Nakazawa, N.
Osaka J. Math.
31 (1994), 291–329

ASYMPTOTIC EXPANSION OF THE BERGMAN KERNEL FOR STRICTLY PSEUDOCONVEX COMPLETE REINHARDT DOMAINS IN C^2

NORIYUKI NAKAZAWA

(Received October 18, 1991)
(Revised July 13, 1993)

Introduction

This paper is concerned with the asymptotic expansion, due to Fefferman [2], of the Bergman kernel for a strictly pseudoconvex domain. Restricting ourselves to the class of complete Reinhardt domains in $\boldsymbol{C}^2$, we consider the symbol of a pseudodifferential operator which represents the singularity of the Bergman kernel. We give an integral representation of that symbol (see Theorem 2 in Section 1). By using that integral representation, we identify six coefficients of Fefferman's asymptotic expansion (see Theorems 1 and 1′ in Section 1).

Given a bounded strictly pseudoconvex domain Ω in $\boldsymbol{C}^N$ with C^∞ boundary $\partial\Omega$, we consider the Bergman kernel $K(z)$ for $z\in\Omega$, which is restricted to the diagonal of $\Omega\times\Omega$. Let $\lambda\in C^\infty(\bar{\Omega})$ be a negatively signed defining function of Ω in the sense that $\lambda<0$ in Ω and $|\text{grada}\ \lambda|>0$ on $\partial\Omega$. Let us recall a classical result of Hörmander [5] asserting that

$$(0.1)\qquad \lim_{z\to z_0}[-\lambda(z)]^{N+1}K(z)=\frac{N!}{\pi^N}J[-\lambda](z_0)>0 \qquad \text{for} \qquad z_0\in\partial\Omega,$$

where $J[-\lambda]$ denotes the Levi determinant defined by

$$J[-\lambda]=(-1)^{N+1}J[\lambda]=-\det\begin{pmatrix}\lambda & \partial\lambda/\partial\bar{z}_k\\ \partial\lambda/\partial z_j & \partial^2\lambda/\partial z_j\partial\bar{z}_k\end{pmatrix}.$$

Fefferman [2] refined this result by showing that

$$(0.2)\qquad K(z)=\frac{N!}{\pi^N}J[\lambda]\left(\frac{\varphi(z)}{\lambda(z)^{N+1}}+\psi(z)\ \log[-\lambda(z)]\right)$$

with φ, $\psi\in C^\infty$ near $\partial\Omega$. If one considers the Taylor expansions

$$\varphi=\sum_{n=0}^{N} L_n^{\varphi}\lambda^n+O(|\lambda|^{N+1}), \qquad \psi\sim\sum_{n=0}^{\infty} L_n^{\psi}\lambda^n$$

with L_n^{φ}, $L_n^{\psi} \in C^{\infty}(\partial\Omega)$, then (0.2) gives rise to an asymptotic expansion of $K(z)$. Conversely, starting from such an expansion, one can construct φ and ψ satisfying (0.2). We thus refer to (0.2) as Fefferman's asymptotic expansion.

We normalize L_n^{φ} and L_n^{ψ} by considering $L_n \in C^{\infty}(\partial\Omega)$ which satisfy

$$\frac{\varphi}{\lambda^{N+1}}+\psi\log[-\lambda]\sim\sum_{n=0}^{\infty} L_n\partial_{\lambda}^{-n}\frac{1}{\lambda^{N+1}} \quad \text{with } \partial_{\lambda}=\frac{\partial}{\partial\lambda},$$

that is, $L_n=(-1)^n\{N!/(N-n)!\}L_n^{\varphi}$ for $n\leqq N$ and $L_n=(-1)^N N!(n-N-1)!L_{n-N-1}^{\psi}$ for $n>N$. We are interested in determining the coefficients L_n. Note that φ, ψ depend on the choice of λ and that L_n depend on λ and the coordinates near the boundary. With an appropriate choice of λ, an invariant theory for φ, ψ were developed by Fefferman [3] and Graham [4] in the context of local biholomorphic geometry. We shall not discuss, in the text, the relation between that invariant theory and our results. We only mention here that our Theorem 1 may be obtained by using a result of Graham [4].

