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A COCHAIN COMPLEX ASSOCIATED TO THE STEENROD ALGEBRA

In Memory of the late Professor José Adem

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0. Introduction

In [8], the author introduced an acyclic, free resolution of the ground ring $\boldsymbol{Z}$ of integers (resp. its localization $\boldsymbol{Z}_{(p)}$ for a prime p) as the trivial module over the Landweber-Novikov algebra S (resp. $S_{(p)}=\boldsymbol{Z}_{(p)}\otimes S$), which is considerably smaller than the bar resolution.

In this paper, the same method of construction is applied to the case of the mod p Steenrod algebra A. The resulted resolution $X=A\otimes\bar{X}\overset{\varepsilon}{\rightarrow}\boldsymbol{Z}/p$ has inductively defined differential d and contracting homotopy σ, and is naturally embedded in the bar resolution $B(A)$ as a direct-summand subcomplex.

The apparent feature of this resolution is that it seems to be an immediate 'lift' of the May resolution [5], while the latter is a resolution over the associated graded algebra E^0A for the augmentation filtration on the Steenrod algebra. In fact, the corresponding filtration on X leads to an equivalent of the May spectral sequence, of which E^1X is isomorphic to the May resolution and E^r-terms are the same as those of the May spectral sequence for $r\geq 2$.

In the case $p=2$, the chain complex $\bar{X}$ will be given as a polynomial ring P, and the dual cochain complex P^* has a non-associative product, which induces the usual associative product in its cohomology $H^*(A)=\mathrm{Ext}_A^{**}(\boldsymbol{Z}/2,\boldsymbol{Z}/2)$, the E_2-term of the Adams spectral sequence [1,2].

May [5] studied extensively his spectral sequence and succeeded to obtain a great deal of information about $H^*(A)$ (See also, Tangora [10] and Novikov [7].).

It is hoped that the present work could be useful for calculating the differentials in the May spectral sequence and the ring structure of $H^*(A)$.

In this paper we shall restrict ourselves to the case $p=2$. A parallel treatment for the odd prime case will be only suggsted in the last section.

1. Notation and results

Let A_* be the dual Hopf algebra ([6],[9]) of the mod 2 Steenrod algebra A. A_* is given as the polynomial algebra $\boldsymbol{Z}/2[\xi_1,\xi_2,\cdots]$ over $\boldsymbol{Z}/2$ on indeterminates $\xi_i (i\geq 1)$ of degree 2^i-1, with comultiplication

$$\psi\xi_k=\sum_{i=0}^{k}\xi_{k-i}^{2^i}\otimes\xi_i \qquad (\xi_0=1).$$

Let $e_{i,k}=(\xi_i^k)^*$ denote the dual element of ξ_i^k with respect to the monomial basis $\{\xi_\omega=\xi_1^{k_1}\cdots\xi_n^{k_n}\}$ of A_*.

Lemma 1.1. (i) *The Steenrod algebra A is multiplicatively generated by the set* $\{e_{i,2^k};\ i\geq 1,k\geq 0\}$, (ii) *the set* $\{1,e_{i_1,2^{k_1}}\cdots e_{i_n,2^{k_n}};\ (i_1,k_1)<(i_2,k_2)<\cdots<(i_n k_n)$ *in the lexicographical order*$\}$ *forms a $\boldsymbol{Z}/2$-basis of A, of which elements* $e_I=e_{i_1,2^{k_1}}\cdots e_{i_n,2^{k_n}}$ *are called admissible monomials.*

Let L denote the $\boldsymbol{Z}/2$-submodule of A spanned by the set $\{e_{i,2^k};\ i\geq 1,k\geq 0\}$, and $sL=\boldsymbol{Z}/2\{\langle e_{i,2^k}\rangle;\ i\geq 1,k\geq 0\}$, the suspension of L, with bideg $\langle e_{i,2^k}\rangle=(1,2^k(2^i-1))$. Denote by $P=P(sL)$ the polynomial algebra (symmetric tensor algebra) on sL. We use the notation

$$\langle e_J\rangle=\langle e_{j_1,2^{l_1}},\cdots,e_{j_s,2^{l_s}}\rangle=\langle e_{j_1,2^{l_1}}\rangle\otimes\cdots\otimes\langle e_{j_s,2^{l_s}}\rangle$$

with the index sequence

$$J:\ (j_1,l_1)\leq(j_2,l_2)\leq\cdots\leq(j_s,l_s),$$

in the lexicographical order and call it a canonical monomial in P.

Theorem 1.2. *$X=A\otimes P$, with an inductively defined differential d gives an acyclic A-free resolution of $\boldsymbol{Z}/2$.*

Proposition 1.3. *There exist natural A-linear chain maps f: $X\to B(A)$ and g: $B(A)\to X$, such that $g\circ f=id$ and $f(P)\subset\bar{B}(A)=\boldsymbol{Z}/2\otimes_A B(A)\subset B(A)$.*

Proposition 1.4. *The chain complex P with the induced differential $\bar{d}=\boldsymbol{Z}/2\otimes_A d$ has a comultiplication $\Delta:P\to P\otimes P$ such that $(\bar{d}\otimes 1+1\otimes\bar{d})\Delta=\Delta\bar{d}$. This is not coassociative in general, but $(\Delta\otimes 1)\Delta$ and $(1\otimes\Delta)\Delta$ are chain homotopic.*

Corollary 1.5. *The dual complex P^* of P with diffeential $\delta=\bar{d}^*$ has a non-associative product, therein δ is a derivation. This product induces the usual product in the cohomology $H^*(P^*,\delta)=H^*(A)$.*

2. Preliminary

The lemma 1.1 may be well-known ([6],[4]), but we will recall its proof, since the resolution (Theorem 1.2) stems from the lemma.

