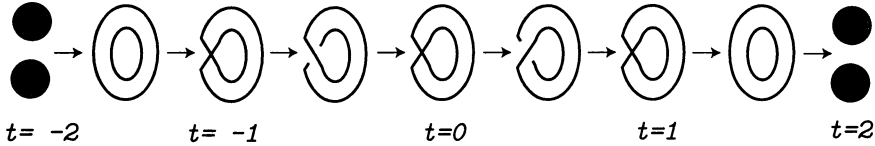


Fig. 11.

elementary quasi-free edge. The Whitney 2-spheres U_+ and U_- are presented by Λ_+ and Λ_- respectively. (The closure of a singular 2-dimensional braid presented by Λ_+ is illustrated in Fig. 11, which is ambient isotopic to U_+ .)

If an $(m + 1)$ -chart Γ' is obtained from an m -chart Γ by a stabilization, then by definition

$$\Gamma' = \iota_0^1(\Gamma) \amalg \iota_{m-1}^0(\Lambda_1).$$

By Lemma 2.3, a surface-link presented by Γ' is a knot sum of surface-links presented by Γ and Λ_1 . Since Λ_1 presents an unknotted embedded 2-sphere in R^4 , we see that a stabilization does not change the ambient isotopy class of the presenting surface-link (Lemma 2.2).

We notice that a surface-link is unknotted if and only if it is obtained from some copies of U_0, T_0, U_+ and U_- by knot sum and split sum operations. Thus the only if part of Theorem 1.3 follows from Lemma 2.3 and Example 3.1.

Lemma 3.2. *If $|i - j| = 1$, then each local operation illustrated in Fig. 12 is C-move equivalence.*

Proof. (a) A C_{III} -move transforms the left hand to the right hand with a hoop. Eliminate the hoop by a C_I -move.

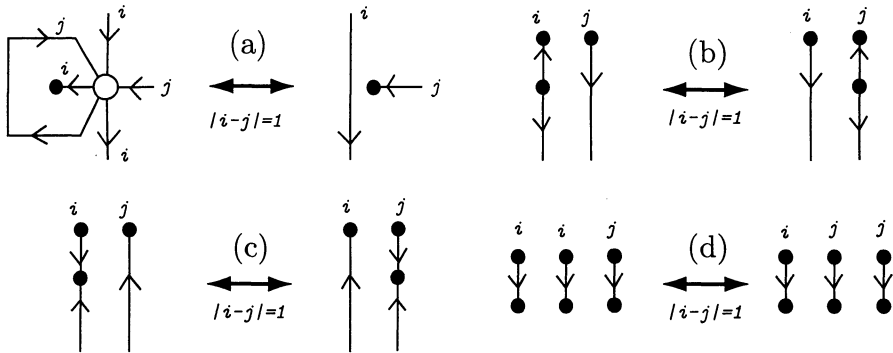


Fig. 12.

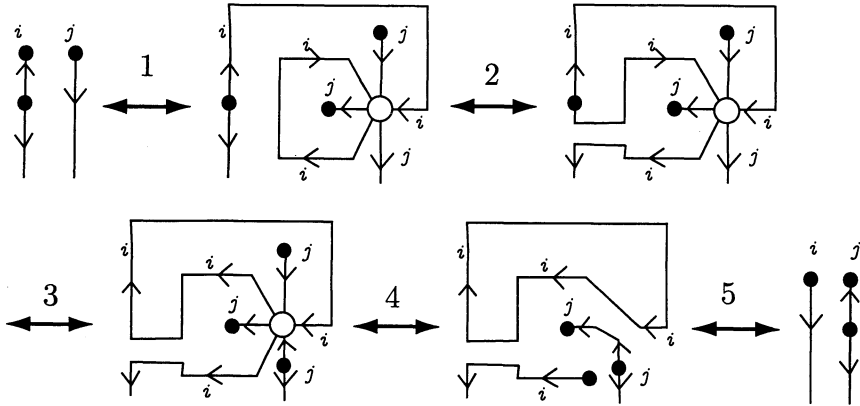


Fig. 13.

(b) See Fig. 13, where the Step 1 is an operation as in (a); the Step 2 is a channel change; the Step 3 is a C_V -move; the Step 4 is a C_{III} -move and the Step 5 is a C_I -move.

