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ON INFINITE DIMENSIONAL STOCHASTIC DIFFERENTIAL GAMES

Dedicated to Professor S. Watanabe on his sixtieth birthday

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1. Introduction

In this paper, we will study the relationship between infinite dimensional stochastic differential games (SDG in short) and Isaacs equations on Hilbert spaces. We deal with SDG for systems governed by some special stochastic partial differential equations (1.1). We define upper and lower semi-discrete approximations and, using a negative norm, show that their limits satisfy the dynamic programming principle [Theorems 3.2 and 3.3] and turn out to be unique viscosity solutions of associated Isaacs equations [Theorem 4.1].

For finite dimensional SDG, Fleming and Souganidis [3] proved that lower and upper value functions, in Elliott-Kalton sense, are unique viscosity solutions of associated Isaacs equations. Moreover, limit functions of upper and lower semi-discrete approximations coincide with upper and lower value functions reapectively. Since in our SDG the relationship between limit functions and value functions is still open, our results are partial extensions of [3] into an infinite dimensional one.

Let W_k, $k = 1, 2, \cdots$ be independent 1 dimensional Brownian motions, defined on a probability space (Ω, F, P), F_t denotes the σ-field generated by $\{\mathrm{W}_k(s), s \leq t, k = 1, 2, \cdots\}$. Let D be a bounded open domain of $\mathbb{R}^n$ with smooth boundary. We put $H = L^2(D)$, $\| \ \|$ = its norm and

$$A\zeta = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a^{ij}(x)\frac{\partial \zeta}{\partial x_j}\right) + \sum_{i,j=1}^{n} r^i(x)\frac{\partial \zeta}{\partial x_i} - c(x)\zeta.$$

Let $\boldsymbol{Y}$ and $\boldsymbol{Z}$ be compact convex subsets of $L^2(D, \mathbb{R}^L)$ and $L^2(D, \mathbb{R}^M)$ respectively. Processes taking vales in $\boldsymbol{Y}$ and $\boldsymbol{Z}$ are called admissible controls of players I and II respectively, if they are F_t-progressively measurable and right continuous processes with left limits. $\mathcal{Y}$ (resp. $\mathcal{Z}$) denotes the set of admissible controls of player I (resp. II).

When players I and II apply admissible controls $Y(\circ)$ and $Z(\circ)$ respectively, the system $X(\circ)$ evolves according to the following stochastic partial differential equation on a fixed time interval $[0,T]$,

$$(1.1)\qquad dX(t,x) = (AX(t,x) + b(x, X(t,x), Y(t,x), Z(t,x)))dt + dM(t,x) \qquad 0 < t < T, \quad x \in D,$$

with intial condition $X(0,x) = \eta(x)$

and boundary condition $X(t,x) = 0, \quad x \in \mathrm{bdy}(D)$

where a random force M is an H-valued colored noise of the form

$$M(t,x) = \sum_{k=1}^{\infty} \sqrt{m_k} e_k(x) W_k(t)$$

with a finite sum $\sum m_k$ ($= \bar{m}$ put) and a smooth orthonormal base $\{e_k, k = 1, 2, \cdots\}$ of H. Defining $\beta; H \times \boldsymbol{Y} \times \boldsymbol{Z} \to H$ by

$$(1.2)\qquad \beta(\zeta, y, z)(x) = b(x, \zeta(x), y(x), z(x)),$$

we can regard (1.1) as the stochastic differential equation (1.3) on the Hilbert space H [2], [5], [7].

$$(1.3)\qquad \begin{aligned} &dX(t) = (AX(t) + \beta(X(t), Y(t), Z(t)))dt + dM(t), \qquad 0 < t < T \\ &X(0) = \eta. \end{aligned}$$

Let us define the pay-off by

$$J(t, \eta; q, Y, Z) = E \int_0^t h(X(s), Y(s), Z(s))ds + q(X(t))$$

where X is a solution of (1.3), (see Definition 2.1).

In our game, player I controls $(Y\circ)$ and wishes to maximize $J(\circ)$. On the other hand, player II controls $Z(\circ)$ and tries to minimize $J(\circ)$. $L(H)$ denotes the space of continuous linear transformations on H with the usual norm (put $|\ |$). Defining $S \in L(H)$ by $Se_k = m_k e_k$, $k = 1, 2, \cdots$, and introducing semi-discrete approximations, from above and below,we will show that their limites, V and v, turn out to be unique viscosity solutions of Isaacs equations (1.4) and (1.5) respectively.

$$(1.4)\qquad \begin{aligned} &\frac{\partial V}{\partial t}(t,\eta) - \langle A^* \partial V(t,\eta), \eta \rangle - \inf_{z \in Z} \sup_{y \in Y} (\langle \partial V(t,\eta), \beta(\eta, y, z) \rangle \\ &\quad + h(\eta, y, z)) - \frac{1}{2} \mathrm{trace} S \partial^2 V(t,\eta) = 0, \qquad 0 < t < T, \quad \eta \in H, \\ &V(0) = q. \end{aligned}$$

$$(1.5)\qquad \frac{\partial v}{\partial t}(t,\eta) - \langle A^* \partial v(t,\eta), \eta \rangle - \sup_{y \in Y} \inf_{z \in Z} (\langle \partial v(t,\eta), \beta(\eta, y, z) \rangle$$

$$+ h(\eta, y, z)) - \frac{1}{2}\mathrm{trace} S\partial^2 v(t,\eta) = 0, \qquad 0 < t < T, \quad \eta \in H,$$
$$v(0) = q.$$

where ∂ = Fréchet derivative, A^* = adjoint of A and $\langle\ ,\ \rangle$ = duality pair between H^{-1} and $H_0^1(D)$.

Section 2 is devoted to study of properties of solutions of (1.3). In section 3, we introduce semi-discrete approximation and show the dynamic programming principle for limit functions. Isaacs equations will be treated in section 4.

2. Preliminaries

Let us assume the following conditions (A1)~(A6),

(A1) a^{ij} and r^i are in $C^3(\bar{D})$

(A2) $n \times n$ matrix $(a^{ij}(x))$ is uniformly positive definite, say,

$$\sum_{i,j=1}^{n} a^{ij}(x) t_i t_j \geq \lambda_0 |t|^2 \qquad \text{for } t = (t_1, \cdots, t_n) \in \mathbb{R}^n, \text{ with } \lambda_0 > 0$$

(A3) $c(\circ)$ is in $C(\bar{D})$

(A4) $\inf_{x \in D} c(x) > \sum_{i=1}^{n} \sup_{x \in D} |r^i(x)|/4\lambda_0$

(A5) $b; \bar{D} \times \mathbb{R}^1 \times \mathbb{R}^L \times \mathbb{R}^M \to \mathbb{R}^1$ is bounded and Lipschitz continuous