We shall take the dual basis $\{\xi_\omega^*\}$ of A (See §1). By definition the product of basis elements is given by

$$\xi_\omega^* \cdot \xi_\sigma^* = \sum_\tau (\xi_\omega^* \otimes \xi_\sigma^*)(\psi \xi_\tau) \cdot \xi_\tau^*.$$

Define the height of ξ_ω^* to be Σk_i, the sum of exponents in the monomial $\xi_\omega = \xi_1^{k_1} \cdots \xi_n^{k_n}$. Then we have the equality

$$\xi_{\omega'}^* \cdot (\xi_n^{k_n})^* = \xi_\omega^* + \sum_\sigma \xi_\sigma^*, \tag{2.1}$$

where $\xi_{\omega'} = \xi_1^{k_1} \cdots \xi_{n-1}^{k_{n-1}}$, $\xi_\omega = \xi_{\omega'} \cdot \xi_n^{k_n}$, and the second summand in the right hand sideis a sum of suitable basis elements of height $h(\xi_\sigma^*) < h(\xi_\omega^*)$. In fact, ξ_σ are so chosen that $\psi \xi_\sigma$ containt $\xi_{\omega'} \otimes \xi_n^{k_n}$ as a summand, and such a ξ_σ must be of the form

$$\xi_\sigma = \xi_1^{u_1} \cdots \xi_{n+1}^{u_{n-1}} \xi_n^{v_0} \xi_{n+1}^{v_1} \cdots \xi_{2n-1}^{v_{n-1}}, \tag{2.2}$$

with $\sum_{i=0}^{n-1} v_i = k_n$, $u_i + 2^n \cdot v_i = k_i$ (for $1 \le i \le n-1$).

Then

$$h(\xi_\sigma^*) = \sum_{i=1}^{n-1} u_i + \sum_{i=0}^{n-1} v_i = \sum_{i=0}^{n} k_i - 2^n \sum_{i=1}^{n-1} v_i < \sum_{i=0}^{n} k_i = h(\xi_\omega^*).$$

Now by induction on height we conclude that any basis element ξ_ω^* of A can be expressed by a sum of products of $e_{i,k} = (\xi_i^k)^*$. But we can see easily that $e_{i,k}$ with k, *not a power of* 2, *is also decomposable into a sum of products of* $e_{i,2^t}$. This proves (i) of Lemma 1.1.

Note further that

$$(\xi_i^k)^* \cdot (\xi_i^l)^* = \binom{k+1}{k} (\xi_i^{k+l})^* + \Sigma \textit{ terms of lower height} \tag{2.3}$$

and

$$[(\xi_i^k)^*, (\xi_j^l)^*] = \Sigma \textit{ terms of lower height} \text{ for } i \neq j. \tag{2.4}$$

It follows then (ii) of Lemma 1.1.

Here are a few examples of (2.3) and (2.4):

$[e_{1,1}, e_{1,2}]=e_{2,1},\ [e_{1,1}, e_{2,1}]=0$
$[e_{1,1}, e_{2,2}]=e_{3,1}=[e_{1,4}, e_{2,1}],$
$[e_{1,2}, e_{2,2}]=e_{1,1}\cdot e_{3,1},$
$e_{1,2}\cdot e_{1,2}=e_{1,1}\cdot e_{2,1},$
$e_{1,4}\cdot e_{1,4}=e_{1,2}\cdot e_{2,2},$
$e_{1,8}\cdot e_{1,8}=e_{1,4}\cdot e_{2,4}+e_{2,1}\cdot e_{2,2}\cdot e_{3,1}$
$[e_{1,1}, e_{1,64}]=e_{1,62}\cdot e_{2,1}+e_{1,58}\cdot e_{3,1}+e_{1,50}\cdot e_{4,1}+e_{1,34}\cdot e_{5,1}+e_{1,2}\cdot e_{6,1}$
$e_{i,1}\cdot e_{i,1}=0\ (i\geq 1)$, etc. (Cf. [4])

It will be another interesting problem to give the explicit formulae expressing (2.3) and (2.4) by *admissible monomials* in the sense of §1, like the Adem relations [3].

3. Resolution

In this section we shall give a detailed proof of Theorem 1.2, since we had remained in showing only a sketchy proof in [8] for the case of the Landweber-Novikov algebra. Clearly the set of canonical monomials $\langle e_J\rangle$ forms a $\boldsymbol{Z}/2$-basis of P. Then $P=\sum_{s\geq 0}P_s$, where the submodule P_s is spanned by $\langle e_J\rangle$ of length $|J|=s$. We call $|J|$ also the homological dimension of $\langle e_J\rangle$.

We shall introduce in $X=A\otimes P$ a boundary operator $d=(d_s)$:

$$d_s\colon X_s=A\otimes P_s\to X_{s-1}$$

and a contracting homotopy $\sigma=(\sigma_s)$:

$$\sigma_s\colon X_s\to X_{s+1},$$

so that X becames an acyclic differential A-module (a chain complex) with augmentation $\varepsilon\colon X\to \boldsymbol{Z}/2$

First define an A-map $d_1\colon X_1=A\otimes sL\to X_0=A$ by

$$d_1(a\langle e_{i,2^k}\rangle)=a\cdot e_{i,2^k}\ (a\langle e_{i,2^k}\rangle \text{ means } a\otimes\langle e_{i,2^k}\rangle), \tag{3.1}$$

and a $\boldsymbol{Z}/2$-map $\sigma_0\colon X_0\to X_1$ by

$$\sigma_0(1)=0 \tag{3.2}$$
$$\sigma_0(e_{i_1,2^{k_1}}\cdots e_{i_n,2^{k_n}})=e_{i_1,2^{k_1}}e_{i_{n-1},2^{k_{n-1}}}\langle e_{i_n,2^{k_n}}\rangle$$

for admissible monomials. Thus we have a direct sum decomposition

(3.3) $$X_1 = \mathrm{Im}\,\sigma_0 \oplus \mathrm{Ker}\, d_1, \ \mathrm{Ker}\, d_1 = \mathrm{Im}(1-\sigma_0 d_1),$$
$$\sigma_0\eta = 0, \ \varepsilon d_1 = 0 \text{ and } d_1\sigma_0 + \eta\varepsilon = 1,$$

where η: $\boldsymbol{Z}/2 \to A$ is the unit. Then d_2 is easily defined by

(3.4) $$d_2\langle e_{j_1,2^{l_1}}, e_{j_2,2^{l_2}}\rangle = (1-\sigma_0 d_1)(e_{j_2,2^{l_2}}\langle e_{j_1,2^{l_1}}\rangle) \quad ((j_1,l_1) \leq (j_2,l_2)).$$

On the other hand, it is laborious to find and formulate a proper candidate of possible contracting homotopy σ_1. In order to overcome this difficulty, we begin with a careful observation of the construction X.

Take the set of elements

(3.5) $$e_I\langle e_J\rangle = e_{i_1,2^{k_1}} \cdots e_{i_n,2^{k_n}} \langle e_{j_1,2^{l_1}}, \cdots e_{j_s,2^{l_s}}\rangle$$

with the index sequences $I = (i_1,k_1) < \cdots < (i_n,k_n)$ and J: $(j_1,l_1) \leq \cdots \leq (j_s,l_s)$ in the lexicographical order, and call it canonical basis of $X = A \otimes P$.

