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ON LAMBEK TORSION THEORIES, II

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In this note, generalizing recent works of Masaike [15] and Hoshino [9], we will provide another approach to the theory of QF-3 rings. We will also provide an explanation to the symmetry established by Masaike [14, Theorem 2].

Recall that a ring R is called left (resp. right) QF-3 if it has a minimal faithful left (resp. right) module, i.e., a faithful left (resp. right) module which appears as a direct summand in every faithful left (resp. right) module (see, e.g., Tachikawa [30] for details). In his recent paper [15], K. Masaike showed that a left QF-3 ring R is also right QF-3 if and only if it contains an idempotent f such that RfR is a minimal dense left ideal and every finitely solvable system of congruences $\{x \equiv fx_\lambda (\mathrm{mod}\ I_\lambda)\}_{\lambda \in \Lambda}$ with each I_λ a left ideal is solvable. Generalizing this, we will provide a characterization of left and right QF-3 rings. To do so, we will introduce the notion of τ-absolutely pure rings in Section 1 and the notion of τ-semicompact modules in Section 2, where "τ-" means "relative to Lambek torsion theory". With those notions, we will show that a ring R is left and right QF-3 if and only if it is τ-absolutely pure, left and right τ-semicompact and contains idempotents e, f such that ReR and RfR are minimal dense right and left ideals, respectively.

Throughout this note, R stands for an associative ring with identity, modules are unitary modules, and torsion theories are Lambek torsion theories. Sometimes, we use the notation ${}_RX$ (resp. X_R) to stress that the module X considered is a left (resp. right) R-module. We denote by $\mathrm{Mod}\,R$ (resp. $\mathrm{Mod}\,R^{\mathrm{op}}$) the category of left (resp. right) R-modules and by $(\ )^*$ both the R-dual functors. For a module X, we denote by $E(X)$ its injective envelope and by ε_X: $X \to X^{**}$ the usual evaluation map. Recall that a module X is said to be torsionless if ε_X is a monomorphism, and to be reflexive if ε_X is an isomorphism. Note that for a submodule X' of a module X, if X/X' is torsionless then $\mathrm{Ker}\,\varepsilon_X \subset X'$. For an $X \in \mathrm{Mod}\,R$, we denote by $\tau(X)$ its Lambek torsion submodule. Namely, $\tau(X)$ denotes a submodule of X such that $\mathrm{Hom}_R(\tau(X), E({}_RR)) = 0$ and $X/\tau(X)$ is cogenerated by $E({}_RR)$. For also an $M \in \mathrm{Mod}\,R^{\mathrm{op}}$, we denote by $\tau(\mathrm{M})$ its Lambek torsion submodule.

Let us recall several definitions. A module X is said to be torsion if $\tau(X)=X$, and to be torsionfree if $\tau(X)=0$. Note that for a submodule X' of X, if X/X' is torsionfree then $\tau(X)\subset X'$, in particular, we have $\tau(X)\subset \operatorname{Ker}\varepsilon_X$. A nonzero torsionfree module X is said to be cocritical if X/X' is torsion for every nonzero submodule X' of X. A submodule X' of a module X is said to be dense if X/X' is torsion, and to be closed if X/X' is torsionfree. A dense left (resp. right) ideal I is called a minimal dense left (resp. right) ideal if it is contained in every dense left (resp. right) ideal. Note that a minimal dense left ideal, if exists, has to be an idempotent two-sided ideal, that a minimal dense left ideal exists if and only if the class of all torsion left modules is closed under taking direct products, and that in case R is right perfect, R contains an idempotent f with RfR a minimal dense left ideal.

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1. τ-Absolute purity of rings

In this section, we introduce the notion of τ-absolutely pure rings. With that notion, we formulate the symmetry established by Masaike [14, Theorem 2].

We have to recall several more definitions. A module X is said to be τ-finitely generated if it contains a finitely generated dense submodule. A finitely generated module X is said to be τ-finitely presented (resp. τ-coherent) if for every epimorphism (resp. homomorphism) $\pi: X'\to X$ with X' finitely generated, $\operatorname{Ker}\pi$ is τ-finitely generated. Note that every finitely generated submodule of a τ-coherent module is τ-finitely presented. Also, a module X is said to be τ-artinian (resp. τ-noetherian) if it satisfies the descending (resp. ascending) chain condition on closed submodules. Finally, a ring R is said to be left (resp. right) τ-artinian if ${}_RR$ (resp. R_R) is τ-artinian, to be left (resp. right) τ-noetherian if ${}_RR$ (resp. R_R) is τ-noetherian, and to be left (resp. right) τ-coherent if ${}_RR$ (resp. R_R) is τ-coherent.

REMARKS. (1) A module X is τ-finitely presented if and only if there exists an exact suquence $0\to X''\to X'\to X\to 0$ with X' finitely presented and X'' torsion.

(2) A module X is τ-noetherian if and only if every submodule of X is τ-finitely generated (see Faith [5, Proposition 3.1]).

(3) A ring R is left τ-noetherian if and only if every finitely generated left module is τ-finitely presented (see, e.g., Sumioka [28]). In particular, a left τ-noetherian ring R is left τ-coherent.

(4) A left τ-artinian ring R is left τ-noetherian (see Miller and Teply [17, Theorem 1.4]).

The next lemma follows immediately from the fact that $\tau(X) \subset \operatorname{Ker} \varepsilon_X$ for every module X.

Lemma 1.1. *For a module X the following are equivalent.*

(a) $\tau(X) = \operatorname{Ker} \varepsilon_X$.

(b) $\operatorname{Ker} \varepsilon_X$ *is torsion.*

(c) $X/\tau(X)$ *is torsionless.*

The next lemma will play a key role in our arguments below.

Lemma 1.2 (Hoshino [9, Theorem A]). *For a ring R the following are equivalent.*

(a) $\tau(X) = \operatorname{Ker} \varepsilon_X$ *for every finitely presented* $X \in \operatorname{Mod} R$.

(a)$^{\mathrm{op}}$ $\tau(M) = \operatorname{Ker} \varepsilon_M$ *for every finitely presented* $M \in \operatorname{Mod} R^{\mathrm{op}}$.

(b) *Every τ-finitely presented torsionfree* $X \in \operatorname{Mod} R$ *is torsionless.*

(b)$^{\mathrm{op}}$ *Every τ-finitely presented torsionfree* $M \in \operatorname{Mod} R^{\mathrm{op}}$ *is torsionless.*

(c) $\operatorname{Ext}_R{}^1(X, R)$ *is torsion for every finitely presented* $X \in \operatorname{Mod} R$.