(A6) $h; H \times \boldsymbol{Y} \times \boldsymbol{Z} \to \mathbb{R}^1$ is bounded and Lipshitz continuous, say

$$\bar{h} = \sup_{\zeta y z} |h(\zeta, y, z)| \text{ and } |h(\zeta, y, z) - h(\tilde{\zeta}, \tilde{y}, \tilde{z})| \leq \ell\{\|\zeta - \tilde{\zeta}\| + |y - \tilde{y}|_1 + |z - \tilde{z}|_2\}$$

where $|\ |_1$ and $|\ |_2$ are norms in $\boldsymbol{Y}$ and $\boldsymbol{Z}$ respectively. Put H^k = Sobolev space $H_0^k(D)$ and $\|\ \|_k$ = its norm. The operator A can be regarded as a linear mapping $H^1 \to H^{-1}$ satisfying the coercive condition,

$$\langle -A\zeta, \zeta\rangle \geq \lambda \|\zeta\|_1^{\,2} \geq 0 \qquad \text{for } \zeta \in H^1 \tag{2.1}$$

with a positive constant λ, by (A2) and (A4). Moreover β of (1.2) is bounded and Lipshitz continuous, by (A5) say

$$\bar{\beta} = \sup_{\zeta y z} \|\beta(\zeta, y, z)\|$$
$$\text{and} \quad \|\beta(\zeta, y, z) - \beta(\tilde{\zeta}, \tilde{y}, \tilde{z})\| \leq \mu\{\|\zeta - \tilde{\zeta}\| + |y - \tilde{y}|_1 + |z - \tilde{z}|_2\}.$$

Denoting by $M^2(0,T;H^1)$ the subset of $L^2([0,T] \times \Omega; H^1)$ consisting of F_t-progressively measurable processes, we will define a solution of (1.3).

DEFINITION 2.1. $X \in M^2(0,T;H^1)$ is called a solution of (1.3), if $X \in$

$C([0,T];H)$ a.s. and, for any t and smooth function ϕ with support in D,

$$\langle X(t),\phi\rangle = \langle \eta,\phi\rangle + \int_0^t \langle AX(s),\phi\rangle + \langle \beta(X(s),Y(s),Z(s)),\phi\rangle ds + \langle M(t),\phi\rangle, \text{ a.s.}$$

Let us show the outline of proof of unique existence of solution, using the usual successive approximation. Since $M(t)$ is a continuous martingale ([2, Proposition 3.5]), we can get the unique solution X_n, $n = 1, 2, \cdots$, of the following stochastic differential equation on H, putting $X_0(t) = 0$,

$$\begin{aligned} &dX_n(t) = (AX_n(t) + \beta(X_{n-1}(t), Y(t), Z(t)))dt + dM(t), \qquad 0 < t < T \\ &X_n(0) = \eta \quad (\in H) \end{aligned}$$

with the following evaluation

$$E\left(\sup_{t\le T} \|X_n(t)\|^2 + \int_0^T \|X_n(t)\|_1^2\, dt\right) \le K(\|\eta\|^2 + \bar{\beta}^2 T^2 + \bar{m}T)$$

where K is independent of n ([7, Theorem 4 in § 3.1]). On the other hand, (2.1) and Lipshitz continuity of β derive

$$\|X_{n+1}(t) - X_n(t)\|^2 \le \frac{\text{const.}}{n!} \|X_1(t)\|^2$$

and

$$\int_0^s \|X_{n+1}(t) - X_n(t)\|_1^2\, dt \le \frac{\text{const.}}{n!} \int_0^s \|X_1(t)\|_1^2\, dt.$$

Therefore we have

$$\sum_{n=1}^{\infty} \left(\sup_{t\le T} \|X_{n+1}(t) - X_n(t)\| + \int_0^T \|X_{n+1}(t) - X_n(t)\|_1\, dt\right) < \infty \quad \text{a.s.}$$

So, $X_n(t)$ converges uniformly in t and its limit $X(t)$ turns out to be a solution. The uniqueness is also proved by the routine.

Proposition 2.1. *There is a unique solution* $X(\ ;\eta,Y,Z)$ *of* (1.3) *having the following property*

$$E\left(\sup_{t\le T} \|X(t;\eta,Y,Z)\|^2 + \int_0^T \|X(s;\eta,Y,Z)\|_1^2\, ds\right) \le K_1(\|\eta\|^2 + \bar{\beta}^2 T^2 + \bar{m}T) \quad (\le K_2(\|\eta\|^2 + 1) \text{ say}) \tag{2.2}$$

where K_1 and K_2 are independent of Y and Z.

The operator $B; H \to H^2$ defined by

$$B = \left[I - \left(A - \sum_{i=1}^{n} r^i \frac{\partial}{\partial x_i} \right) \right]^{-1} \qquad \text{with boundary value } 0$$

is a compact operator on H. Moreover, A^*B is a bounded operator on H and the following structural condition holds,

$$\langle -A^*B\phi, \phi \rangle \geq \frac{1}{2} \|\phi\|^2 - p\,|\phi|_B{}^2$$

with a constant $p \geq 0$, where $|\phi|_B{}^2 = \langle B\phi, \phi \rangle$.

Since the dynamics of $X(t; \eta, Y, Z) - X(t; \tilde{\eta}, Y, Z)$ does not depend on the random noise $M(\circ)$, we can see the following propositions, employing standard arguments.

Proposition 2.2 ([6, Theorems 1 and 2]). *With probability* 1

$$\text{(2.3)} \quad \sup_{t \leq T} \|X(t; \eta, Y, Z) - X(t; \tilde{\eta}, Y, Z)\| \leq K_3 \|\eta - \tilde{\eta}\|$$

$$\text{(2.4)} \quad \sup_{t \leq T} |X(t; \eta, Y, Z) - X(t; \tilde{\eta}, Y, Z)|_B^2 + \int_0^T \|X(s; \eta, Y, Z) - X(s; \tilde{\eta}, Y, Z)\|^2 \, ds \leq K_4 \,|\eta - \tilde{\eta}|_B{}^2$$

hold, where K_3 and K_4 are independent of Y, Z and $\omega \in \Omega$. Moreover the solution depends on admissible controls continuously.

Proposition 2.3. *With probability* 1

$$\text{(2.5)} \quad \left|X(t; \eta, Y, Z) - X(t; \eta, \tilde{Y}, \tilde{Z})\right|_B^2 \leq \left\|X(t; \eta, Y, Z) - X(t; \eta, \tilde{Y}, \tilde{Z})\right\|^2 \leq K_5 \int_0^t \left|Y(s) - \tilde{Y}(s)\right|_1^2 + \left|Z(s) - \tilde{Z}(s)\right|_2^2 ds$$

holds, with a constant K_5 independent of η, t and $\omega \in \Omega$.

Next we will study the continuity w.r.to time of $X(t) = X(t; \eta, Y, Z)$. For fixed s, we have

$$d\,|X(t) - X(s)|_B{}^2 = \langle dX(t), B(X(t) - X(s)) \rangle + \frac{1}{2}\,|dX(t)|_B{}^2$$

$$= \Big(\langle X(t) - X(s), A^*B(X(t) - X(s)) \rangle + \langle X(s), A^*B(X(t) - X(s)) \rangle$$

$$+ \langle \beta(X(t), Y(t), Z(t)), B(X(t) - X(s)) \rangle + \frac{1}{2} \sum m_i \, |e_i|_B{}^2 \Big) \, dt$$

$$+ \sum \sqrt{m_i} \langle e_i, dB(X(t) - X(s)) \rangle dW_i(t).$$

Therefore, the structural condition yields

Proposition 2.4. *There are two constants K_6 and K_7 independent of s, η, Y and Z, such that*

$$E(|X(t;\eta, Y, Z) - X(s;\eta, Y, Z)|_B{}^2 / F_s) \le K_6(\|X(s;\eta, Y, Z)\|^2 + 1)|t - s|$$
$$E(|X(t;\eta, Y, Z) - X(s;\eta, Y, Z)|_B{}^2) \le K_7(\|\eta\|^2 + 1)|t - s|.$$