Classify the canonical basis elements (*c.b.e.'s*) into the following types:

(3.6) *Type* 1: $\max I < \max J$ (*i.e.* $(i_n,k_n) < (J_s,l_s)$)

and

Type 2: $\max I \geq \max J$.

Put

(3.7) $$C_{1,s} = \boldsymbol{Z}/2\{\text{c.b.e. of } \textit{Type } 1 \text{ in } X_s\}$$

and

$$C_{2,s} = \boldsymbol{Z}/2\{\text{c.b.e. of } \textit{Type } 2 \text{ in } X_s\}$$

Then we have

(3.8) $$X_s = C_{1,s} \oplus C_{2,s},$$

as a $\boldsymbol{Z}/2$-module, with obvious isomorphisms

$$C_{1,s} \underset{\sigma'_{s-1}}{\overset{\tau_s}{\rightleftarrows}} C_{2,s-1}, \ \sigma'_{s-1} = \tau_s^{-1},$$

defined by

(3.9) $$\tau_s(e_I\langle e_J\rangle) = e_{I+(j_s,l_s)}\langle e_{J-(j_s,l_s)}\rangle \text{ for } e_I\langle e_J\rangle \in C_{1,s},$$

$$\sigma'_{s-1}(e_I\langle e_J\rangle)=e_{I-(i_n,k_n)}\langle e_{J+(i_n,k_n)}\rangle \text{ for } e_I\langle e_J\rangle\in C_{2,s-1}.$$

We shall introduce here a partial order in the set of index sequences J of the same length $|J|=s$ as follows:

(3.10) $J'\leq J$ if $(j'_i,l'_i)\leq(j_i,l_i)$ for all i, and

$J'<J$ if, moreover, $(j'_i,l'_i)<(j_i,l_i)$ for at least one i.

Now assume that (d_i,σ_{i-1}) are defined for $1\leq i\leq s-1$ and satisfy the following conditions (for convenience, put $d_0=\varepsilon$ and $\sigma_{-1}=\eta$):

(3.11) (A_i) $\sigma_{i-1}\sigma_{i-2}=0$ and $\operatorname{Im}\sigma_{i-1}=C_{1,i}$,

(B_i) $X_i=\operatorname{Im}\sigma_{i-1}\oplus\operatorname{Ker}d_i$,

(C_i) $d_i\sigma_{i-1}+\sigma_{i-2}d_{i-1}=1$ and $d_{i-1}d_i=0$,

(D_i) (i) There is a $\boldsymbol{Z}/2$-isomorphism φ_i: $C_{2,i}\to\operatorname{Ker}d_i$, defined by $\varphi_i(e_I\langle e_J\rangle)=e_{I'}\cdot(l-\sigma_{i-1}d_i)(e_{i_n,2^{k_n}}\langle e_J\rangle)$ for $e_I\langle e_J\rangle\in C_{2,i}$ and $e_I=e_{I'}\cdot e_{i_n,2^{k_n}}$,

(ii) Further, we have $\varphi_i(e_I\langle e_J\rangle)=e_I\langle e_J\rangle+\Sigma_\alpha e_{I_\alpha}\langle e_{J_\alpha}\rangle$, where $e_{I_\alpha}\langle e_{J_\alpha}\rangle$ are suitable c.b.e.'s with conditions $J_\alpha>J$ and $\max J_\alpha\geq\max I$ (See (3.10)).

We temporally assume (D_1), of which proof is reasonably postponed.

Under this induction hypothesis $(3.11)_{s-1}$ we shall define (d_s,σ_{s-1}) as follows.

First define d_s: $X_s\to X_{s-1}$, as an A-map, by

(3.12) $$d_s\langle e_J\rangle=\varphi_{s-1}\cdot\tau_s\langle e_J\rangle=(1-\sigma_{s-2}d_{s-1})\cdot e_{j_s,2^{l_s}}\langle e_{J-(j_s,l_s)}\rangle$$

where $|J|=s$ and $(j_s,l_s)=\max J$.

It follows immediately, from (C_{s-1})

$$d_{s-1}d_s=0.$$

Next define

(3.13) $$\sigma_{s-1}=0 \text{ on } \operatorname{Im}\sigma_{s-2}=C_{1,s-1}.$$

To define σ_{s-1} on $\operatorname{Ker}d_{s-1}$, take the set $\{\varphi_{s-1}(e_I\langle e_J\rangle);\ e_I\langle e_J\rangle$ c.b.e. of Type 2 in $X_{s-1}\}$ as a fixed basis of $\operatorname{Ker}d_{s-1}$, by virtue of (D_{s-1}), and put

(3.14) $$\sigma_{s-1}(\varphi_{s-1}(e_I\langle e_J\rangle))=\sigma'_{s-1}((e_I\langle e_J\rangle)=e_{I'}\langle e_{J+(i_n,k_n)}\rangle$$

where $(i_n,k_n)=\max I$ (See (3.9)). Then σ_{s-1} is naturally extended to a $\boldsymbol{Z}/2$-map and gives an isomorphism

(3.15) $$\sigma_{s-1}\colon \operatorname{Ker} d_{s-1} \overset{\cong}{\rightarrow} C_{1,s} = \operatorname{Im}\sigma_{s-1}.$$

Thus we have

$$d_s = \begin{cases} \varphi_{s-1}\tau_s & \text{on } C_{1,s} \\ 0 & \text{on } \operatorname{Ker} d_s \end{cases}$$

$$\sigma_{s-1} = \begin{cases} 0 & \text{on } C_{1,s-1} \\ \sigma'_{s-1}\varphi_{s-1}^{-1} & \text{on } \operatorname{Ker} d_{s-1} \end{cases}$$

$$d_s\sigma_{s-1} + \sigma_{s-2}d_{s-1} = 1 \text{ on } X_{s-1}$$

$$X_s = \operatorname{Im}\sigma_{s-1} \oplus \operatorname{Ker} d_s,\ \operatorname{Ker} d_s = \operatorname{Im}(1-\sigma_{s-1}d_s),$$

and verify (A_s), (B_s) and (C_s) for (d_s, σ_{s-1}). From (3.11), (D_{s-1}) and (3.14), it follows that

(3.16) $$\sigma_{s-1}(e_I\langle e_J\rangle) = e_{I'}\langle e_{J+\max I}\rangle + \sum_{\substack{J_\alpha > J, \max J_\alpha \geq \max I \\ \max I_\alpha \geq \max J_\alpha}} \sigma_{s-1}(e_{I_\alpha}\langle e_{J_\alpha}\rangle)$$

for $e_I\langle e_J\rangle \in C_{2,s-1}$,

where the added conditions on the summand come from those of $e_{I_\alpha}\langle e_{J_\alpha}\rangle \in C_{2,s-1}$, and as well