(c)$^{\mathrm{op}}$ $\operatorname{Ext}_R{}^1(M, R)$ *is torsion for every finitely presented* $M \in \operatorname{Mod} R^{\mathrm{op}}$.

Proof. (a)$\Leftrightarrow$(a)$^{\mathrm{op}}$. See Hoshino [9, Theorem A].

(a)$\Rightarrow$(b). Let $X \in \operatorname{Mod} R$ be τ-finitely presented. Then there exists an epimorphism $\pi: X' \to X$ with X' finitely presented and $\operatorname{Ker} \pi$ torsion. Since π^{**} is an isomorphism, π induces an epimorphism $\operatorname{Ker} \varepsilon_{X'} \to \operatorname{Ker} \varepsilon_X$. Hence by Lemma 1.1 the assertion follows.

(b)$\Rightarrow$(a). Let $X \in \operatorname{Mod} R$ be finitely presented. Then $X/\tau(X)$ is τ-finitely presented. Hence by Lemma 1.1 the assertion follows.

(a)$\Leftrightarrow$(c)$^{\mathrm{op}}$. Let $P_1 \to P_0 \to M \to 0$ be a finite presentation in $\operatorname{Mod} R^{\mathrm{op}}$ and put $X = \operatorname{Cok}(P_0{}^* \to P_1{}^*)$. Then we have a finite presentation $P_0{}^* \to P_1{}^* \to X \to 0$ in $\operatorname{Mod} R$ with $\operatorname{Cok}(P_1{}^{**} \to P_0{}^{**}) \cong M$. Note that $\operatorname{Ext}_R{}^1(M,R) \cong \operatorname{Ker} \varepsilon_X$ by Auslander [1, Proposition 6.3]. Hence by Lemma 1.1 the assertion follows.

In the following, a ring R will be called τ-absolutely pure if it satisfies the equivalent conditions of Lemma 1.2. We notice the following.

REMARK. In Lemma 1.2, the conditions (a) and (c) are equivalent to the following conditions (a)′ and (c)′, respectively:

(a)′ $\tau(X)=\mathrm{Ker}\ \varepsilon_X$ for every τ-finitely presented $X\in\mathrm{Mod}\,R$.

(c)′ $\mathrm{Ext}_R{}^1(X, R)$ is torsion for every τ-finitely presented $X\in\mathrm{Mod}\,R$.

Lemma 1.3. *For a ring R the following are equivalent.*

(a) $\tau(X)=\mathrm{Ker}\ \varepsilon_X$ *for every finitely generated* $X\in\mathrm{Mod}\,R$.

(b) *Every finitely generated torsionfree* $X\in\mathrm{Mod}\,R$ *is torsionless.*

(c) *Every finitely generated submodule of* $E({}_RR)$ *is torsionless.*

Proof. (a)$\Rightarrow$(b). By Lemma 1.1.

(b)$\Rightarrow$(c). Obvious.

(c)$\Rightarrow$(a). See Hoshino [9, Lemma 5].

Lemma 1.4. *The implications (a)$\Rightarrow$(b)$\Rightarrow$(d) and (a)$\Rightarrow$(c)$\Rightarrow$(d) hold among the following conditions*:

(a) *Every finitely generated submodule of* $E({}_RR)$ *embeds in a projective module.*

(b) $E({}_RR)$ *is flat.*

(c) *Every finitely generated submodule of* $E({}_RR)$ *is torsionless.*

(d) *R is τ-absolutely pure.*

Proof. (a)$\Rightarrow$(b). See, e.g., Rutter [22, Lemma 2].

(a)$\Rightarrow$(c). Obvious.

(b)$\Rightarrow$(d). See the proof of Hoshino [9, Proposition B].

(c)$\Rightarrow$(d). By Lemma 1.3.

Lemma 1.5. *Assume that R is τ-absolutely pure. Then the following are equivalent.*

(a) *R is left τ-noetherian.*

(b) *R satisfies the ascending chain condition on annihilator left ideals.*

Proof. (a)$\Rightarrow$(b). Obvious.

(b)$\Rightarrow$(a). Let I be a left ideal of R. We claim that I is τ-finitely generated. By Faith [5, Proposition 3.1] I contains a finitely generated subideal I' such that $(R/I)^* \tilde{\rightarrow} (R/I')^*$. Hence $(R/I')^{**} \tilde{\rightarrow} (R/I)^{**}$ and I/I' embeds in $\mathrm{Ker}\ \varepsilon_{R/I'}$. Since R/I' is finitely presented, $\mathrm{Ker}\ \varepsilon_{R/I'}$ is torsion, so is I/I'.

The next proposition generalizes results of Morita [18, Theorem 1] and Sumioka [28, Lemma 7].

Proposition 1.6 (cf. Hoshino [9, Proposition B]). *Assume that R is right τ-coherent. Then the following are equivalent.*

(a) *R is τ-absolutely pure.*
(b) *Every torsionfree injective $E \in \operatorname{Mod} R$ is flat.*
(c) *$E({}_R R)$ is flat.*

Proof. (a) $\Rightarrow$ (b). Let $E \in \operatorname{Mod} R$ be torsionfree injective and let $M \in \operatorname{Mod} R^{\mathrm{op}}$ be finitely presented. We claim that $\operatorname{Tor}^R{}_1(M, E) = 0$. Let $0 \to N \to F \to M \to 0$ be an exact sequence in $\operatorname{Mod} R^{\mathrm{op}}$ with F free of finite rank. Since N is a finitely generated submodule of a τ-coherent module F, it follows that N is τ-finitely presented (see Jones [11]). Thus there exists an exact sequence $0 \to K \to L \to N \to 0$ in $\operatorname{Mod} R^{\mathrm{op}}$ with L finitely presented and K torsion. Let π denote the composite $L \to N \to F$. It suffices to show that $\pi \otimes_R E$ is monic. Note that $\operatorname{Ker}(\pi \otimes_R E) \cong \operatorname{Ker}(\operatorname{Hom}_R(\pi^*, E))$ because both L and F are finitely presented (see Cartan and Eilenberg [3, Chap. VI, Proposition 5.3]). Since $\operatorname{Cok} \pi^* \cong \operatorname{Ext}_R{}^1(M, R)$ is torsion, it follows that $\operatorname{Hom}_R(\pi^*, E)$ is monic.

(b) $\Rightarrow$ (c). Obvious.
(c) $\Rightarrow$ (a). By Lemma 1.4.