We need a finer evaluation in the case of $s = 0$. Let us divide (1.3) into two parts, (2.6) and (2.7)

$$d\xi(t) = A\xi(t)dt + dM(t), \qquad \xi(0) = 0 \tag{2.6}$$

$$d\zeta(t) = (A\zeta(t) + \beta(X(t), Y(t), Z(t)))dt, \qquad \zeta(0) = \eta. \tag{2.7}$$

Since $X(t)$ is a known process, both equations have unique solutions. Moreover we have

$$\zeta(t) = e^{tA}\eta + \int_0^t e^{(t-s)A} \beta(X(s), Y(s), Z(s)) ds.$$

Therefore there is a constant K_8 independent of η, Y, Z and ω, such that

$$\sup_{t \le \theta} \|\zeta(t) - \eta\| \le \sup_{t \le \theta} \|e^{tA}\eta - \eta\| + K_8 \bar{\beta} \theta. \tag{2.8}$$

On the other hand, Ito's formula says

$$\|\xi(t)\|^2 \le \bar{m}t + 2 \int_0^t \langle \xi(s), dM(s) \rangle$$

by the condition (2.1). Hence

$$\|\xi(t)\|^4 \le 2\bar{m}^2 t^2 + 8 \left(\int_0^t \langle \xi(s), dM(s) \rangle \right)^2.$$

Now martingale inequality [4] yields

$$E(\sup_{t \le \theta} \|\xi(t)\|^4) \le K_9 \theta^2. \tag{2.9}$$

Noting $X(t) = \xi(t) + \zeta(t)$, we can easily see

$$E(\sup_{t\le\theta} \|X(t;\eta,Y,Z) - \eta\|^4) \le K_{10}(\sup_{t\le\theta} \|e^{tA}\eta - \eta\|^4 + \theta^2)$$

with K_{10} independent of Y and Z. Setting $\tau(\eta,d)$ = exit time from the ball, $\{\zeta \in H; \|\zeta - \eta\| \le d\}$, and fixing small $\tilde{\theta} = \tilde{\theta}(\eta,d)$ such that

$$\tilde{\theta} < d/(3\bar{\beta}K_8) \quad \text{and} \quad \sup_{t\le\tilde{\theta}} \|e^{tA}\eta - \eta\| < \frac{d}{3},$$

we get, by (2.8) and (2.9)

$$\begin{aligned} P(\tau(\eta,d) < s) &= P(\sup_{t<s} \|X(t;\eta,Y,Z) - \eta\| > d) \\ &\le P\left(\sup_{t<s} \|\xi(t)\| > \frac{d}{3}\right) \le 4K_9 s^2/d^4 \qquad \text{for} \quad s < \tilde{\theta}. \end{aligned}$$

Proposition 2.5.

$$P(\tau(\eta,d) < s) \le K_{11}s^2/d^4 \qquad \text{for} \quad s < \tilde{\theta}(\eta,d)$$

with a constant K_{11} independent of η, d, Y and Z.

3. Semi-discrete approximation

According to [3], we will define a semi-descretization of game with equipartition of $[0,T]$. An admissible control Y for player I is called Δ-step, if $Y(t) = y$ for $t \in [0,\Delta)$ with $y \in \boldsymbol{Y}$ and $Y(s) = Y(k\Delta)$ for $s \in [k\Delta,(k+1)\Delta)$. For $\Delta = 2^{-N}T$, the set of Δ-step admissible controls for player I is denoted by $\mathcal{Y}_N$. The Δ-step admissible control for player II is defined in a similar way and their collection is denoted by $\mathcal{Z}_N$. Hereafter we put $\Delta = 2^{-N}\mathrm{T}$.

DEFINITION 3.1.

(i) Δ-step strategy for player I is a mpping $\alpha : \mathcal{Z} \to \mathcal{Y}_N$ such that
  (1) $\alpha(Z)(t), t \in [0,\Delta)$, does not depend on Z and t.
  (2) if $P(Z(s) = \tilde{Z}(s)) = 1$ for $s \in [0,k\Delta)$, then $\alpha(Z)(k\Delta) = \alpha(\tilde{Z})(k\Delta)$, a.s. for $k = 1,2,\cdots,2^N$.

(ii) $\alpha; \mathcal{Z} \to \mathcal{Y}$ is called an elementary strategy (e-strategy in short) of player I, if
  (1) α is non-anticipative, namely "$P(Z(s) = \tilde{Z}(s)) = 1$ for $s < t$" implies $P(\alpha(Z)(t) = \alpha(\tilde{Z})(t)) = 1$
  (2) for any $\varepsilon > 0$, there is an approximate step strategy α_ε such that

$$\sup_{s\le T} \sup_{Z\in\mathcal{Z}} E|\alpha(Z)(s) - \alpha_\varepsilon(Z)(s)|_1^2 < \varepsilon. \tag{3.1}$$

For player II, Δ-step straregy $\gamma; \mathcal{Y} \to \mathcal{Z}_N$ and e-strategy $\gamma; \mathcal{Y} \to \mathcal{Z}$ are defined in a similar way. $\mathcal{A}_N$ and $\mathcal{A}$ (resp. $\mathcal{R}_N$ and $\mathcal{R}$) denote the sets of Δ-step strategies and e-strategies of player I (resp. II) respectively.

Proposition 3.1 (See Proof of (2.3) in [3]). *For any $\alpha \in \mathcal{A}$ and $\gamma \in \mathcal{R}_N$, there exist $\tilde{Y} \in \mathcal{Y}$ and $\tilde{Z} \in \mathcal{Z}_N$ such that*

$$\alpha(\tilde{Z})(t) = \tilde{Y}(t) \quad \textit{and} \quad \gamma(\tilde{Y})(t) = \tilde{Z}(t) \qquad \textit{on } [0,T].$$

Let us set

$$\begin{aligned} Q = \{q; H \to \mathbb{R}^1, &\textit{bounded and Lipschitz continuous w. r. to } |\ |_B, \\ &\textit{say } \bar{q} = \sup_{\eta} |q(\eta)| \quad \textit{and} \quad |q(\eta) - q(\tilde{\eta})| \leq L_q |\eta - \tilde{\eta}|_B\}. \end{aligned}$$

For a given $q \in Q$, the pay-off J satisfies

$$|J(t,\eta;q,Y,Z)| \leq \bar{h}t + \bar{q}$$

$$|J(t,\eta;q,Y,Z) - J(t,\tilde{\eta};q,Y,Z)| \leq c_1(1+L_q)|\eta - \tilde{\eta}|_B \tag{3.2}$$

$$|J(t,\eta;q,Y,Z) - J(s,\eta;q,Y,Z)| \leq c_2(\|\eta\| + 1 + L_q)\sqrt{|t-s|} \tag{3.3}$$

$$\begin{aligned} &|J(t,\eta;q,Y,Z) - J(t,\eta;q,\tilde{Y},\tilde{Z})| \\ &\quad \leq c_3(1+L_q)[E\int_0^t (|Y(s) - \tilde{Y}(s)|_1{}^2 + |Z(s) - \tilde{Z}(s)|_2{}^2)ds]^{1/2}, \end{aligned}$$

where c_i, $i = 1, 2, 3$, are independent of t, η ; q, Y and Z, by (2.3)~(2.5).