(c) This is similar to (b). For (d), see Fig. 14, where the Step 1 is (a); the Step 2 and 3 are channel changes; and the Step 4 is (a) again. \square

Proof of Theorem 1.3. As mentioned before, the only if part follows from Lemma 2.3 and Example 3.1. We prove the if part. Suppose that a surface-link F is presented by an m -chart Γ which is empty or the union of some quasi-free edges. If there exists an integer i in $\{1, \dots, m - 1\}$ such that there is no edge of Γ labeled i , then by Lemma 2.3 the surface-link F is the split sum of two closed singular 2-dimensional braids of degree m_1 and m_2 (with $m = m_1 + m_2$) presented by m_1 -charts Γ_1 and m_2 -charts Γ_2 respectively such that each Γ_i is empty or the union of some quasi-free edges. Thus, it suffices to prove that F is unknotted in a special case

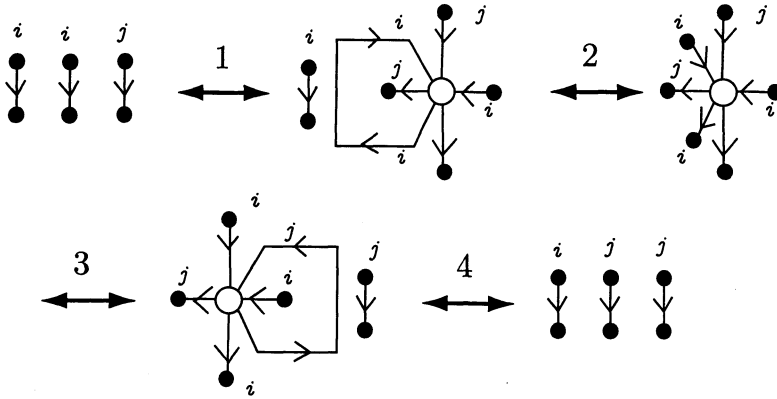


Fig. 14.

that all $1, \dots, m - 1$ appear as labels of the edges of Γ (equivalently F is connected). If $m = 1$, then Γ is empty and presents an unknotted 2-sphere. Assume that $m > 1$. In the case Γ has no nodes, by Lemma 3.2(d) we change Γ so that it is the union of n free edges each labeled 1 (for some $n > 0$) and $m - 2$ free edges each labeled $2, \dots, m - 1$. The chart is obtained from Λ_n by iteration of $m - 2$ stabilizations. Thus F is an unknotted surface-link, which is a connected surface of genus $n - 1$. Assume that Γ has $s (\neq 0)$ nodes. Let Γ' be an $(m + s)$ -chart obtained from Γ by stabilization s times, and then Γ' is $\iota_0^s(\Gamma)$ together with s free edges whose labels are $m, m + 1, \dots, m + s - 1$. By Lemma 3.2(b) and (c), all nodes are transferred from $\iota_0^s(\Gamma)$ to the additional free edges such that they become positive/negative elementary quasi-free edges. The result is an $(m + s)$ -chart obtained from Γ_0 by knot sum operation s times as in Lemma 2.3 with Λ_+ or Λ_- , where Γ_0 is obtained from Γ by removing its all nodes. Since Γ_0 presents an unknotted surface-link (by the previous argument) and Λ_+, Λ_- present U_+, U_- (Example 3.1), we see that F is an unknotted surface-link. \square

4. Unknotting procedure for surface-links

Lemma 4.1 ([12]). *Let S and S' be singular 2-dimensional braids presented by m -charts Γ and Γ' . If Γ' is obtained from Γ by inserting a free edge, then S' is obtained from S by surgery along a 1-handle.*

Lemma 4.2. *By C -moves and insertion of free edges, every non-empty chart is transformed into one that is the union of some quasi-free edges.*

Proof. For a white vertex W , let e be a non-middle edge of W . Insert a free edge f near e whose label is the same with e . Apply a channel change between f and

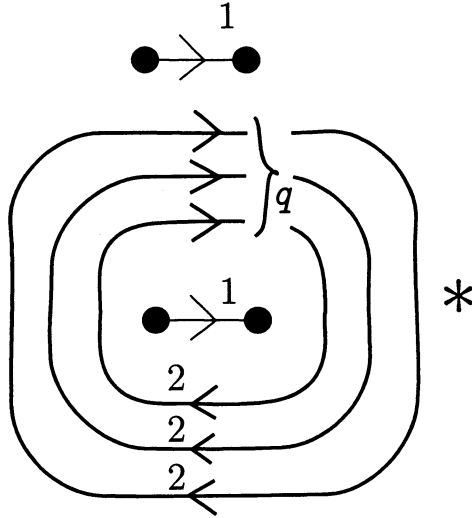


Fig. 15.

e so that the new edge e has a black vertex as the endpoint opposite to W . Delete the white vertex by a C_{III} -move. Remove each white vertex by repeating this procedure. By C_I -, C_{II} - and C_{IV} -moves, change Γ to the union of some quasi-free edges and quasi-hoops. Insert $m - 1$ free edges f_1, \dots, f_{m-1} with labels $1, \dots, m - 1$ far from Γ . Let α be an outermost quasi-hoop of Γ . We apply a channel change to α and f_i , where i is the label of α . The result is a quasi-free edge labeled i , which we denote by f_i again. In this way, all quasi-hoops are removed and we have a desired chart. (This trick is used in [12] on embedded 2-dimensional braids.) \square

REMARK. If there is a non-middle edge of a white vertex W whose endpoint opposite to W is a black vertex, then we can remove W by a C_{III} -move, without inserting a free edge. When we remove quasi-hoops from a chart Γ that is the union of some quasi-free edges and quasi-hoops, it is sufficient to insert free edges whose labels are the same with the quasi-hoops. Moreover, we can often use quasi-free edges of Γ to remove the quasi-hoops.