Proposition 1.7 (cf. Hoshino [9, Proposition C]). *Assume that R is left τ-noetherian. Then the following are equivalent.*

(a) *R is τ-absolutely pure.*
(b) *Every finitely generated submodule of $E({}_R R)$ is torsionless.*
(c) *$E(R_R)$ is flat.*

Proof. (a) $\Leftrightarrow$ (b). Since R is left τ-noetherian, every finitely generated left module is τ-finitely presented (see Sumioka [28]). By Lemma 1.3 the assertion follows.

(a) $\Leftrightarrow$ (c). By Proposition 1.6.

Finally, we formulate the symmetry established by Masaike [14, Theorem 2] as follows (cf. Sumioka [27, Proposition 1]).

Theorem 1.8. *Assume that R is τ-absolutely pure. Then the following are equivalent.*

(a) *R is left and right τ-noetherian.*
(b) *R is left and right τ-artinian.*

(c) *R is left τ-artinian.*
(c)^{op} *R is right τ-artinian.*

Proof. We have only to prove (a)$\Leftrightarrow$(c).

(a)$\Rightarrow$(c) By Lemma 1.3 and Proposition 1.7 every finitely generated torsionfree left module is torsionless. Also, R satisfies the descending chain condition on annihilator left ideals. Hence R is left τ-artinian.

(c)$\Rightarrow$(a). By Miller and Teply [17, Theorem 1.4] R is left τ-noetherian. Also, since R satisfies the acending chain condition on annihilator right ideals, R is right τ-noetherian by Lemma 1.5.

2. τ-Semicompactness of modules

In this section, we introduce the notion of τ-semicompact modules, which is closely related to the notion of reflexive modules.

Recall that a homomorphism $\pi: X' \to X$ is called a τ-epimorphism if $\operatorname{Cok} \pi$ is torsion. In the following, a module X will be called τ-semicompact if for every inverse system of τ-epimorphisms $\{\pi_\lambda: X \to Y_\lambda\}_{\lambda \in \Lambda}$ with each Y_λ torsionless, $\varprojlim \pi_\lambda$ is τ-epic. A ring R will be called left (resp. right) τ-semicompact if ${}_RR$ (resp. R_R) is τ-semicompact.

REMARKS. (1) Every epimorphic image of a τ-semicompact module is τ-semicompact.

(2) The τ-semicompactness is just the R-linear compactness, in the sense of Gómez Pardo [7], relative to Lambek torsion theory.

(3) Even if R is commutative, the τ-semicompactness differs from the semicompactness, in the sense of Matlis [16], relative to Lambek torsion theory in general. However, for modules ${}_RR$ and R_R, the τ-semicompactness coincides with the semicompactness, in the sense of Stenström [25], relative to Lambek torsion theory.

The next lemma is due essentially to Müller [19, Lemma 1] (cf. also Sandomierski [24, Lemma 3.4]).

Lemma 2.1. *Assume that every finitely generated submodule of $E({}_RR)$ is torsionless. Let $X \in \operatorname{Mod} R$ and let $j: M \to X^*$ be monic in $\operatorname{Mod} R^{op}$ with M finitely generated. Then $j^* \circ \varepsilon_X$* is τ-epic.

Proof. Let $m_1, \cdots, m_n \in M$ be generators over R and put $\alpha = {}^t(j(m_1), \cdots, j(m_n)): X \to F = {}_RR^{(n)}$. Then we have an epic $\pi: F^* \to M$ such that $\alpha^* = j \circ \pi$. Put $Y = \operatorname{Cok} \alpha$. Then we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} & X & \overset{\alpha}{\to} & F & \to & Y & \to & 0 \\ j^*\circ\varepsilon_X\downarrow & & & \downarrow\wr & & \downarrow\varepsilon_Y & & \\ 0 \to & M^* & \overset{\pi^*}{\to} & F^{**} & \to & Y^{**} & & \end{array}$$

Hence $\mathrm{Cok}(j^*\circ\varepsilon_X)\cong \mathrm{Ker}\ \varepsilon_Y$ is torsion by Lemma 1.3.

Corollary 2.2. *Assume that every finitely generated submodule of $E({}_RR)$ is torsionless. Then ε_X is τ-epic for every τ-semicompact $X\in \mathrm{Mod}\,R$.*

Proof. Let $X\in\mathrm{Mod}\,R$ be τ-semicompact. Take a direct system of monomorphisms $\{j_\lambda\colon M_\lambda\to X^*\}_{\lambda\in\Lambda}$ with each M_λ finitely generated such that $\varinjlim j_\lambda\colon \varinjlim M_\lambda\to X^*$. Then by Lemma 2.1 we get an inverse system of τ-epimorphisms $\{j_\lambda{}^*\circ\varepsilon_X\colon X\tilde{\to} M_\lambda{}^*\}_{\lambda\in\Lambda}$ with each $M_\lambda{}^*$ torsionless. Thus $(\varprojlim j_\lambda{}^*)\circ\varepsilon_X=\varprojlim\,(j_\lambda{}^*\circ\varepsilon_X)$ is τ-epic. Hence, since $\varprojlim j_\lambda{}^*\cong(\varinjlim j_\lambda)^*$ is an isomorphism, ε_X is τ-epic.

REMARK. The argument above yields that if R is τ-absolutely pure then ε_X is τ-epic for every finitely generated τ-semicompact $X\in\mathrm{Mod}\,R$.

Lemma 2.3. *Assume that R has a minimal dense left ideal. Then for an $X\in\mathrm{Mod}\,R$ the following are equivalent.*

(a) *X is τ-semicompact.*

(b) *For every inverse system of epimorphisms $\{\pi_\lambda\colon X\to Y_\lambda\}_{\lambda\in\Lambda}$ with each Y_λ torsionless, $\varprojlim \pi_\lambda$ is τ-epic.*

In particular, every $X\in\mathrm{Mod}\,R$ which satisfies the descending chain condition on submodules X' with X/X' torsionless is τ-semicompact.

Proof. (a)$\Rightarrow$(b). Obvious.