Putting $J(t,\eta;q,Y,\gamma) = J(t,\eta;q,Y,\gamma Y)$ and $J(t,\eta;q,\alpha,Z) = J(t,\eta;q,\alpha Z,Z)$ for simplicity, we define semi-discrete approximations, V_N and v_N, by

$$\begin{aligned} V_N(t,\eta;q) &= \inf_{\gamma \in \mathcal{R}_N} \sup_{Y \in \mathcal{Y}} J(t,\eta;q,Y,\gamma) \\ v_N(t,\eta;q) &= \sup_{\alpha \in \mathcal{A}_N} \inf_{Z \in \mathcal{Z}} J(t,\eta;q,\alpha,Z). \end{aligned}$$

From the definitions, we can easily see that V_N (resp. v_N) is decreasing (resp. increasing), as $N \to \infty$, and

$$\begin{aligned} \lim_{N\to\infty} V_N(t,\eta;q) &= \inf_{\gamma \in \mathcal{R}} \sup_{Y \in \mathcal{Y}} J(t,\eta;q,Y,\gamma) \quad (= V(t,\eta;q) \text{ say}) \\ \lim_{N\to\infty} v_N(t,\eta;q) &= \sup_{\alpha \in \mathcal{A}} \inf_{Z \in \mathcal{Z}} J(t,\eta;q,\alpha,Z) \quad (= v(t,\eta;q) \text{ say}) \end{aligned}$$

Moreover, we have, by (3.2) and (3.3), for $N = 1, 2, \cdots$,

$$|V_N(t,\eta;q) - V_N(t,\tilde{\eta};q)| \leq c_1(1+L_q)|\eta - \tilde{\eta}|_B \tag{3.4}$$

$$
\begin{aligned}
|V(t,\eta;q) - V(t,\tilde{\eta};q)| &\le c_1(1+L_q)|\eta-\tilde{\eta}|_B \\
|V_N(t,\eta;q) - V_N(s,\eta;q)| &\le c_2(\|\eta\|+1+L_q)\sqrt{|t-s|} \\
|V(t,\eta;q) - V(s,\eta;q)| &\le c_2(\|\eta\|+1+L_q)\sqrt{|t-s|}.
\end{aligned}
\tag{3.5}
$$

Hereafter we will consider V_N and V, because v_N and v are treated by similar methods. Putting

$$\psi(\Delta,\eta;q,z) = \sup_{Y\in\mathcal{Y}} J(\Delta,\eta;q,Y,z) \quad \text{for} \quad z\in Z,$$

we define $S = S_N; Q \to Q$ by

$$Sq(\eta) = \inf_{z\in\mathcal{Z}} \psi(\Delta,\eta;q,z). \tag{3.6}$$

Then we have the following proposition, which is useful for the proof of dynamic programming principle.

Proposition 3.2. *For any k and a positive ε, there exist $\alpha \in \mathcal{A}$ and $\gamma \in \mathcal{R}_N$ such that*

$$
\begin{aligned}
J(k\Delta,\eta;q,Y,\gamma) - \varepsilon \le {S_N}^k q(\eta) &\le J(k\Delta,\eta;q,\alpha,Z)+\varepsilon \\
\text{for any} \quad Y\in\mathcal{Y} \quad &\text{and} \quad Z\in\mathcal{Z}_N.
\end{aligned}
\tag{3.7}
$$

Proof. We will apply similar arguments as [3]. For $c>0$, we take a positive $\delta = \delta(c,q)$ such that

$$|J(\Delta,\eta;q,Y,Z) - J(\Delta,\tilde{\eta};q,Y,Z)| < c, \quad \text{whenever} \quad |\eta-\tilde{\eta}|_B < \delta \tag{3.8}$$

and

$$|J(\Delta,\eta;q,Y,z) - J(\Delta,\eta;q,Y,\tilde{z})| < c, \quad \text{whenever} \quad |z-\tilde{z}|_2 < \delta$$

Dividing $H = \bigcup_{j=1}^{\infty} A_j$ and $\boldsymbol{Z} = \bigcup_{\ell=1}^{L} C_\ell$ with $|\ \ |_B -$ diam. $(A_j) < \delta$ and diam. $(C_\ell) < \delta$, we fix $\eta_j \in A_j$ and $z_\ell \in C_\ell$ arbitrarily. Since there is $z^* = z^*(\eta;q)$ such that

$$\psi(\Delta,\eta;q,z^*) < Sq(\eta) + c$$

putting ${z^*}_j = z^*(\eta_j;q)$, we can see, from (3.8)

$$J(\Delta,\eta;q,Y,{z^*}_j) \le \psi(\Delta,\eta;q,{z^*}_j) < Sq(\eta)+3c \quad \text{for} \quad \eta\in A_j. \tag{3.9}$$

Since $\boldsymbol{Y}$ is compact and convex, we can take a step admissible control $Y_{j\ell} = Y_{j\ell}(q)$, say $Y_{j\ell} \in \mathcal{Y}_m$ with $N \leq m$, such that

$$J(\Delta, \eta_j; q, Y_{j\ell}, z_\ell) > \psi(\Delta, \eta_j; q, z_\ell) - c.$$

Therefore (3.8) again yields

$$\text{(3.10)} \quad J(\Delta, \eta; q, Y_{j\ell}, z) > \psi(\Delta, \eta; q, z) - 5c \quad \text{for} \quad \eta \in A_j \quad \text{and} \quad z \in C_\ell.$$

Putting $q_i = S^i q$, $z^*{}_{ji} = z^*(\eta_j, q_i)$ and $Y_{j\ell i} = Y_{j\ell}(q_i)$, we define $\gamma \in \mathcal{R}_N$ and $\alpha \in \mathcal{A}$ as follows,

$$\gamma(Y)(s) = \sum_j z^*{}_{j,k-1} I_{A_j}(\eta) \quad \text{for} \quad s < \Delta,$$

where $I_A =$ indicator of A, namely $\gamma(Y)(s) = z^*{}_{j,k-1}$ for $\eta \in A_j, s < \Delta$. Using the unique solution $X(s) = X(s; \eta, Y, \gamma)$ on $[0, \Delta]$, we define $\gamma(Y)$ on $[\Delta, 2\Delta)$ by

$$\gamma(Y)(s) = \sum_j z^*{}_{j,k-2} I_{A_j}(X(\Delta)) \quad \text{for} \quad s \in [\Delta, 2\Delta).$$

Since we have a unique solution $X(s) = X(s; \eta, Y, \gamma)$ on $[0, 2\Delta]$, repeating the same procedure, we get the following $\gamma \in \mathcal{R}_N$ on $[0, k\Delta)$.

$$\begin{aligned} \gamma(Y)(s) &= I_{[0,\Delta)}(s) z^*{}_{p,k-1} \\ &+ \sum_{i=1}^{k-1} I_{[i\Delta,(i+1)\Delta)}(s) \left(\sum_{j=1}^{\infty} z^*{}_{j,k-i} I_{A_j}(X(i\Delta)) \right), \quad \text{for} \quad \eta \in A_p. \end{aligned}$$