Proof of Theorem 1.4 and Proposition 1.5. Let S be a singular 2-dimensional braid presented by a chart Γ . If Γ is empty, then S is unknotted by definition. Otherwise, by Lemma 4.2 the chart is transformed into the one that is the union of some quasi-free edges by C -moves and by insertion of $w(\Gamma) + m - 1$ free edges. Recall that insertion of a single free edge corresponds to surgery along a 1-handle (Lemma 4.1) and that C -moves do not change the equivalence class of the singular 2-dimensional braid (Lemma 2.1). Thus we have the results. \square

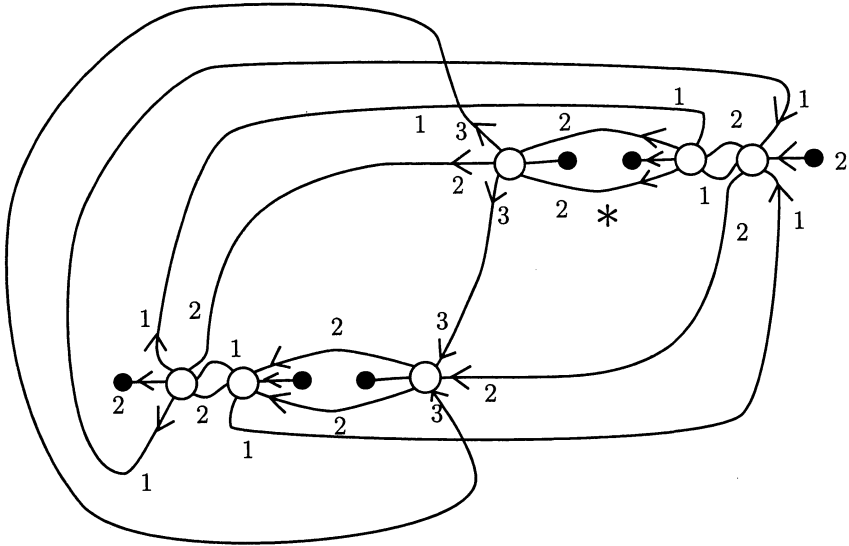


Fig. 16.

5. Examples

EXAMPLE 5.1. Let Γ be a 3-chart illustrated in Fig. 15 and F a surface-link presented by Γ . If the number q of the hoops is an odd integer, then F is a 2-knot which is the spun $(2, q)$ -torus knot [11]. Insert a free edge with label 2 at the asterisk; apply a channel change to the free edge and the outermost hoop. This changes the hoop to a free edge with label 2 again. Inductively all hoops are removed and the result consists of three free edges with labels 1, 1 and 2. By the inverse of a stabilization, it changes to Λ_2 , which presents a standard torus T_0 .

EXAMPLE 5.2. The 2-twist spun trefoil knot [25] is presented by a 4-chart as in Fig. 16, [11]. Insert a free edge with label 2 at the asterisk; apply a channel change to the free edge and an edge close to it, as shown in Fig. 17. Then there are two white vertices that are removed by C_{III} -moves. The result has such vertices again. We finally have a 4-chart consisting of four free edges labeled 1, 1, 2 and 3 (Fig. 17). It presents T_0 .

EXAMPLE 5.3. The Fenn-Rolfsen 2-link [7] is a 2-component surface-link that is not homotopically trivial, and its components are Whitney 2-spheres U_- and U_+ . This surface-link is presented by a 4-chart illustrated in Fig. 18, [16]. Insert a free edge with label 1 at the asterisk in the figure. Then the given surface-link is transformed into a chart presenting an unknotted surface-link. Verifying this is left to the readers as an exercise.