(b)$\Rightarrow$(a). Let $\{\pi_\lambda\colon X\to Y_\lambda\}_{\lambda\in\Lambda}$ be an inverse system of τ-epimorphisms with each Y_λ torsionless. For each $\lambda\in\Lambda$, let $X\overset{\alpha_\lambda}{\twoheadrightarrow}X_\lambda\overset{\beta_\lambda}{\rightarrowtail}Y_\lambda$ be an epic-monic factorization of π_λ. Since $\varprojlim\pi_\lambda=(\varprojlim\beta_\lambda)\circ(\varprojlim\alpha_\lambda)$ with $\varprojlim\beta_\lambda$ monic, we get the following exact sequence:

$$0\to\mathrm{Cok}(\varprojlim\alpha_\lambda)\to\mathrm{Cok}(\varprojlim\pi_\lambda)\to\mathrm{Cok}(\varprojlim\beta_\lambda)\to 0.$$

Note that the class of all torsion left modules is closed under taking direct products. Since we have a sequence of embeddings $\mathrm{Cok}(\varprojlim\beta_\lambda)$

$\subsetneq \varprojlim \operatorname{Cok} \beta_\lambda \subsetneq \Pi_{\lambda\in\Lambda} \operatorname{Cok} \beta_\lambda$, $\operatorname{Cok}(\varprojlim \beta_\lambda)$ is torsion. Also, $\operatorname{Cok}(\varprojlim \alpha_\lambda)$ is torsion by hypothesis. Hence $\operatorname{Cok}(\varprojlim \pi_\lambda)$ is torsion.

Proposition 2.4. *Assume that R contains an idempotent f with RfR a minimal dense left ideal and fR an injective right ideal. Then every $X \in \operatorname{Mod} R$ with ε_X τ-epic is τ-semicompact. In particular, every finitely generated $X \in \operatorname{Mod} R$ is τ-semicompact.*

Proof. Let $X \in \operatorname{Mod} R$ with ε_X τ-epic. Let $\{\pi_\lambda : X \to Y_\lambda\}_{\lambda\in\Lambda}$ be an inverse system of epimorphisms with each Y_λ torsionless. Since $(\varprojlim \varepsilon_{Y_\lambda}) \circ (\varprojlim \pi_\lambda) = (\varprojlim \pi_\lambda^{**}) \circ \varepsilon_X$, $\varprojlim \varepsilon_{Y_\lambda}$ induces homomorphisms α: $\operatorname{Im}(\varprojlim \pi_\lambda) \to \operatorname{Im}(\varprojlim \pi_\lambda^{**})$ and β: $\operatorname{Cok}(\varprojlim \pi_\lambda) \to \operatorname{Cok}(\varprojlim \pi_\lambda^{**})$. Since $\varprojlim \varepsilon_{Y_\lambda}$ is monic, $\operatorname{Ker} \beta$ embeds in $\operatorname{Cok} \alpha$. On the other hand, $\operatorname{Cok} \alpha$ is an epimorphic image of $\operatorname{Cok} \varepsilon_X$. Thus $\operatorname{Ker} \beta$ is torsion. Next, since $\varinjlim \pi_\lambda^*$ is monic, $fR \otimes_R (\varprojlim \pi_\lambda^{**}) \cong \operatorname{Hom}_R(\varinjlim \pi_\lambda^*, fR)$ is epic. Hence $\operatorname{Cok}(\varprojlim \pi_\lambda^{**})$ is torsion, so is $\operatorname{Im} \beta$. Therefore $\operatorname{Cok}(\varprojlim \pi_\lambda)$ is torsion and by Lemma 2.3 X is τ-semicompact. Finally, we claim that ε_X is τ-epic for every finitely generated $X \in \operatorname{Mod} R$. Let π: $F \to X$ be epic in $\operatorname{Mod} R$ with F free of finite rank. Put $M = \operatorname{Cok} \pi^*$. Since F is reflexive, $\operatorname{Cok} \varepsilon_X \cong \operatorname{Cok} \pi^{**} \cong \operatorname{Ext}_R{}^1(M, R)$. Thus $fR \otimes_R \operatorname{Cok} \varepsilon_X \cong \operatorname{Ext}_R{}^1(M, fR) = 0$, so that ε_X is τ-epic.

Remark. Let $X \in \operatorname{Mod} R$ be torsionless with ε_X τ-epic. Then ε_X is an essential monomorphism, so that $\cap_{\lambda\in\Lambda} \operatorname{Ker} \alpha_\lambda^{**} = 0$ for every family $\{\alpha_\lambda\}_{\lambda\in\Lambda}$ of homomorphisms $\alpha_\lambda \in X^*$ with $\cap_{\lambda\in\Lambda} \operatorname{Ker} \alpha_\lambda = 0$. Thus, if X embeds in a direct product of copies of ${}_RR$ as a closed submodule, then X is reflexive. Hence, putting Corollary 2.2 and Proposition 2.4 together, one can obtain an extension of a result of Masaike [15, Theorem 3].

Lemma 2.5. *Assume that every finitely generated submodule of $E({}_RR)$ is torsionless, and that R has a minimal dense left ideal. Then for a finitely generated $X \in \operatorname{Mod} R$ the following are equivalent.*

(a) *X is τ-semicompact.*

(b) *For every inverse system of epimorphisms $\{\pi_\lambda : X \to Y_\lambda\}_{\lambda\in\Lambda}$, $\varprojlim \pi_\lambda$ is τ-epic.*

(c) *For every inverse system of* τ-epimorphisms $\{\pi_\lambda : X \to Y_\lambda\}_{\lambda\in\Lambda}$, $\varprojlim \pi_\lambda$ *is τ-epic.*

Proof. (a) $\Rightarrow$ (b). Let $\{\pi_\lambda : X \to Y_\lambda\}_{\lambda\in\Lambda}$ be an inverse system of epimorphisms. For each $\lambda \in \Lambda$, let $\alpha_\lambda : Y_\lambda \to Y_\lambda/\tau(Y_\lambda)$ denote the canonical epimorphism. Then $\varprojlim \alpha_\lambda$ induces the following exact sequence:

$$\operatorname{Ker}(\varprojlim \alpha_\lambda) \to \operatorname{Cok}(\varprojlim \pi_\lambda) \to \operatorname{Cok}(\varprojlim \alpha_\lambda \circ \pi_\lambda).$$

Since the class of all torsion left modules is closed under taking direct products, $\mathrm{Ker}(\varprojlim \alpha_\lambda) \cong \varprojlim \tau(Y_\lambda)$ is torsion. On the other hand, each $Y_\lambda/\tau(Y_\lambda)$ is finitely generated torsionfree and thus torsionless by Lemma 1.3. Hence $\mathrm{Cok}(\varprojlim \alpha_\lambda \circ \pi_\lambda)$ is torsion by hypothesis. Therefore $\mathrm{Cok}(\varprojlim \pi_\lambda)$ is torsion.

(b) ⇒ (c). By the same argument as in the proof of (b) ⇒ (a) in Lemma 2.3.