Next, putting $w_\theta^+(t) = w(t+\theta) - w(\theta)$ and $\hat{Y}_{j\ell i}(w)(s) = Y_{j\ell,k-1-i}(w_{i\Delta}^+)(s - i\Delta)$ for $s \in [i\Delta, (i+1)\Delta)$ and using the same procedure as γ, we define α by (3.11),

$$\begin{aligned} \text{(3.11)} \quad \alpha(Z)(s) &= I_{[0,\Delta)}(s) \sum_{\ell=1}^{L} Y_{p\ell,k-1}(s) I_{C_\ell}(Z(0)) \\ &+ \sum_{i=1}^{k-1} I_{[i\Delta,(i+1)\Delta)}(s) \sum_{j=1}^{\infty} \sum_{\ell=1}^{L} \hat{Y}_{j\ell i}(s) I_{A_j}(X(i\Delta; \eta, \alpha, Z)) I_{C_\ell}(Z(i\Delta)), \\ &\text{for} \quad \eta \in A_p. \end{aligned}$$

We shall prove that $\alpha \in \mathcal{A}$. For a small $\delta = 2^{-p}T$, $p > N$, we can take a large $m = m(\eta, \delta)$, by (2.2), such that

$$P(X(i\Delta; \eta, \alpha, Z)) \notin F) < \delta \quad \text{for} \quad i = 1, 2, \cdots, Y \in \mathcal{Y}, \quad Z \in \mathcal{Z},$$

where $F = \bigcup_{j=1}^{m} A_j$. Fixing $\tilde{y} \in \boldsymbol{Y}$ arbitrarily, we define an approximate δ-step strategy $\tilde{\alpha}$ by

$$\begin{aligned}\tilde{\alpha}(Z)(s) &= I_{[0,\delta)}(s)\tilde{y} + I_{[\delta,\Delta)}(s)\sum_{\ell=1}^{L} Y_{p\ell,k-1} I_{C_\ell}(Z(0)) \\ &+ \sum_{i=1}^{k-1} I_{[i\Delta,(i+1)\Delta)}(s)\left[\sum_{j=1}^{m}\sum_{\ell=1}^{L}\hat{Y}_{j\ell i}(s) I_{A_j}(X(i\Delta;\eta,\alpha,Z)) I_{C_\ell}(Z(i\Delta)),\right. \\ &\qquad \left. + \tilde{y} I_{F^c}(X(i\Delta;\eta,\alpha,Z))\right].\end{aligned}$$

Then we get

$$|\alpha(Z)(s) - \tilde{\alpha}(Z)(s)|_1 \le 2\delta(T+1)\mathrm{diam}.\boldsymbol{Y}.$$

This concludes $\alpha \in \mathcal{A}$.

We will now prove the inequality (3.7).

$$\begin{aligned}&S^k q(\eta) - J(k\Delta,\eta;q,Y,\gamma) \\ &\quad = \sum_{i=0}^{k-1} J(i\Delta,\eta;q_{k-i},Y,\gamma) - J((i+1)\Delta,\eta;q_{k-i-1},Y,\gamma).\end{aligned} \tag{3.12}$$

Using $\gamma Y \in \mathcal{Z}_N$ and (3.9), we have

$$\begin{aligned}&J((i+1)\Delta,\eta;q_{k-i-1},Y,\gamma) \\ &\le E\left[\int_0^{i\Delta} h(X(s),Y(s),\gamma Y(s))ds + \psi(\Delta,X(i\Delta);q_{k-i-1},\gamma Y(i\Delta))\right] \\ &\le J(i\Delta,\eta;q_{k-i},Y,\gamma) + 5c.\end{aligned} \tag{3.13}$$

Hence (3.12) and (3.13) yield

$$S^k q(\eta) - J(k\Delta,\eta;q,Y,\gamma) \ge -5kc.$$

For the right inequality of (3.7), we can see, from (3.6), (3.10) and (3.11)

$$J((i+1)\Delta,\eta;q_{k-i-1},\alpha,Z) \ge J(i\Delta,\eta;q_{k-i},\alpha,Z) - 5c. \tag{3.14}$$

Inserting (3.14) into (3.12), we have

$$S^k q(\eta) - J(k\Delta,\eta;q,\alpha,Z) \le 5kc.$$

Replacing c with $\varepsilon/5k$, we complete the proof of Proposition. □

Now we get

$$\inf_{\gamma\in\mathcal{R}_N}\sup_{Y\in\mathcal{Y}} J(k\Delta,\eta;q,Y,\gamma) \le S^k q(\eta) \le \sup_{\alpha\in\mathcal{A}}\inf_{Z\in\mathcal{Z}_N} J(k\Delta,\eta;q,\alpha,Z). \tag{3.15}$$

Proposition 3.1 however derives, for any $\alpha\in\mathcal{A}$ and $\gamma\in\mathcal{R}_N$,

$$\begin{aligned}\inf_{Z\in\mathcal{Z}_N} J(t,\eta;q,\alpha,Z) &\le J(t,\eta;q,\alpha,\tilde{Z})\\ &= J(t,\eta;q,\tilde{Y},\gamma) \le \sup_{Y\in\mathcal{Y}} J(t,\eta;q,Y,\gamma)\end{aligned}$$

with some $\tilde{Y}\in\mathcal{Y}$ and $\tilde{Z}\in\mathcal{Z}_N$. Therefore, for any $q\in Q$,

$$\sup_{\alpha\in\mathcal{A}}\inf_{Z\in\mathcal{Z}_N} J(t,\eta;q,\alpha,Z) \le \inf_{\gamma\in\mathcal{R}_N}\sup_{Y\in\mathcal{Y}} J(t,\eta;q,Y,\gamma) \tag{3.16}$$

holds. Consequently, both inequalities of (3.15) turn out to be equalities. We have

$$V_N(k\Delta,\eta;q) = S^k q(\eta).$$

that means

Theorem 3.1 (Discrete dynamic programming principle for V_N).

$$\begin{aligned}&V_N((k+j)\Delta,\eta;q)\\ &= \inf_{\gamma\in\mathcal{R}_N}\sup_{Y\in\mathcal{Y}} E\left[\int_0^{k\Delta} h(X(s),Y(s),\gamma Y(s))ds + V_N(j\Delta,X(k\Delta);q)\right]\end{aligned}$$

where $X(t)=X(t;\eta,Y,\gamma Y)$.

Proposition 3.3. *As* $N\to\infty$, $V_N(\ ;q)$ *is decreasing to* $V(\ ;q)$ *uniformly on any bounded set of* $[0,T]\times H$.