(3.17) $$d_s\langle e_J\rangle = e_{j_s,2^{l_s}}\langle e_{J'}\rangle + \sum_{J_\gamma > J', \max J_\gamma \geq (j_s,l_s) = \max J} e_{I_\gamma}\langle e_{J_\gamma}\rangle.$$

Lemma 3.18.

$$\sigma_{s-1}(e_I\langle e_J\rangle) = e_{I'}\langle e_{J+\max I}\rangle + \sum_{J_\sigma > J+\max I} e_{I_\alpha}\langle e_{J_\alpha}\rangle \quad \textit{for } e_I\langle e_J\rangle \in C_{2,s-1}$$

or, we write simply

$$\sigma_{s-1}(e_I\langle e_J\rangle) = e_{I'}\langle e_{J+\max I}\rangle + \Sigma \textit{ higher terms.}$$

Proof. In the right hand side of (3.16), using itself again, we have

$$\sigma_{s-1}(e_{I_\alpha}\langle e_{J_\alpha}\rangle) = e_{I'_\alpha}\langle e_{J_\alpha+\max I_\alpha}\rangle + \sum_{\substack{J_\beta > J_\alpha, \max J_\beta \geq \max I_\alpha \\ \max I_\beta \geq \max J_\beta}} \sigma_{s-1}(e_{I_\beta}\langle e_{J_\beta}\rangle)$$

here $J_\alpha > J$ and $\max I_\alpha \geq \max J_\alpha \geq \max I$ so that $J_\alpha + \max I_\alpha > J + \max I$. Repeating this process, we obtain Lemma 3.18.

Now φ_s: $C_{2,s}\to \operatorname{Ker} d_s$ will be defined just as before:

$$\varphi_s(e_I\langle e_J\rangle)=e_{I'}(1-\sigma_{s-1}d_s)(e_{i_n,2^{k_n}}\langle e_J\rangle). \tag{3.19}$$

To prove (D_s), (ii) it is sufficient to consider the special case $|I|=1$:

$$\varphi_s(e_{i,2^k}\langle e_J\rangle)=(1-\sigma_{s-1}d_s)(e_{i,2^k}\langle e_J\rangle)\quad ((i,k)\geq \max J).$$

In view of (3.17), we have

$$\begin{aligned}\varphi_s(e_{i,2^k}\langle e_J\rangle)&=e_{i,2^k}\langle e_J\rangle+\sigma_{s-1}(e_{i,2^k}\cdot d_s\langle e_J\rangle)\\&=e_{i,2^k}\langle e_J\rangle+\sigma_{s-1}(e_{i,2^k}e_{j_s,2^{l_s}}\langle e_J\rangle)\\&\quad+\sum_{J_\gamma>J',\max J_\gamma\geq\max J}\sigma_{s-1}(e_{i,2^k}\cdot e_{I_\gamma}\langle e_{J_\gamma}\rangle)\end{aligned}\tag{3.20}$$

Rewriting $e_{i,2^k}e_{j_s,2^{l_s}}$ and $e_{i,2^k}\cdot e_{I_\gamma}$ in the admissible form:

$$e_{i,2^k}e_{j_s,2^{l_s}}=\sum_{\max I_\varepsilon\geq(i,k)}e_{I_\varepsilon}$$

$$e_{i,2^k}\cdot e_{I_\gamma}=\sum_{\max I_{\gamma,\delta}\geq(i,k)}e_{I_{\gamma,\delta}}$$

we have, from Lemma 3.18,

(3.21)

$$\begin{aligned}&\varphi_s(e_{i,2^k}\langle e_J\rangle)\\&=e_{i,2^k}\langle e_J\rangle+\sum_{\max I_\varepsilon\geq(i,k)}\sigma_{s-1}(e_{I_\varepsilon}\langle e_{J'}\rangle)+\sum_{\substack{J_\gamma>J',\max J_\gamma\geq\max J\\ \max I_{\gamma,\delta}\geq(i,k)}}\sigma_{s-1}(e_{I_{\gamma,\delta}}\langle e_{J_\gamma}\rangle)\\&=e_{i,2^k}\langle e_J\rangle+\sum_{\max I_\varepsilon\geq(i,k)}\left(e_{I'_\varepsilon}\langle e_{J'+\max I_\varepsilon}\rangle+\Sigma\ \textit{higher terms}\right)\\&\quad+\sum_{J_\gamma+\max I_{\gamma,\delta}>J'+(i,k)\geq J}\left(e_{I'_{\gamma,\delta}}\langle e_{J_\gamma+\max I_{\gamma,\delta}}\rangle+\Sigma\ \textit{higher terms}\right).\end{aligned}$$

Then we have in general

$$\begin{aligned}\varphi_s(e_I\langle e_J\rangle)&=e_{I'}\cdot\varphi_s(e_{i_n,2^{k_n}}\langle e_J\rangle)\\&=e_I\langle e_J\rangle+\sum_{\substack{J_\alpha>J\\ \max J_\alpha\geq\max I}}e_{I_\alpha}\langle e_{J_\alpha}\rangle\ \text{ for c.b.e. } e_I\langle e_J\rangle\in C_{2,s}.\end{aligned}\tag{3.22}$$

Thus we have proved (3.11), (D_s), (ii).

To show (D_s), (i), first note that $\varphi_s(e_I\langle e_J\rangle)\in \operatorname{Ker} d_s$ and the set $\{\varphi_s(e_I\langle e_J\rangle);\ \text{c.b.e. } e_I\langle e_J\rangle\in C_{2,s}\}$ are linearly independent in virtue of

(3.22). This means that φ_s is injective. To show the surjectivity of φ_s, we replace each higher term $e_{I_\alpha}\langle e_{J_\alpha}\rangle$ of Type 2 in (3.22) by $\varphi_s(e_{I_\alpha}\langle e_{J_\alpha}\rangle)$. Repeating this process, we should finally obtain

$$\varphi_s(e_I\langle e_J\rangle)=e_I\langle e_J\rangle+\Sigma\varphi_s(e_{I_\beta}\langle e_{J_\beta}\rangle)+u_{I,J}, \tag{3.23}$$

where $u_{I,J}\in C_{1,s}$ and $e_I\langle e_J\rangle+u_{I,J}\in \operatorname{Im}\varphi_s$.
The difference $(1-\sigma_{s-1}d_s)(e_I\langle e_J\rangle)-(e_I\langle e_J\rangle+u_{I,J})$ belongs to $\operatorname{Ker} d_s\cap \operatorname{Im}\sigma_{s-1}=0$. Therefore we have

$$(1-\sigma_{s-1}d_s)(e_I\langle e_J\rangle)=e_I\langle e_J\rangle+u_{I,J}\in \operatorname{Im}\varphi_s. \tag{3.24}$$

Since $(1-\sigma_{s-1}d_s)(C_{1,s})=0$ and $(1-\sigma_{s-1}d_s)(C_{2,s})=(1-\sigma_{s-1}d_s)(X_s)$, we have $\operatorname{Im}\varphi_s=\operatorname{Im}(1-\sigma_{s-1}d_s)=\operatorname{Ker} d_s$.
This proves (3.11), (D_s), (i).