(c) ⇒ (a). Obvious.

Corollary 2.6. *Let* $0 \to X' \xrightarrow{\alpha} X \xrightarrow{\beta} X'' \to 0$ *be an exact sequence of finitely generated modules in* Mod *R. Assume that every finitely generated submodule of* $E({}_RR)$ *is torsionless, and that R has a minimal dense left ideal. Then the following are equivalent.*

(a) *X is τ-semicompact.*
(b) *Both X′ and X″ are τ-semicompact.*

Proof. (a) ⇒ (b). Let $\{\pi_\lambda' : X' \to Y_\lambda'\}_{\lambda \in \Lambda}$ be an inverse system of epimorphisms. For each $\lambda \in \Lambda$, take a push-out of π_λ' along with α:

$$\begin{array}{ccccccccc} 0 & \to & X' & \xrightarrow{\alpha} & X & \to & X'' & \to & 0 \\ & & \pi_\lambda' \downarrow & & \downarrow \pi_\lambda & & \| & & \\ 0 & \to & Y_\lambda' & \to & Y_\lambda & \to & X'' & \to & 0. \end{array}$$

Then $\mathrm{Cok}(\varprojlim \pi_\lambda') \cong \mathrm{Cok}(\varprojlim \pi_\lambda)$ is torsion. Next, let $\{\pi_\lambda'' : X'' \to Y_\lambda''\}_{\lambda \in \Lambda}$ be an inverse system of epimorphisms. Then $\mathrm{Cok}(\varprojlim \pi_\lambda'') \cong \mathrm{Cok}(\varprojlim \beta \circ \pi_\lambda'')$ is torsion.

(b) ⇒ (a). Let $\{\pi_\lambda : X \to Y_\lambda\}_{\lambda \in \Lambda}$ be an inverse system of epimorphisms. For each $\lambda \in \Lambda$, let $X' \overset{\pi_\lambda'}{\twoheadrightarrow} Y_\lambda' \overset{\alpha_\lambda}{\rightarrowtail} Y_\lambda$ be an epic-monic factorization of $\pi_\lambda \circ \alpha$, let $\beta_\lambda : Y_\lambda \to Y_\lambda''$ denote a cokernel of α_λ, and let $\pi_\lambda'' : X'' \to X_\lambda''$ satisfy $\pi_\lambda'' \circ \beta = \beta_\lambda \circ \pi_\lambda$. Then we get the following exact sequence:

$$\mathrm{Cok}(\varprojlim \pi_\lambda') \to \mathrm{Cok}(\varprojlim \pi_\lambda) \to \mathrm{Cok}(\varprojlim \pi_\lambda'').$$

Since both $\mathrm{Cok}(\varprojlim \pi_\lambda')$ and $\mathrm{Cok}(\varprojlim \pi_\lambda'')$ are torsion, so is $\mathrm{Cok}(\varprojlim \pi_\lambda)$.

Lemma 2.7. *For a ring R the following are equivalent.*

(a) *R is τ-absolutely pure and left τ-semicompact.*
(b) $\mathrm{Ext}_R{}^1(R/I, R)$ *is torsion for every right ideal I.*

Proof. (a) $\Rightarrow$ (b). Let I be a right ideal. Take a direct system of inclusions $\{j_\lambda : I_\lambda \to I\}_{\lambda\in\Lambda}$ with each I_λ a finitely generated subideal of I such that $\varinjlim j_\lambda : \varinjlim I_\lambda \xrightarrow{\sim} I$. Let $j: I \to R$ denote the inclusion. Since $\varprojlim j_\lambda{}^* \cong (\varinjlim j_\lambda)^*$ is an isomorphism, $\mathrm{Ext}_R{}^1(R/I,R) \cong \mathrm{Cok}\, j^* \cong \mathrm{Cok}(\varprojlim j_\lambda{}^* \circ j^*)$. For each $\lambda\in\Lambda$, $I_\lambda{}^*$ is torsionless, and $\mathrm{Cok}(j_\lambda{}^* \circ j^*) \cong \mathrm{Ext}_R{}^1(R/I_\lambda,R)$ is torsion. Hence $\mathrm{Cok}(\varprojlim j_\lambda{}^* \circ j^*)$ is torsion.

(b) $\Rightarrow$ (a). By induction on the number of generators, it follows that $\mathrm{Ext}_R{}^1(M,R)$ is torsion for every finitely generated $M \in \mathrm{Mod}\, R^{\mathrm{op}}$. In particular, R is τ-absolutely pure. Next, let $\{\pi_\lambda : {}_RR \to Y_\lambda\}_{\lambda\in\Lambda}$ be an inverse system of τ-epimorphisms with each Y_λ torsionless. Since $\varprojlim \varepsilon_{Y_\lambda}$ is monic, $\mathrm{Cok}(\varprojlim \pi_\lambda)$ embeds in $\mathrm{Cok}(\varprojlim \pi_\lambda{}^{**})$. Identify $({}_RR)^*$ with R_R and put $I = \mathrm{Im}(\varinjlim \pi_\lambda{}^*)$. Since $\varinjlim \pi_\lambda{}^*$ is monic, and since $\varprojlim Y_\lambda{}^{**} \cong (\varinjlim Y_\lambda{}^*)^*$, $\mathrm{Cok}(\varprojlim \pi_\lambda{}^{**}) \cong \mathrm{Ext}_R{}^1(R/I,R)$. Thus $\mathrm{Cok}(\varprojlim \pi_\lambda{}^{**})$ is torsion, so is $\mathrm{Cok}(\varprojlim \pi_\lambda)$.

3. Idempotent generated minimal dense ideals

In this section, we collect several basic results on idempotent generated minimal dense ideals which we use in the next section.

Remarks. (1) For an idempotent $f \in R$, RfR is a minimal dense left ideal if and only if $\mathrm{Ker}(fR\otimes_R -) = \mathrm{Ker}(\mathrm{Hom}_R(-, E({}_RR)))$. Thus, if RfR is a minimal dense left ideal with f an idempotent, then $fR\otimes_R -$: $\mathrm{Mod}\, R \to \mathrm{Mod}\, fRf$ induces $\mathrm{Mod}\, R/\tau \cong \mathrm{Mod}\, fRf$, where $\mathrm{Mod}\, R/\tau$ denotes the quotient category of $\mathrm{Mod}\, R$ over the full subcategory $\mathrm{Ker}(\mathrm{Hom}_R(-,E({}_RR)))$.

(2) Assume that R is right perfect. Then R contains an idempotent f with RfR a minimal dense left ideal (see Storrer [26]).