Proof. For $\gamma\in\mathcal{R}$ and $\varepsilon>0$, we can take a step strategy $\tilde{\gamma}$ ($\in\mathcal{R}_N$ say) such that

$$\sup_{t\le T}\sup_{Y\in\mathcal{Y}} |J(t,\eta;q,Y,\gamma) - J(t,\eta;q,Y,\tilde{\gamma})| < \varepsilon$$

by (2.6) and (3.1). Hence we have

$$\sup_{Y\in\mathcal{Y}} J(t,\eta;q,Y,\gamma) \ge \sup_{Y\in\mathcal{Y}} J(t,\eta;q,Y,\tilde{\gamma}) - \varepsilon \ge V_N(t,\eta;q) - \varepsilon.$$

Therefore, for any t and η, we can take e-strategy $\gamma^* = \gamma^*(t,\eta)$ and $N^* = N^*(t,\eta)$ such that

$$V(t,\eta;q) > \sup_{Y\in\mathcal{Y}} J(t,\eta;q,Y,\gamma^*) - \varepsilon > V_N(t,\eta;q) - 2\varepsilon$$

whenever $N \geq N^*$. Moreover, for a bounded set Λ of $[0,T]\times H$, there is a finite set $\{(t_i,\eta_j), i,j = 1,2,\cdots,m\}$ such that, for any $(t,\eta)\in\Lambda$

$$\min_{i,j=1,\cdots,m} |V_N(t,\eta;q) - V_N(t_i,\eta_j;q)| < \varepsilon \quad N = 1,2,\cdots$$
$$\min_{i,j=1,\cdots,m} |V(t,\eta;q) - V(t_i,\eta_j;q)| < \varepsilon,$$

by virtue of (3.4) and (3.5). Hence, putting $M = \max\{N^*(t_i\eta_j), i,j = 1,\cdots,m\}$, we get

$$V(t,\eta;q) > V_M(t,\eta;q) - 4\varepsilon \quad \text{for} \quad (t,\eta)\in\Lambda. \tag{3.17}$$

Since $V_N(t,\eta;q)$ is decreasing to $V(t,\eta;q)$, (3.17) completes the proof of Proposition. We are now ready to state the dynamic programming principle. □

Theorem 3.2.

$$V(t+s,\eta;q) = \inf_{\gamma\in\mathcal{R}} \sup_{Y\in\mathcal{Y}} E\left[\int_0^t h(X(\theta),Y(\theta).\gamma Y(\theta))d\theta + V(s,X(t);q)\right]$$

where $X(t) = X(t;\eta,Y,\gamma Y)$. *Namely*

$$V(t+s,\eta;q) = V(t,\eta;V(s,\eta;q)). \tag{3.18}$$

Proof. First of all, we show (t,s)-continuity of the right hand side of (3.18). Recalling (3.5), we have

$$\begin{aligned}&|V(t,\eta;V(s,\circ;q)) - V(t,\eta;V(\tilde{s},\circ;q))|\\ &\leq \sup_{\gamma\in\mathcal{R}}\sup_{Y\in\mathcal{Y}} E|V(s,X(t);q) - V(\tilde{s},X(t);q)| \leq c_4(1+L_q+\|\eta\|)\sqrt{|s-\tilde{s}|}\end{aligned}$$

and

$$|V(t,\eta;V(\tilde{s},\circ;q)) - V(\tilde{t},\eta;V(\tilde{s},\circ;q))| \leq c_5(1+L_q+\|\eta\|)\sqrt{|t-\tilde{t}|}.$$

Hence it is enough to prove (3.18) for dense points t and s, say $t = k2^{-p}$ and $s = j2^{-p}$. Theorem 3.1 yields

$$V_N(t+s,\eta;q) = V_N(t,\eta;V_N(s,\eta;q)) \quad \text{for} \quad N \leq p. \tag{3.19}$$

Moreover Proposition 3.3 says that, for $\varepsilon > 0$, there is a large N_0 such that

$$|V_N(s,\eta;q) - V(s,\eta;q)| < \varepsilon \quad \text{for} \quad \|\eta\| < \frac{1}{\varepsilon}$$

whenever $N \geq N_0$. Therefore

$$\begin{aligned} &E|V_N(s, X(t;\eta,Y,Z);q) - V(s,X(t;\eta,Y,Z);q)| \\ &\quad < \varepsilon + 2(\bar{h}T + \bar{q})P\left(\|X(t;\eta,Y,Z)\| > \frac{1}{\varepsilon}\right) \\ &\quad < \varepsilon + 2\varepsilon^2(\bar{h}T + \bar{q})K_2(1 + \|\eta\|^2) \quad \text{for} \quad N \geq N_0. \end{aligned}$$

So we get

$$\begin{aligned} &|V_N(t,\eta;V_N(s,\circ;q)) - V_N(t,\eta;V(s,\circ;q))| \\ &\qquad < \varepsilon + 2\varepsilon^2(\bar{h}T + \bar{q})K_2(1 + \|\eta\|^2) \quad \text{for } N \geq N_0. \end{aligned} \tag{3.20}$$

Since $V_N(t,\eta;V(s,\circ;q))$ is decreasing to $V(t,\eta;V(s,\circ;q))$, (3.19) and (3.20) complete the proof of Theorem. □

Employing similar arguments, we can prove

Theorem 3.3. $v(\ ;q)$ *satisfies the dynamic programming principle,*

$$v(t+s,\eta;q) = \sup_{\alpha\in\mathcal{A}} \inf_{Z\in\mathcal{Z}} E\left[\int_0^t h(X(\theta), \alpha Z(\theta), Z(\theta))d\theta + v(s, X(t);q)\right].$$

We can easily see, from (3.16), the following proposition.

Proposition 3.4. $v(t,\eta;q) \leq V(t,\eta;q)$.

4. Viscosity solutions

We shall define a viscosity solution of the nonlinear equation (4.1) below, according to Crandall and Lions [1], [8].

$\phi \in C^{12}((0,T) \times H)$ is called a test function, if (i) ϕ is weakly lower semi-continuous and bounded from below and (ii) $\partial\phi(t,\eta) \in H^2$ and both of $\partial\phi$ and $A^*\partial\phi$ are continuous. $g \in C^2(H)$ is called radial, if $g(\eta) = \tilde{g}(\|\eta\|)$ with $\tilde{g} \in C^2[0,\infty)$ increasing from 0 to ∞.

Let us consider the following equation

$$\begin{aligned} 0 &= \frac{\partial V}{\partial t}(t,\eta) - \langle A^*\partial V(t,\eta), \eta\rangle + F(t,\eta,V(t,\eta),\partial V(t,\eta),\partial^2 V(t,\eta)) \\ &\text{for} \quad t \in (0,T), \quad \eta \in H, \quad V(0,\eta) = \Psi(\eta), \end{aligned} \tag{4.1}$$

where $F; [0,T]\times H\times\mathbb{R}^1\times H\times L(H)\to\mathbb{R}^1$ is uniformly continuous on any bounded set.

DEFINITION 4.1. $V\in C([0,T]\times H)$ is called a subsolution (resp. super solution) of (4.1), if $V(0,\eta)=\Psi(\eta)$ and the following condition (i) (resp. (ii)) holds for any test function ϕ and radial function g,

(i) If $V-\phi-g$ has a local maximum at $(\hat{t},\hat{\eta})\in(0,T)\times H$, then

$$\frac{\partial\phi}{\partial t}(\hat{t},\hat{\eta})-\langle A^*\partial\phi(\hat{t},\hat{\eta}),\hat{\eta}\rangle+F(\hat{t},\hat{\eta},V(\hat{t},\hat{\eta}),\partial(\phi+g)(\hat{t},\hat{\eta}),\partial^2(\phi+g)(\hat{t},\hat{\eta}))\leq 0.$$

(ii) If $V+\phi+g$ has a local minimum at $(\hat{t},\hat{\eta})\in(0,T)\times H$, then

$$\begin{aligned}&-\frac{\partial\phi}{\partial t}(\hat{t},\hat{\eta})+\langle A^*\partial\phi(\hat{t},\hat{\eta}),\hat{\eta}\rangle\\&\qquad+F(\hat{t},\hat{\eta},V(\hat{t},\hat{\eta}),-\partial(\phi+g)(\hat{t},\hat{\eta}),-\partial^2(\phi+g)(\hat{t},\hat{\eta}))\geq 0.\end{aligned}$$

V is called a viscosity solution, if it is both a subsolution and a super solution.