Now, for the remaining case of $n=1$, a proof of (D_1) can be performed in a literally parallel way as just described, so it will be ommited.
Thus we have completed the induction process and a proof of the theorem 1.2.

Here we shall show some simple examples of boundaries and contracting homotopies:

(3.25)
$$\begin{aligned}
&d\langle e_{1,1},e_{1,1}\rangle=e_{1,1}\langle e_{1,1}\rangle\\
&d\langle e_{1,1},e_{1,2}\rangle=e_{1,2}\langle e_{1,1}\rangle+e_{1,1}\langle e_{1,2}\rangle+\langle e_{2,1}\rangle\\
&d\langle e_{j,2^l},e_{j,2^l}\rangle=e_{j,2^l}\langle e_{j,2^l}\rangle+\sigma_0(e_{j,2^l}\cdot e_{j,2^l})\\
&d\langle e_{i,2^k},e_{j,2^l}\rangle=e_{j,2^l}\langle e_{i,2^k}\rangle+e_{i,2^k}\langle e_{j,2^l}\rangle+\sigma_0[e_{i,2^k},e_{j,2^l}]\ \text{ for } (i,k)<(j,l),
\end{aligned}$$

where [,] means the commutator.

$$\begin{aligned}
&d\langle e_{1,2},e_{1,2},e_{1,2}\rangle=e_{1,2}\langle e_{1,2},e_{1,2}\rangle+e_{1,1}\langle e_{1,2},e_{2,1}\rangle+\langle e_{2,1},e_{2,1}\rangle\\
&\sigma(e_{i,2^k}\langle e_J\rangle)=\langle e_{J+(i,k)}\rangle \quad \text{for} \quad (i,k)\geq \max J\\
&\sigma(e_{2,2}\cdot e_{3,1}\langle e_{1,4}\rangle)=e_{2,2}\langle e_{1,4},e_{3,1}\rangle+e_{2,1}\langle e_{3,1},e_{3,1}\rangle
\end{aligned}$$

where the last example shows that $\sigma_i\neq\sigma_i'$ in general.

4. Chain complex P and its dual

The construction P defined in §1 with the induced differential

(4.1) $$\bar{d}=\boldsymbol{Z}/2\otimes d\colon P\to P$$

becomes a chain complex.

Define natural A-linear chain maps $f\colon X\to B(A)$ and $g\colon B(A)\to X$ in the usual way ([2]), using contracting homotopy σ of X resp. S of $B(A)$:

(4.2) $$\begin{aligned} &f_0=\text{id.} \quad : X_0=A\to A=B(A)_0,\\ &f_s\langle e_J\rangle=Sf_{s-1}d\langle e_J\rangle \quad \text{for} \quad s\geq 1,\\ &f_s(e_I\langle e_J\rangle)=e_I\cdot f_s\langle e_J\rangle \end{aligned}$$
and similar for g.

By induction on dimension, we see easily that

(4.3) $$g\circ f=\text{id} \quad \text{on} \quad X \quad \text{and} \quad f_s\langle e_J\rangle\in\bar{B}(A).$$

This proves Prop. 1.3.
Similarly define a diagonal $\psi\colon X\to X\otimes X$ by

(4.4) $$\psi_0\colon X_0=A\to A\otimes A=(X\otimes X)_0, \quad \text{the diagonal of} \quad A$$
$$(i.e.\ \psi_0(e_{i,k})=\sum_j e_{i,k-j}\otimes e_{i,j}),$$
$$\psi_s\langle e_J\rangle=\tilde{\sigma}\psi_{s-1}d\langle e_J\rangle \quad \text{for} \quad s\geq 1,$$

where $\tilde{\sigma}=\sigma\otimes 1+\varepsilon\otimes\sigma$ is the induced contracting homotopy of $X\otimes X$.
This ψ is a chain map, and there is a natural chain homotopy:

(4.5) $$\begin{aligned} &(\psi\otimes 1)\psi-(1\otimes\psi)\psi=d^{(3)}H+Hd,\\ &\text{with} \quad d^{(3)}=d\otimes 1\otimes 1+1\otimes d\otimes 1+1\otimes 1\otimes d, \end{aligned}$$

where $H\colon X\to X\otimes X\otimes X$ is a $\boldsymbol{Z}/2$-map of degree (1,0).

The following example shows non-coassociativity of ψ.

(4.6) $$\psi\langle e_{1,4}\rangle=\langle e_{1,4}\rangle\otimes 1+e_{1,1}\langle e_{1,2}\rangle\otimes e_{1,1}+\langle e_{1,2}\rangle\otimes e_{1,2}+\langle e_{1,1}\rangle\otimes e_{1,3}$$
$$+1\otimes\langle e_{1,4}\rangle$$
$$((\psi\otimes 1)\psi-(1\otimes\psi)\psi)\langle e_{1,4}\rangle=e_{1,1}\otimes\langle e_{1,2}\rangle\otimes e_{1,1}.$$

The diagonal ψ induces a diagonal Δ: $P\to P\otimes P$,

(4.7) $$\Delta=(\rho\otimes\rho)\circ\psi,\ \rho=\varepsilon_A\otimes 1_P\colon X\to P,\quad \text{with}\quad \bar{d}^{(2)}\Delta=\Delta\bar{d}.$$

From (4.5), it follows that Δ is also homotopy coassociative.