Lemma 3.1 (Rutter [23, Theorem 1.4]). *For an idempotent $f \in R$ the following are equivalent.*

(a) *RfR is a minimal dense left ideal.*

(b) *fR_R is faithful and every simple homomorphic image of ${}_RRf$ is torsionless.*

Corollary 3.2. *Let $f \in R$ be an idempotent with RfR a minimal dense left ideal and fR an injective right ideal, and let $f_1 \in fRf$ be a local idempotent. Then $(Rf_1/Jf_1)^*$ is cocritical and embeds in f_1R_R, where J denotes the Jacobson radical of R.*

Proof. Note that ${}_RRf_1/Jf_1$ and ${}_{fRf}fR\otimes_R(Rf_1/Jf_1)$ are simple.

Thus by Lemma 3.1 $(Rf_1/Jf_1)^* \neq 0$. Since fR_R is injective and faithful by Lemma 3.1, it is sufficient for $(Rf_1/Jf_1)^*$ to be cocritical that ${}_{fRf}\mathrm{Hom}_R((Rf_1/Jf_1)^*, fR) \cong {}_{fRf}fR \otimes_R (Rf_1/Jf_1)^{**}$ is simple. Let $\pi: Rf_1 \to Rf_1/Jf_1$ denote the canonical epimorphism. Then, since Rf_1 is reflexive, $\mathrm{Cok}\ \varepsilon_{Rf_1/Jf_1} \cong \mathrm{Cok}\ \pi^{**}$. Thus, since $fR \otimes_R \pi^{**} \cong \mathrm{Hom}_R(\pi^*, fR)$ is epic, so is $fR \otimes_R \varepsilon_{Rf_1/Jf_1}$. Hence ${}_{fRf}fR \otimes_R (Rf_1/Jf_1)^{**}$ is simple. The last statement is obvious.

Corollary 3.3 (cf. Rutter [23. Corollary 1.2]). *Let $f = f_1 + \cdots + f_n$ be an orthogonal sum of local idempotents f_i in R. Assume that fR_R is faithful and injective, and that each f_iR_R contains a cocritical submodule M_i. Then RfR is a minimal dense left ideal. In particular, R is left τ-semicompact.*

Proof. Let J denote the Jacobson radical of R. We claim that each ${}_RRf_i/Jf_i$ is torsionless. Since every nonzero $h \in \mathrm{Hom}_R(M_i, fR)$ is monic, it follows that ${}_{fRf}\mathrm{Hom}_R(M_i, fR) \cong {}_{fRf}fRf_i/fJf_i$. Thus $\mathrm{Hom}_{fRf}(fRf_i/fJf_i, fR) \neq 0$, which implies $\mathrm{Hom}_R(Rf_i/Jf_i, R) \neq 0$. Hence by Lemma 3.1 RfR is a minimal dense left ideal. It then follows by Proposition 2.4 that R is left τ-semicompact.

Lemma 3.4. *Let $f \in R$ be an idempotent with RfR a minimal dense left ideal. Then ${}_RRfX$ is simple for every $X \in \mathrm{Mod}\, R$ with $X/\tau(X)$ cocritical. In particular, every cocritical $X \in \mathrm{Mod}\, R$ has a nonzero socle.*

Proof. Let $X \in \mathrm{Mod}\, R$ with $X/\tau(X)$ cocritical. We may assume that $\tau(X) = 0$. Let X' be a nonzero submodule of X. Then X/X' is torsion, so that $RfX \subset X'$. Hence ${}_RRfX$ is simple.

As pointed out by Stenström [25, Proposition 2.5], the argument of Matlis [16, Propositions 2 and 3] would yield the following.

Proposition 3.5. *Let $f \in R$ be an idempotent with RfR a minimal dense left ideal. Then the following are equivalent.*

(a) *fR is an injective right ideal.*

(b) *R is τ-absolutely pure and left τ-semicompact.*

Proof. (a) $\Rightarrow$ (b). For an $M \in \mathrm{Mod}\, R^{\mathrm{op}}$, $fR \otimes_R \mathrm{Ext}_R{}^1(M, R) \cong \mathrm{Ext}_R{}^1(M, fR) = 0$ implies $\mathrm{Ext}_R{}^1(M, R)$ torsion. Thus R is τ-absolutely pure. Also, R is left τ-semicompact by Proposition 2.4.

(b) $\Rightarrow$ (a). By Lemma 2.7 $\mathrm{Ext}_R{}^1(R/I, fR) \cong fR \otimes_R \mathrm{Ext}_R{}^1(R/I, R) = 0$ for every right ideal I.

Proposition 3.6. *Let $f \in R$ be an idempotent with RfR a minimal dense left ideal. Assume that every finitely generated submodule of $E({}_RR)$ is torsionless, and that R is left τ-semicompact. Then every $X \in \mathrm{Mod}\, fRf$ with ${}_RRf \otimes_{fRf} X$ finitely generated is linearly compact in the usual sense. In particular, fRf is a semiperfect ring.*

Proof. Let $X \in \mathrm{Mod}\, fRf$ with ${}_RRf \otimes_{fRf} X$ finitely generated. Let $\{\pi_\lambda : X \to Y_\lambda\}_{\lambda \in \Lambda}$ be an inverse system of epimorphisms in $\mathrm{Mod}\, fRf$. Then $\{Rf \otimes_{fRf} \pi_\lambda : Rf \otimes_{fRf} X \to Rf \otimes_{fRf} Y_\lambda\}_{\lambda \in \Lambda}$ is an inverse system of epimorphisms in $\mathrm{Mod}\, R$. It follows by Corollary 2.6 that every free left R-module of finite rank is τ-semicompact. Thus every finitely generated left R-module is τ-semicompact. Hence by Lemma 2.5

$$\begin{aligned} \mathrm{Cok}(\varprojlim \pi_\lambda) &\cong \mathrm{Cok}(\varprojlim \mathrm{Hom}_R(Rf, Rf \otimes_{fRf} \pi_\lambda)) \\ &\cong \mathrm{Cok}(\mathrm{Hom}_R(Rf, \varprojlim Rf \otimes_{fRf} \pi_\lambda)) \\ &\cong \mathrm{Hom}_R(Rf, \mathrm{Cok}(\varprojlim Rf \otimes_{fRf} \pi_\lambda)) \\ &= 0, \end{aligned}$$

so that X is linearly compact in the usual sense (see, e.g., Gómez Pardo [7, Proposition 1]). Since ${}_RRf \otimes_{fRf} fRf$ is finitely generated, it follows that fRf is a semiperfect ring (see Kasch and Mares [12] and Sandomierski [24]).