This section is devoted to the proof of Theorem 4.1.

Theorem 4.1. $V(\ ;q)$ *is the unique viscosity solution of Isaacs equation* (1.4), *in the set of bounded and weakly continuous functions.*

Proof. Suppose that V-ϕ-g has a local maximum at $(\hat{t},\hat{\eta})\in(0,T)\times H$, say

$$V(\hat{t},\hat{\eta})-\phi(\hat{t},\hat{\eta})-g(\hat{\eta})\geq V(t,\eta)-\phi(t,\eta)-g(\eta)\quad\text{for}\quad(t,\eta)\in\Lambda \tag{4.2}$$

where $\Lambda=\{(t,\eta);|t-\tilde{t}|<\delta^*\text{ and }\|\eta-\tilde{\eta}\|<\delta^*\}$. Moreover, for $\hat{\varepsilon}>0$, there is $\hat{\delta}>0$, such that

$$\begin{aligned}&|f_1(t,\eta)-f_1(\hat{t},\hat{\eta})|<\hat{\varepsilon}&&\text{for}\quad f_1=\phi,\frac{\partial\phi}{\partial t},g\\&\|f_2(t,\eta)-f_2(\hat{t},\hat{\eta})\|<\hat{\varepsilon}&&\text{for}\quad f_2=\partial\phi,A^*\partial\phi,\partial g\\&|f_3(t,\eta)-f_3(\hat{t},\hat{\eta})|<\hat{\varepsilon}&&\text{for}\quad f_3=\partial^2\phi,\partial^2 g,\end{aligned}$$

whenever $|t-\hat{t}|<\hat{\delta}$ and $\|\eta-\hat{\eta}\|<\hat{\delta}$.

First of all, we evaluate $E[V(\hat{t}-\theta,X(\theta);q)-V(\hat{t},\hat{\eta});q)]$, where $X(\theta)=X(\theta;\hat{\eta},Y,\gamma Y)$. Let us set $\delta=\min(\delta^*,\hat{\delta})$ and $\tau=$ exit time from the closed ball with center $\hat{\eta}$ and radius δ. Applying (4.2) and Ito's formula, we get, for $\theta<\delta$,

$$\begin{aligned}&E(V(\hat{t}-\theta,X(\theta);q)-V(\hat{t},\hat{\eta};q);\tau\geq\theta)\\&\leq E(\phi(\hat{t}-\theta,X(\theta))-\phi(\hat{t},\hat{\eta})+g(X(\theta))-g(\hat{\eta});\tau\geq\theta)\end{aligned}\tag{4.3}$$

$$\le E\left[\int_0^\theta\left(-\frac{\partial\phi}{\partial t}(\hat{t}-s,X(s))+\langle A^*\partial\phi(\hat{t}-s,X(s)\right),X(s)\rangle\right.$$
$$+\langle\partial g(X(s)),AX(s)\rangle$$
$$+\langle\partial(\phi+g)(\hat{t}-s,X(s)),\beta(X(s),Y(s),\gamma Y(s))\rangle$$
$$\left.+\frac{1}{2}\mathrm{trace}S\partial^2(\phi+g)(\hat{t}-s,X(s)))ds;\tau\ge\theta\right]$$
$$+E\left[\int_0^\theta\langle\partial(\phi+g)(\hat{t}-s,X(s)),dM(s)\rangle;\tau\ge\theta\right].$$

where $X(\theta)=X(\theta;\hat{\eta},Y,\gamma Y)$. Denoting the last term by I, we have

$$\begin{aligned}I&=E\int_0^{\tau\Lambda\theta}\langle\partial(\phi+g)(\hat{t}-s,X(s)),dM(s)\rangle\\&\quad-E\left[\int_0^{\tau\Lambda\theta}\langle\partial(\phi+g)(\hat{t}-s,X(s)),dM(s)\rangle;\tau<\theta\right]\\&=I_1-I_2.\end{aligned}$$

$$\begin{aligned}(I_2)^2&\le\bar{m}E\left[\int_0^{\tau\Lambda\theta}\|\partial(\phi+g)(\hat{t}-s,X(s))\|^2dsP(\tau<\theta)\right]\\&\le\bar{m}K_{11}(\|\partial(\phi+g)(\hat{t},\hat{\eta})\|^2+1)\theta^3/\delta^4\end{aligned}$$

for a small θ. Hence

$$|I_2|\le k_1\sqrt{\theta^3}/\delta^2\quad\text{for}\quad\theta\in(0,\delta)$$

where k_1 is independent of Y and γ. Hereafter k_i stands for a constant independent of Y and γ. Since (2.1) yields $\langle\partial g(\zeta),A\zeta\rangle\le 0$,

$$\begin{aligned}&E(V(\hat{t}-\theta,X(\theta);q)-V(\hat{t},\hat{\eta};q);\tau\ge\theta)\\&\le\left(-\frac{\partial\phi}{\partial t}(\hat{t},\hat{\eta})+\langle A^*\partial\phi(\hat{t},\hat{\eta}),\hat{\eta}\rangle+\frac{1}{2}\mathrm{trace}S\partial^2(\phi+g)(\hat{t},\hat{\eta})\right)\theta\\&\quad+E\int_0^\theta\langle\partial(\phi+g)(\hat{t},\hat{\eta}),\beta(\hat{\eta},Y(s),\gamma Y(s))\rangle ds+k_2\hat{\varepsilon}\theta+k_3\sqrt{\theta^3}/\delta^2.\end{aligned}\tag{4.4}$$

holds. Again Proposition 2.5 says

$$E[V(\hat{t}-\theta,X(\theta);q)-V(\hat{t},\hat{\eta};q);\tau<\theta]\le k_4\theta^2/\delta^4.\tag{4.5}$$

Combining (4.4) with (4.5), we get

$$J(Y,\gamma)=E\left[V(\hat{t}-\theta,X(\theta);q)-V(\hat{t},\hat{\eta};q)+\int_0^\theta h(X(s),Y(s),\gamma Y(s))ds\right]\tag{4.6}$$

$$\leq \left(-\frac{\partial \phi}{\partial t}(\hat{t},\hat{\eta}) + \langle A^* \partial\phi(\hat{t},\hat{\eta}),\hat{\eta}\rangle + \frac{1}{2}\text{trace} S\partial^2(\phi+g)(\hat{t},\hat{\eta}) \right)\theta$$
$$+ E\int_0^\theta \langle \partial(\phi+g)(\hat{t},\hat{\eta}), \beta(\hat{\eta}, Y(s), \gamma Y(s))\rangle + h(\hat{\eta}, Y(s), \gamma Y(s))ds$$
$$+ k_5\hat{\varepsilon}\theta + k_6\sqrt{\theta^3}/\delta^2, \quad \text{for} \quad \text{small } \theta.$$

Let us put

$$F(y,z) = \langle \partial(\phi+g)(\hat{t},\hat{\eta}), \beta(\hat{\eta},y,z)\rangle + h(\hat{\eta},y,z).$$

Since a constant strategy, $\gamma Y(s) = z$ for any Y and s, is in $\mathcal{R}$, we see