We shall show a few examples of $\Delta\langle e_J\rangle$:

(4.8)
$$\begin{aligned}
&\Delta\langle e_{i,2^k}\rangle=\langle e_{i,2^k}\rangle\otimes 1+1\otimes\langle e_{i,2^k}\rangle\\
&\Delta\langle e_{1,2},e_{1,2}\rangle=\langle e_{1,2},e_{1,2}\rangle\otimes 1+\langle e_{1,2}\rangle\otimes\langle e_{1,2}\rangle+1\otimes\langle e_{1,2},e_{1,2}\rangle\\
&\Delta\langle e_{1,1},e_{1,4}\rangle=\langle e_{1,1},e_{1,4}\rangle\otimes 1+\langle e_{1,1}\rangle\otimes\langle e_{1,4}\rangle+\langle e_{1,4}\rangle\otimes\langle e_{1,1}\rangle\\
&\quad+1\otimes\langle e_{1,1},e_{1,4}\rangle+\underline{\langle e_{1,2}\rangle\otimes\langle e_{2,1}\rangle}\\
&\Delta\langle e_{1,4},e_{2,2},e_{3,1}\rangle=\langle e_{1,4},e_{2,2},e_{3,1}\rangle\otimes 1+\langle e_{1,4}\rangle\otimes\langle e_{2,2},e_{3,1}\rangle\\
&\quad+\langle e_{2,2}\rangle\otimes\langle e_{1,4},e_{3,1}\rangle\\
&\quad+\langle e_{3,1}\rangle\otimes\langle e_{1,4},e_{2,2}\rangle+\langle e_{2,2},e_{3,1}\rangle\otimes\langle e_{1,4}\rangle+\langle e_{1,4},e_{3,1}\rangle\otimes\langle e_{2,2}\rangle\\
&\quad+\langle e_{1,4},e_{2,2}\rangle\otimes\langle e_{3,1}\rangle+1\otimes\langle e_{1,4},e_{2,2},e_{3,1}\rangle+\underline{\langle e_{2,1},e_{3,1}\rangle\otimes\langle e_{3,1}\rangle}
\end{aligned}$$

and, in general

$$\Delta\langle e_J\rangle=\mathit{shuffle}+\Sigma\ \mathit{extra\ terms},$$

where an extra term $\langle e_{J_1}\rangle\otimes\langle e_{J_2}\rangle$, with $\langle e_{J_1}\rangle\cdot\langle e_{J_2}\rangle\neq\langle e_J\rangle$, is indicated by the underline.

Now the dual cochain complex P^*, with differential $\delta=\bar{d}^*$, has a product Δ^*: $P^*\otimes P^*\to P^*$, which is 'homotopy associative' and δ is a derivation there.

The product Δ^* of P^* induces the usual associative product in the cohomology $H^*(P^*)=\mathrm{Ext}_A^{**}(\boldsymbol{Z}/2,\boldsymbol{Z}/2)$ as stated in Corollary 1.5.

A few examples of boundaries are given by

(4.9)
$$\begin{aligned}
&\bar{d}\langle e_{1,1},e_{1,4},e_{1,4}\rangle=\langle e_{2,2},e_{2,1}\rangle\\
&\bar{d}\langle e_{1,2},e_{1,4},e_{2,1}\rangle=\langle e_{3,1},e_{1,2}\rangle+\langle e_{2,2},e_{2,1}\rangle\\
&\bar{d}\langle e_{1,1},e_{1,2},e_{2,2}\rangle=\langle e_{3,1},e_{1,2}\rangle+\langle e_{2,2},e_{2,1}\rangle\\
&\delta\langle e_{2,2},e_{2,1}\rangle^*=\langle e_{1,1},e_{1,4},e_{1,4}\rangle^*+\langle e_{1,2},e_{1,4},e_{2,1}\rangle^*+\langle e_{1,1},e_{1,2},e_{2,2}\rangle^*\\
&\delta\langle e_{3,1},e_{1,2}\rangle^*=\langle e_{1,2},e_{1,4},e_{2,1}\rangle^*+\langle e_{1,1},e_{1,2},e_{2,2}\rangle^*\\
&\delta(\langle e_{2,2},e_{2,1}\rangle^*+\langle e_{3,1},e_{1,2}\rangle^*)=\langle e_{1,1},e_{1,4},e_{1,4}\rangle^*\\
&\bar{d}\langle e_{1,1},e_{1,1},e_{1,4}\rangle=\langle e_{2,1},e_{2,1}\rangle
\end{aligned}$$

$$\bar{d}\langle e_{1,2},e_{1,2},e_{1,2}\rangle=\langle e_{2,1},e_{2,1}\rangle$$

$$\bar{d}\langle e_{1,1},e_{1,2},e_{2,1}\rangle=0$$

$$\delta\langle e_{2,1},e_{2,1}\rangle^*=\langle e_{1,1},e_{1,1},e_{1,4}\rangle^*+\langle e_{1,2},e_{1,2}e_{1,2}\rangle^*, \ etc.$$

5. Spectral sequence

We shall define a filtration on X which corresponds to May's filtration on $B(A)$ ([5]). This leads to a spectral sequence, essentially the same as the May spectral sequence.

Define a weight function w on X by

$$w(e_I\langle e_J\rangle)=\sum_{h=1}^{n} i_h+\sum_{m=1}^{s} j_m, \text{ for a } c.b.e. \ e_I\langle e_J\rangle, \tag{5.1}$$

where $I=\{(i_1,k_1)<\cdots<(i_n,k_n)\}$ and $J=\{(j_1,l_1)\leq\cdots\leq(j_s,l_s)\}$, and put $w(x+y)=\max(w(x),w(y))$.

Define a filtration F_u on X, for $u\leq 0$, by

$$e_I\langle e_J\rangle\in F_u, \quad \text{if} \quad |J|-w(e_I\langle e_J\rangle)\leq u. \tag{5.2}$$

Then we have

(5.3)

$$X=F_0\supset F_{-1}\supset\cdots F_u\supset F_{u-1}\supset\cdots$$

and

$$dF_u\subset F_u.$$

Putting $Z_u^r=\mathrm{Ker}(F_u\xrightarrow{d}F_u\rightarrow F_u/F_{u-r})$ for $r\geq 0$, we get a spectral sequence $\{E_u^r\}$:

(5.4)

$$E_u^r=Z_u^r+F_{u-1}/dZ_{u+r-1}^{r-1}+F_{u-1},$$

$$d^r\colon E_u^r\rightarrow E_{u-r}^r, \quad \text{induced by} \quad d.$$

It follows that

(5.5)

$$E^0X=\sum_{u\leq 0} F_u/F_{u-1}\cong E^0A\otimes E^0P,$$

$$d^0=0.$$

Here E^0A is the primitively generated Hopf algebra, isomorphic to the enveloping algebra $V(E^0L)$ of restricted Lie algebra E^0L (in [5] and [10],

E^0L is simply denoted by L).