4. QF-3 rings

In this section, generalizing a result of Masaike [15, Theorem 5], we provide a characterization of left and right QF-3 rings.

To point out the difference between "one-sided QF-3 rings" and "two-sided QF-3 rings", we first provide a characterization of right QF-3 rings.

Proposition 4.1. *For a ring R the following are equivalent.*

(1) *R is right QF-3.*

(2) (a) *R is τ-absolutely pure.*

(b) *R is left τ-semicompact.*

(c) *R contains an idempotent f such that RfR is a minimal dense left ideal and fRf is a semiperfect ring.*

(d) *Every cocritical right module has a nonzero socle.*

Proof. (1) $\Rightarrow$ (2). Let $f \in R$ be an idempotent with fR a minimal faithful right module. Then by Rutter [21, Theorem 1] fR_R is faithful, injective of finite Goldie dimension and has an essential socle. Also, by Rutter [23, Corollary 1.2] RfR is a minimal dense left ideal. Thus by Proposition 3.5 (a) and (b) hold. Since fR_R is injective of finite Goldie dimension, $fRf \cong \mathrm{End}(fR_R)$ is semiperfect, so (c) holds. It is obvious that every cocritical right module embeds in fR_R. Since fR_R has an essential socle, (d) holds.

(2)$\Rightarrow$(1). We may assume that fRf is a selfbasic ring. Note that fR_R is faithful by Lemma 3.1 and injective by Proposition 3.5. Let $f=f_1+\cdots+f_n$ be an orthogonal sum decomposition into local idempotents. Then by Corollary 3.2 each f_iR_R contains a cocritical submodule, so that each f_iR_R has a nonzero socle. Hence by Rutter [21, Theorem 1] $fR_R \cong f_1R_R \oplus \cdots \oplus f_nR_R$ is a minimal faithful right module.

Theorem 4.2. *For a ring R the following are equivalent.*

(1) *R is left and right QF*-3.

(2) (a) *R is τ-absolutely pure.*

(b) *R is left and right τ-semicompact.*

(c) *R contains idempotents e, f such that ReR and RfR are minimal dense right and left ideals, respectively.*

Proof. (1)$\Rightarrow$(2). By Proposition 4.1.

(2)$\Rightarrow$(1). By symmetry, it suffices to show that R is right QF-3. By Lemma 3.1 and Proposition 3.5 ${}_RRe$ is faithful and injective. Hence every torsionfree left module is torsionless, so that by Proposition 3.6 fRf is a semiperfect ring. Also, by Lemma 3.4 every cocritical right module has a nonzero socle. Hence by Proposition 4.1 R is right QF-3.

REMARK. Assume that R is left and right perfect. Then in Proposition 4.1 (c) and (d) of (2) are satisfied. Thus R is right QF-3 if and only if R is τ-absolutely pure and left τ-semicompact.

Corollary 4.3 (cf. Sumioka [28, Theorem 8]). *For a ring R the following are equivalent.*

(1) *R is semiprimary, left and right QF*-3.

(2) (a) *R is τ-absolutely pure.*

(b) *R is left perfect.*

(c) *R is either left τ-noetherian or right τ-coherent.*

Proof. (1) ⇒ (2). It only remains to see that (c) holds. We claim that R is left and right τ-artinian. Let $f \in R$ be an idempotent with fR a minimal faithful right module. By Rutter [23, Corollary 1.2] RfR is a minimal dense left ideal. Also, by Colby and Rutter [4, Theorem 1.3] ${}_{fRf}fR$ is artinian. Thus R is left τ-artinian, since $fR \otimes_R -$: Mod $R \rightarrow$ Mod fRf induces Mod $R/\tau \cong$ Mod fRf, where Mod R/τ denotes the quotient category of Mod R over tha full subcategory Ker(Hom$_R$(–, $E({}_RR)$)). By symmetry, R is also right τ-artinian.

(2) ⇒ (1). It suffices to show that R is semiprimary, left and right τ-semicompact. In case R is right τ-coherent, by Proposition 1.6 every torsionfree injective left module is projective and by Masaike [14, Theorem 1] R is left τ-artinian. So we may restrict ourselves to the case where R is left τ-noetherian. Then by Faith [5, Proposition 4.1] R is semiprimary and thus left τ-artinian. By Theorem 1.8 R is also right τ-artinian. It now follows by Lemma 2.3 that R is left and right τ-semicompact.

5. Maximal quotient rings

In this section, we deal with the case where R has a maximal two-sided quotient ring. Recall that a maximal left (resp. right) quotient ring Q_l (resp. Q_r) of R is defined as a biendomorphism ring of $E({}_RR)$ (resp. $E(R_R)$), and that R is said to have a maximal two-sided quotient ring if $Q_l \cong Q_r$ as ring extensions of R.

In the following, we denote by Mod R/τ the quotient category of Mod R over the full subcategory Ker(Hom$_R$(–, $E({}_RR)$)). Also, Mod R^{op}/τ denotes the quotient category of Mod R^{op} over the full subcategory Ker(Hom(–, $E(R_R)$)).

REMARKS. (1) Let ${}_RQ$ be a maximal rational extension of ${}_RR$. Then Q has a ring structure such that the inclusion $R \rightarrow \mathrm{Q}$ is a ring homomorphism. Furthermore, as a ring extension of R, Q is isomorphic to a maximal left quotient ring of R.

(2) Let Q be a maximal left quotient ring of R, and let $\mathscr{L}$: Mod $R \rightarrow$ Mod R denote the localization functor associated with Lambek torsion theory. Then the correspondence $I \mapsto \mathscr{L}(I)$ gives rise to an isomorphism from the lattice of all closed left ideals of R to the lattice of all closed left ideals of Q. Hence R is left τ-artinian (resp. τ-noetherian) if and only if so is Q.

(3) Let Q be a maximal left quotient ring of R. Then Hom$_Q({}_QQ_R$,–):2 Mod $Q \rightarrow$ Mod R induces Mod $Q/\tau \cong$ Mod R/τ.

In his proof of [14, Lemma 1], K. Masaike showed the following.

Proposition 5.1 (cf. Vinsonhaler [31, Theorem A]). *Assume that $E({}_RR)$ is τ-noetherian. Then R is left τ-artinian.*

Proof. Let Q be a maximal left quotient ring of R. It sufficies to show that Q is left τ-artinian. Since ${}_R\mathrm{Hom}_Q({}_QQ_R, E({}_QQ)) \cong E({}_RR)$ is τ-noetherian, it follows that $E({}_QQ)$ is τ-noetherian. In particular, Q is left τ-noetherian. On the other hand, it follows by the argument of Masaike [14, Lemma 1] that Q is semiprimary. Hence Q is left τ-artinian.