(4.7)
$$\inf_{\gamma\in\mathcal{R}} \sup_{Y\in\mathcal{Y}} E\int_0^\theta F(Y(s),\gamma Y(s))ds$$
$$\leq \inf_{z\in Z} \sup_{Y\in\mathcal{Y}} E\int_0^\theta F(Y(s),z)ds$$
$$\leq \inf_{z\in Z} \sup_{Y\in\mathcal{Y}} E\int_0^\theta \sup_{y\in Y} F(y,z)ds \leq \inf_{z\in Z}\sup_{y\in Y} F(y,z)\theta.$$

For any $\varepsilon > 0$, there is $\tilde{\delta} > 0$ such that

$$|F(y,z) - F(\tilde{y},\tilde{z})| < \varepsilon, \quad \text{whenever } |y-\tilde{y}|_1 < \tilde{\delta} \quad \text{and} \quad |z-\tilde{z}|_2 < \tilde{\delta}.$$

Dividing $Y = \bigcup_{i=1}^{j} Y_i$ and $Z = \bigcup_{p=1}^{m} Z_p$ with $\text{diam}.Y_i < \tilde{\delta}$ and $\text{diam}.Z_p < \tilde{\delta}$

respectively and fixing $y_i \in Y_i$ and $z_p \in Z_p$ arbitrarily, we define $G; Z \to Y$ by

$$Gz = y_{\ell(p)} \quad \text{for} \quad z \in Z_p,$$

where $\ell(p) = min.\{k; \max_{i=1,\cdots,j} F(y_i,z_p) = F(y_k,z_p)\}$. Then, for any $z \in Z_p$

(4.8)
$$F(Gz,z) \geq F(y_{\ell(p)},z_p) - \varepsilon \geq \max_{i=1,\cdots,j} F(y_i,z) - 2\varepsilon$$
$$\geq \sup_{y\in Y} F(y,z) - 3\varepsilon.$$

Fixing a step strategy γ arbitrarily, say $\gamma \in \mathcal{R}_N$ we define $\hat{Y} \in Y_N$ and $Z \in Z_N$ as follows. Noting $\gamma Y(s)$, $s \in [0,\Delta)$ is independent of Y and s for $\gamma \in \mathcal{R}_N$, we put $Z(s) = \gamma Y(s)$ and $\hat{Y}(s) = GZ(0)$ for $s \in [0,\Delta)$. For $s \in [\Delta, 2\Delta)$, we put $Z(s) = \gamma\hat{Y}(\Delta)$ and $\hat{Y}(s) = GZ(\Delta)$. Repeating this argument, we get $Z \in \mathcal{Z}_N$ and

$\hat{Y} \in \mathcal{Y}_N$ such that $Z = \gamma\hat{Y}$ and $\hat{Y} = GZ$. Therefore, for $s \in [k\Delta, (k+1)\Delta)$,

$$
\begin{aligned}
F(\hat{Y}(s), \gamma\hat{Y}(s)) &= F(G(\gamma\hat{Y}(k\Delta)), \gamma\hat{Y}(k\Delta)) \\
&\geq \sup_{y\in Y} F(y, \gamma\hat{Y}(k\Delta)) - 3\varepsilon \geq \inf_{z\in Z}\sup_{y\in Y} F(y,z) - 3\varepsilon
\end{aligned}
$$

holds, by (4.8). Hence for any step strategy γ,

$$
\begin{aligned}
&\sup_{Y\in\mathcal{Y}} E\int_0^\theta F(Y(s), \gamma Y(s))ds \geq E\int_0^\theta F(\hat{Y}(s), \gamma\hat{Y}(s))ds \\
&\geq (\inf_{z\in Z}\sup_{y\in Y} F(y,z) - 3\varepsilon)\theta
\end{aligned}
$$

holds. Since step strategies are dense in $\mathcal{R}$, we have

$$
\inf_{\gamma\in\mathcal{R}}\sup_{Y\in\mathcal{Y}} E\int_0^\theta F(Y(s), \gamma Y(s))ds \geq (\inf_{z\in Z}\sup_{y\in Y} F(y,z) - 3\varepsilon)\theta.
$$

Since ε is arbitrary, we get, recalling (4.7),

$$
\inf_{\gamma\in\mathcal{R}}\sup_{Y\in\mathcal{Y}} E\int_0^\theta F(Y(s), \gamma Y(s))ds = \inf_{z\in Z}\sup_{y\in Y} F(y,z)\theta. \tag{4.9}
$$

Inserting (4.6) and (4.9) into (4.3) and dividing by θ, we obtain, as $\theta \to 0$,

$$
\begin{aligned}
0 \leq & -\frac{\partial\phi}{\partial t}(\hat{t},\hat{\eta}) + \langle A^*\partial\phi(\hat{t},\hat{\eta}), \hat{\eta}\rangle + \frac{1}{2}\mathrm{trace} S\partial^2(\phi+g)(\hat{t},\hat{\eta}) \\
& + \inf_{z\in Z}\sup_{y\in Y}(\langle\partial(\phi+g)(\hat{t},\hat{\eta}), \beta(\hat{\eta},y,z)\rangle + h(\hat{\eta},y,z)) + \hat{\varepsilon}k_5.
\end{aligned}
$$

Since $\hat{\varepsilon}$ is arbitrary, V turns out to be a subsolution of (1.4).

Employing similar arguments, we can prove that V is a super solution. Hence V is a viscosity solution. Now the uniqueness theorem [8] completes the proof, since V is bounded and weakly continuous. □

In the same way, we can see the following theorem,

Theorem 4.2. *$v(\ ;q)$ is the unique viscoity solution of Isaacs equation* (1.5) *in the set of bounded and weakly countinuous functions.*

Hence we have

Corollary. *$V(\ ;q) = v(\ ;q)$ holds, under the following Isaacs' condition*;

$$
\sup_{y\in Y}\inf_{z\in Z}\langle\xi, \beta(\eta,y,z)\rangle = \inf_{z\in Z}\sup_{y\in Y}\langle\xi, \beta(\eta,y,z)\rangle, \quad \textit{for any } \xi \in H.
$$

References

[1] M.G. Crandall and P.L. Lions: *Hamilton-Jacobi equations in infinite dimensions*, Part 4. J. Funct. Anal. **90** (1990), 237–283.

[2] G. Da Prato and J. Zabczyk: Stochastic Equations in Infinite Dimentions, Encyclopedia of Math. Appl, Cambridge Univ. Press, 1992.

[3] W.H. Fleming and P.E. Souganidis: *On the existence of value functions of two-player, zero-sum stochastic differential games*, Indiana Univ. Math. J. **38** (1989), 293–314.

[4] N. Ikeda and S. Watanabe: Stochastic Differential Equations and Diffusion Processes, North-Holland Publ.- Kodansha, 1981.

[5] K. Itô: Foundations of Stochastic Differential Equations in Infinite Dimensional Spaces, SIAM Reg. Conf. Series 47, 1984.

[6] M. Nisio: *On sensitive control for stochastic partial differential equations*, Pitman Res. Notes Math. **310** (1994), 231–241.

[7] B.L. Rozovskii: Stochastic Evolution Systems. Kluwer, 1990.

[8] A. Swiech: Viscosity solutions of fully nonlinear partial differential equations with unbounded terms in infinitedimensions, Ph. D. Dissertation, Univ. of Calif, 1993.

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