From (5.5), we have

$$(5.6)\qquad E^1X = E^0X \quad \text{as} \quad E^0A\text{-module},$$

$$d^1\langle e_J\rangle = \sum_{(j,l)} e_{j,2^l}\langle e_{J-(j,l)}\rangle,$$

where (j,l) run over the index sequence J without dublication.

Thus we have an isomorphism:

$$(5.7)\qquad (E^1P, \bar{d}^1 = E^1(\bar{d})) \cong (\Gamma(sE^0L), d),$$

the May complex (being divided polynomial algebra)

as a commutative DGA-coalgebra, in which $\langle e_{j,2^l}\rangle^n = \langle e_{j,2^l}, \cdots, e_{j,2^l}\rangle$ corresponds to $\gamma_n(\bar{P}^l_j) \in \Gamma(sE^0L)$. Thus we have $E^1X \cong E^0A \otimes \Gamma(sE^0L)$, the May resolution.

Dualizing the above things, we shall have a filtration $\mathscr{F}_u$ on $X^* = A_* \otimes P^*$ such that

$$(5.8)\qquad \mathscr{F}_u = (X/F_{u-1})^*, \quad \text{for} \quad u \leq 0,$$

$$0 = \mathscr{F}_1 \subset \mathscr{F}_0 \subset \mathscr{F}_{-1} \subset \cdots \subset \mathscr{F}_u \subset \mathscr{F}_{u-1} \subset \cdots \subset \mathscr{F}_{-\infty} = X^*,$$

$$\delta\mathscr{F}_u \subset \mathscr{F}_u,$$

$$Z^u_r = \mathrm{Ker}\,(\mathscr{F}_u \xrightarrow{\delta} \mathscr{F}_u \to \mathscr{F}_u/\mathscr{F}_{u+r}),$$

$$E^u_r = Z^u_r + \mathscr{F}_{u+1}/\delta Z^{u-r+1}_{r-1} + \mathscr{F}_{u+1},$$

$$\delta_r\colon E^u_r \to E^{u+r}_r.$$

Thus we have

$$E_0X^* = E_0(A_*) \otimes E_0(P^*), \ \delta_0 = 0,$$

$$E_1X^* = E_0X^* \text{ as a module},$$

$$(5.9)\qquad E_1(P^*) \cong \Gamma(sE^0L)^* = \mathfrak{R} \text{ as a DGA-polynomial algebra ([5],[10])},$$

$$E_2X^* \cong H^*(E^0A),$$

and E_rX^* coincide with those of the May spectral sequence for $r \geq 2$. Here $\langle e_J\rangle^* \in E_1(P^*)$ corresponds to $R^{l_1}_{j_1} \cdots R^{l_s}_{j_s} \in \mathfrak{R}$ ([5],[10]).

Returning to the complex P^*, we denote $\langle e_{j,2^l}\rangle^*$ by $\varepsilon_{j,2^l}$. Then we have

$$\delta\varepsilon_{j,2^l}=\sum_{i=1}^{j-1}\varepsilon_{j-i,2^{i+l}}\cdot\varepsilon_{i,2^l},\ [\varepsilon_{j-i,2^{i+l}},\varepsilon_{i,2^l}]=0,$$

$$\text{(5.10)}\qquad \text{and}\quad \langle e_J\rangle^*=\varepsilon_{j_s,2^{l_s}}\cdot\langle e_{J'}\rangle^*,\quad \text{for}\quad J=J'+(j_s,l_s)$$

$$\text{and}\quad (j_s,l_s)=\max J,$$

because $\langle e_{j_s,2^{l_s}}\rangle\otimes\langle e_{J'}\rangle$ appears, with non-zero coefficient, only in $\Delta\langle e_J\rangle$, and not in $\Delta\langle e_{\tilde{J}}\rangle$ for other $\tilde{J}$.

P^* has no zero-divisor and contains the polynomial ring $\boldsymbol{Z}/2[\varepsilon_{1,2^i};\ i\geq 1]$.

6. Appendix

Consider the case of the mod p Steenrod algebra A for an odd prime p. We shall sketch similar argument as in the preceeding sections.

Lemma 6.1. (i) *A is multiplicatively generated by $\{e_{i,p^k}, f_j;\ i\geq 1, k\geq 0$ and $j\geq 0\}$, $e_{i,p^k}=(\xi_i^{p^k})^*$ (resp. $f_j=\tau_j^*$) the dual element $\xi_i^{p^k}$ (resp. τ_j) with respect to the Milnor monomial basis of the dual Hopf algebra A_* of A.* (ii) *The set $\{1, e_I^L\cdot f_J=e_{i_1,p^{k_1}}^{l_1}\cdots e_{i_m,p^{k_m}}^{l_m}\cdot f_{j_1}\cdots f_{j_n};$ with index sequences I: $(i_1,k_1)<\cdots<(i_m,k_m)$, $L=(l_1,\cdots,l_m)$ with $1\leq l_i<p$, and J: $j_1<\cdots<j_n\}$ forms a basis of A.*

Put $L^+=\boldsymbol{Z}/p\{e_{i,p^k};\ (i,k)\geq(1,0)\}$, $L^-=\boldsymbol{Z}/p\{f_j\}$. Let $sL^+=\boldsymbol{Z}/p\{\langle e_{i,p^k}\rangle\}$, $sL^-=\boldsymbol{Z}/p\{\langle f_j\rangle\}$ be the suspensions with bideg $\langle e_{i,p^k}\rangle=(1,2p^k(p^i-1))$, bideg $\langle f_j\rangle=(1,2p^j-1)$ respectively. And let $s^2\pi L^+$ denote a vector space $\boldsymbol{Z}/p\{y_{i,p^k};\ (i,k)\geq(1,0)\}$ spanned by indeterminates y_{i,p^k} of bidegree $(2,2p^{k+1}(p^i-1))$.

Define

$$E(sL^+)=\ \text{the exterior algebra on } sL^+,$$

$$P(sL^-)=\ \text{the polynomial algebra on } sL^-,$$

and

$$P(s^2\pi L^+)=\ \text{the polynomial algebra on } s^2\pi L^+.$$

Theorem 6.2. *The A-module $X=A\otimes E(sL^+)\otimes P(s^2\pi L^+)\otimes P(sL^-)$ with an inductively defined differential d gives an acyclic, A-free resolution of $\boldsymbol{Z}/p$: $X\xrightarrow{\varepsilon}\boldsymbol{Z}/p$.*

Corollary 6.3. *A suitable filtration on X induces a spectral sequence in which $E^1\bar{X}\cong E(sE^0L^+)\otimes\Gamma(sE^0L^-)\otimes\Gamma(s^2\pi E^0L^+)$, the May's construction,* *as a cocommutative DGA-coalgebra* ([5]) *and the E^r-terms are the same as those of May S.S.* $(r\geq 2)$.