The next lemma seems to be known.

Lemma 5.2. *Assume that R is left τ-artinian. Then* $\mathrm{Mod}\, R/\tau \cong \mathrm{Mod}\, A$ *with A left artinian.*

Proof. Let Q be a maximal left quotient ring of R. It is known that Q is semiprimary (see Faith [6, Part I, Corollary 7.5]). However, for the benefit of the reader, we provide an elementary proof of this fact. Let $H = \mathrm{End}(E({}_RR))^{\mathrm{op}}$, the opposite ring of $\mathrm{End}(E({}_RR))$, operate on $E({}_RR)$ by the right hand side. We claim that $E({}_RR)_H$ has a finite composition length. Note that R is also left τ-noetherian by Miller and Teply [17, Therem 1.4]. Thus there exists a chain of left ideals of R:

$$0 = I_0 \subset I_1 \subset \cdots \subset I_n = R$$

such that $(I_{i+1}/I_i)/\tau(I_{i+1}/I_i)$ is cocritical for $0 \le i < n$ (see, e.g., Sumioka [28]). Hence it suffices to show that $\mathrm{Hom}_R(X, E({}_RR))_H$ is simple for every $X \in \mathrm{Mod}\, R$ with $X/\tau(X)$ cocritical. Let $X \in \mathrm{Mod}\, R$ with $X/\tau(X)$ cocritical. Since $\mathrm{Hom}_R(X/\tau(X), E({}_RR))_H \cong \mathrm{Hom}_R(X, E({}_RR))_H$, we may assume that $\tau(X) = 0$. Let $\alpha, \beta \in \mathrm{Hom}_R(X, E({}_RR))$ with $\alpha \neq 0$. Then α is monic, so that $\beta = \alpha h$ for some $h \in H$. Hence $\mathrm{Hom}_R(X, E({}_RR))_H$ is simple. Therefore Q is semiprimary. Note that $\mathrm{Mod}\, Q/\tau \cong \mathrm{Mod}\, R/\tau$. Since Q contains an idempotent f with QfQ a minimal dense left ideal of Q, $\mathrm{Mod}\, Q/\tau \cong \mathrm{Mod}\, fQf$. Consequently, $\mathrm{Mod}\, R/\tau \cong \mathrm{Mod}\, fQf$. Finally, since ${}_QQ$ is τ-artinian, ${}_{fQf}fQ$ is artinian. In particular, fQf is left artinian.

After completing the first version of this note, the authors found that the next proposition had been observed by Gómez Pardo and Guil Asensio [8].

Proposition 5.3. *Assume that R is τ-absolutely pure and left*

τ-artinian. Then there exist a left artinian ring A and a right artinian ring B such that $\operatorname{Mod} R/\tau \cong \operatorname{Mod} A$, $\operatorname{Mod} R^{\mathrm{op}}/\tau \cong \operatorname{Mod} B^{\mathrm{op}}$ *and A is left Morita dual to B.*

Proof. By Proposition 1.7 and Masaike [14, Theorem 2], R has a maximal two-sided quotient ring Q which is semiprimary, left and right QF-3. Let $e, f \in Q$ be idempotents such that ${}_QQe$ and fQ_Q are minimal faithful left and right Q-modules, respectively. Then by Tachikawa [30, Theorem 5.1] fQf is left Morita dual to eQe, and then by Osofsky [20, Theorem 3] fQf is left artinian and eQe is right artinian. Finally, by Lemma 5.2 $\operatorname{Mod} R/\tau \cong \operatorname{Mod} fQf$ and $\operatorname{Mod} R^{\mathrm{op}}/\tau \cong \operatorname{Mod} eQe^{\mathrm{op}}$.

In case R is commutative, the next proposition is well known (see Bass [2, Proposition 6.1]).

Proposition 5.4. *Assume that R is left and right noetherian. Then the following are equivalent.*

(1) *$E({}_RR)$ is flat.*
(2) (a) *R has a maximal two-sided quotient ring.*
(b) *X^* is reflexive for every finitely generated $X \in \operatorname{Mod} R$.*

Proof. (1) $\Rightarrow$ (2). By Proposition 1.7 and Masaike [14, Theorem 2], (a) holds. Also, by Jans [10, Corollary 1.5] and Cartan and Eilenberg [3, Chap. VI, Proposition 5.3], (b) holds.

(2) $\Rightarrow$ (1). By Hoshino [9, Proposition F], it suffices to show that weak dim $E({}_RR) \leq 1$. Let $M \in \operatorname{Mod} R^{\mathrm{op}}$ be finitely generated. By Jans [10, Corollary 1.5] $\operatorname{Ext}_R^2(M, R)^* = 0$, thus by Sumioka [29, Proposition 3] $\operatorname{Ext}_R^2(M, R)$ is torsion. Hence by Cartan and Eilenberg [3, Chap. VI, Proposition 5.3] $\operatorname{Tor}^R{}_2(M, E({}_RR)) \cong \operatorname{Hom}_R(\operatorname{Ext}_R{}^2(M, R), E({}_RR)) = 0$. Therefore weak dim $E({}_RR) \leq 1$.

Proposition 5.5. *Assume that R is τ-absolutely pure, left and right τ-semicompact. Then R has a maximal two-sided quotient ring.*

Proof. By Lemma 2.7 and Sumioka [28, Proposition 6].

Proposition 5.6. *Let Q be a maximal left quotient ring of R. Assume that R has a minimal dense right ideal. Then the following are equivalent.*

(*a*) *${}_RQ$ is torsionless.*
(b) *Every finitely generated submodule of ${}_RQ$ is torsionless.*

Proof. (a) ⇒ (b). Obvious.

(b) ⇒ (a). It suffices to show that for each nonzero $q \in Q$ there exists an $r \in R$ such that $Qr \subset R$ and $qr \neq 0$. Note that by Masaike [13, Proposition 2] the inclusion $R_R \to Q_R$ is a rational extension. Put $I = \{r \in R \mid Qr \subset R\}$. Since R/I embeds in a direct product of copies of $(Q/R)_R$, I is a dense right ideal and $\mathrm{Hom}_R(R/I, Q) = 0$. Hence for each nonzero $q \in Q$ there exists an $r \in I$ with $qr \neq 0$.

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