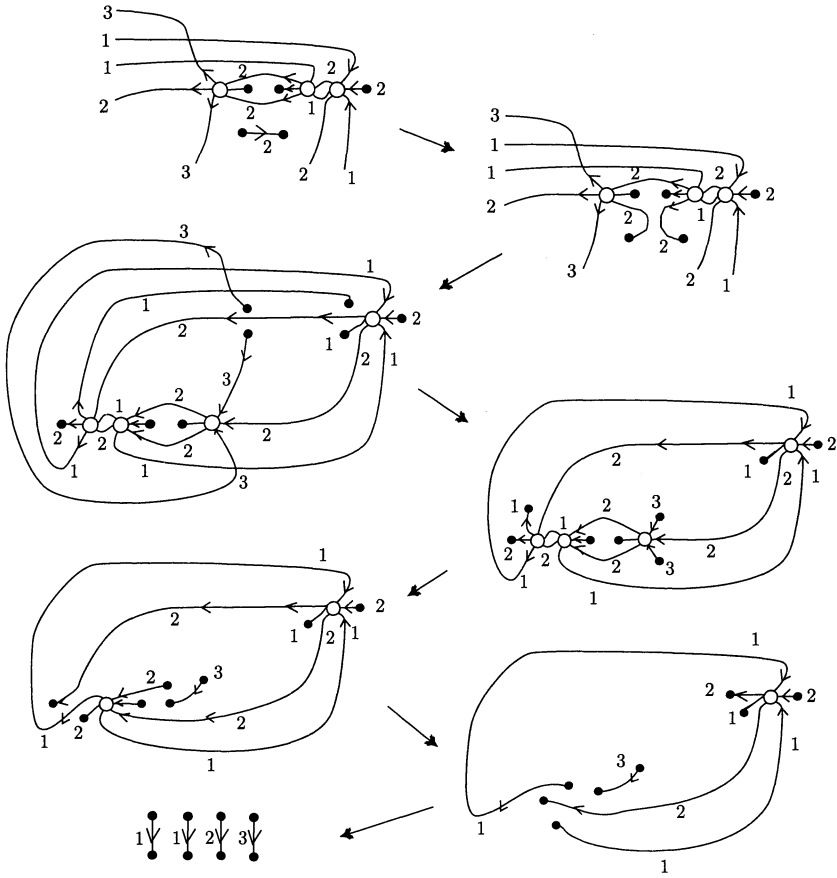


Fig. 17.

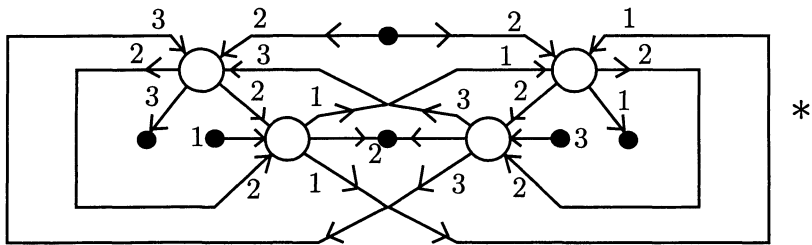


Fig. 18.

EXAMPLE 5.4. Let Γ be a 2-chart illustrated in Fig. 19(1), which presents a surface-link called a Montesinos twin (cf. [16]). Insert a free edge with label 1 and apply a channel change. By the proof of Theorem 1.3, we see that the result presents

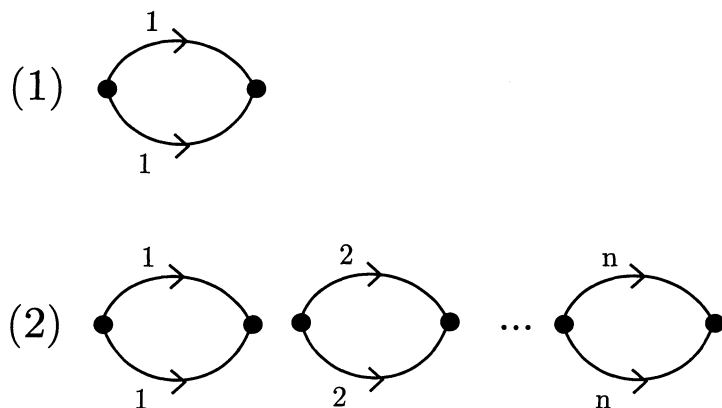


Fig. 19.

an immersed 2-sphere that is the knot sum of U_+ and U_- . Let Γ_n be an $(n+1)$ -chart illustrated in Fig. 19(2) and S_n the singular 2-dimensional braid of degree $n+1$ presented by Γ_n . This singular 2-dimensional braid presents a surface-link, say F_n , that is a knot sum of n copies of the Montesinos twin (Lemma 2.3). By an insertion of n free edges with labels $1, \dots, n$ and C-moves, the chart Γ_n changes to an $(n+1)$ -chart presenting an unknotted singular 2-dimensional braid. Thus $u(F_n) \leq u(S_n) \leq n$. On the other hand, $u(F_n) \geq n$. (Let F' be any surface-link obtained from F_n by surgery along less than n 1-handles. Then there are at least two components, say K_1 and K_2 , of F' such that K_1 intersects with K_2 . Thus F' is not unknotted.) Therefore we have $u(F_n) = u(S_n) = n$. This implies that the estimation in Proposition 1.5 is best possible.

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