We can prove this theorem quite similarly as in the mod 2 case, although we need here a more fine classification of the canonical basis elements $e_I^L\cdot f_J\cdot\langle e_G\rangle\cdot y_M\cdot\langle f_K\rangle$ as follows.

Introduce first the following notation on the index sequences:

(6.4) $a_1(I)=\max I=(i_m,k_m)$ for $I=(i_1,k_1)<\cdots<(i_m,k_m)$,
and $a_1(\phi)=(0,0),\phi$ being the empty set.
$b(G)=\max G$ for $G=(g_1,h_1)<\cdots<(g_t,h_t)$,
and $b(\phi)=(0,0)$,
$c(M)=\max M$ for $M=(m_1,q_1)\leq\cdots\leq(m_u,q_u)$,
and $c(\phi)=(0,0)$,

and

$a_2(J)=\max J$ for $J=(j_1<\cdots<j_n)$,
and $a_2(\phi)=-1$,
$d(K)=\max K$ for $K=(k_1\leq\cdots\leq k_v)$,
and $d(\phi)=-1$.

A *c.b.e.* $e_I^L\cdot f_J\cdot\langle e_G\rangle\cdot y_M\cdot\langle f_K\rangle$ belongs to one of the following types:

Provided that $J=K=\phi$ the empty set,

(6.5)
$$\begin{cases} I_1:\ a_1\leq b\geq c \text{ and,} \\ \text{if } a_1=b, l_m<p-1, \end{cases} \qquad II_1: b<a_1\geq c,$$
$$I_2: a_1<c>b, \qquad II_2:\ a_1=b\geq c, \text{ and } l_m=p-1,$$

Otherwise, if J or $K\neq\phi$, put
$$I_3: a_2<d, \qquad II_3:\ a_2\geq d.$$

Thus we have a direct sum decomposition

(6.6) $X=C_I\oplus C_{II},\ C_I=C_{I_1}\oplus C_{I_2}\oplus C_{I_3},\ C_{II}=C_{II_1}\oplus C_{II_2}\oplus C_{II_3}$,
where
$C_{I_i}=\boldsymbol{Z}/p\{c.b.e. \text{ of type } I_i\}$ and $C_{II_i}=\boldsymbol{Z}/p\{c.b.e. \text{ of type } II_i\}$
for $i=1,2,3$,
with linear isomorphisms

$$C_{I_i,s} \underset{\sigma'_{s-1}}{\overset{\tau_s}{\rightleftarrows}} C_{II_i,s-1}$$

defined by

(6.7)
$$\tau_s(e_I^L\langle e_G\rangle y_M)=(-1)^{|G|-1}e_I^L\cdot e_{g_t,p^{h_t}}\langle e_{G-(g_t,h_t)}\rangle y_M$$
on $c.b.e.$ of type I_1 $((g_t,h_t)=\max G)$
$$\tau_s(e_I^L\langle e_G\rangle y_M)=e_I^L\cdot e_{m_u,p^{q_u}}^{p-1}\ \langle e_{G+(m_u,q_u)}\rangle y_{M-(m_u,q_u)}$$
on $c.b.e.$ of type I_2 $((m_u,q_u)=\max M)$
$$\tau_s(e_I^L f_J\langle e_G\rangle y_M\langle f_K\rangle)=(-1)^{|G|+|K|-1}\,e_I^L\cdot f_J\cdot f_{k_v}\cdot\langle e_G\rangle\cdot y_M\langle f_{K-(k_v)}\rangle$$
on $c.b.e.$ of type I_3 $(k_v=\max K)$

where $|G|$ denotes the length of the index sequence G and similarly for others, and $s=|G|+2|M|+|K|$ the homology dimension.
The inverse σ'_{s-1} of τ_s will be defined obviously.

Then, starting from

$$d_1\langle e_{j,p^l}\rangle=e_{j,p^l},\ d_1\langle f_j\rangle=f_j,$$

$$\sigma_0(e_I^L)=(e_I^L)'\cdot\langle e_{i_m,p^{k_m}}\rangle, \text{ with } (i_m,k_m)=\max I \text{ and}$$

$$(e_I^L)'=\begin{cases} e_{i_1,p^{k_1}}^{l_1}\cdots e_{i_m,p^{k_m}}^{l_m-1} & \text{if } l_m>1\\ e_{i_1,p^{k_1}}^{l_1}\cdots e_{i_{m-1},p^{k_{m-1}}}^{l_m-1} & \text{if } l_m=1\end{cases}$$

$$\sigma_0(e_I^L\cdot f_J)=e_I^L\cdot f_{J'}\langle f_{j_n}\rangle \quad \text{with} \quad j_n=\max J \quad \text{and} \quad J'=J-\{j_n\},$$

we could define differential d and contracting homotopy σ inductively in X as before, and as well carry out all the parallel discussion.

References

[1] J.F. Adams: *On the structure and applications of the Steenrod algebra*, Comment. Math. Helv. **32** (1958), 180–214.

[2] J.F. Adams: *On the non-existence of elements of Hopf invariant one*, Ann. of Math. **72** (1960), 20–104.

[3] J. Adem: *The iteration of the Steenrod squares in algebraic topology*, Proc. Nat. Acad. Sci. USA **38** (1952), 720–726.

[4] H.R. Margolis: Spectra and the Steenrod Algebra, North-Holland, 1983.

[5] J.P. May: The cohomology of restricted Lie algebras and of Hopf algebras,

Application to the Steenrod algebra, Dissertation, Princeton Univ. 1964.
[6] J. Milnor: *The Steenrod algebra and its dual*, Ann. of Math. **67** (1958), 150–171.
[7] S.P. Novikov: *On the cohomology of the Steenrod algebra* (Russian), Doklady Acad. Nauk, SSSR **131** (1959), 893–895.
[8] N. Shimada: *Some resolutions for the Landweber-Novikov algebra*, Q. & A. in General Topology **8** (1990), Special issue, 201–206.
[9] N.E. Steenrod and D.B.A. Epstein: Cohomology operations, Ann. of Math. Studies 50, Princeton, 1962.
[10] M.C. Tangora: *On the cohomology of the Steenrod algebra*, Math. Z. **116** (1970), 18–64.

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