We now assume that $N=2$ and that Ω is a complete Reinhardt domain. Then, the mapping

(0.3) $\qquad x=\log|z_1|,\ y=\log|z_2| \qquad \text{for } z=(z_1, z_2)\in\Omega$

defines an unbounded domain $\log|\Omega|\subset \boldsymbol{R}^2$, called the logarithmic real representation domain of Ω, which takes the form $\log|\Omega|=\{y<f(x),\ x\in I\}$, where $f\in C^{\infty}(I)$ with $I=(-\infty, x_+)$ satisfies $f'<0$, $f''<0$ and

$$\lim_{x\downarrow-\infty} f(x)<+\infty, \quad \lim_{x\uparrow x_+} f(x)=-\infty.$$

The function $\lambda=y-f(x)$ is a defining function of $\log|\Omega|$, and its pull-back by (0.3), denoted again by λ, defines $\partial\Omega\cap\{z_1z_2\neq 0\}$. Thus one can consider Fefferman's asymptotic expansion (0.2) for this local defining function λ of Ω.

Fefferman's asymptotic expansion for the class of complete Reinhardt domains $\Omega\subset \boldsymbol{C}^2$ was previously studied by Boichu and Coeuré [1]. They showed that the coefficient L_3, the boundary value of 2ψ, is a polynomial of $f^{(2+k)}$ $(0\leqq k\leqq 6)$ divided by $(f'')^9$. Analyzing that polynomial, they further tried to prove that if $\psi|_{\partial\Omega}=0$ then Ω is biholomorphic to a ball. Our results are obtained by modifying their ideas.

Roughly speaking, Boichu and Coeuré considered a one-parameter family of Reinhardt domains Ω_x^t $(0 \leqq t \leqq 1)$, with $x \in I$ arbitrarily fixed, such that $\Omega_x^1 = \Omega$, that Ω_x^0 is locally biholomorphic to a ball, and that the real domain $\log|\Omega_x^t|$ for each t is tangent to the curve $\partial \log|\Omega|$ to second order at the point $(x, f(x))$. Using the Bergman kernel K_x^t of Ω_x^t, they obtained an asymptotic expansion

$$(0.4) \qquad K(z) \sim \sum_{m=0}^{\infty} \frac{1}{m!} \partial_t^m K_x^t(z)|_{t=0} \qquad \text{with } x = \log|z_1|,$$

which we refer to as the asymptotic expansion via boundary variations. It should be mentioned that we have not explained preliminary procedures such as localization of the boundary. Con-sequently, the notation used here is slightly different from that in the text; it is also different from that of Boichu and Coeuré [1].

By using (0.4), Boichu and Coeuré obtained a polynomial $\mathscr{L} = \mathscr{L}(f'', f''', \cdots, f^{(8)})$ such that $\mathscr{L}/(f'')^9 = \psi|_{\partial\Omega}$. However, they did not give the explicit form of $\mathscr{L}$. Also, the relation between the asymptotic expansions (0.4) and (0.2) was not clear. We modify the construction of the one-parameter family Ω_x^t so that the relation between (0.4) and (0.2) becomes obvious. Specifically, we get

$$\frac{1}{(2n)!} \partial_t^{2n} K_x^t(z)|_{t=0} \equiv \frac{J[\lambda]}{\pi^2} L_n \partial_\lambda^{-n} \frac{2}{\lambda^3}, \qquad \partial_t^{2n+1} K_x^t(z)|_{t=0} \equiv 0$$

modulo C^∞ as functions of (λ, x). Then we are naturally led to a fairly simple formula of giving all the coefficients L_n (see Theorem 2). Using that formula, we identify L_n for $n \leqq 5$ (see Theorem 1 for $n \leqq 3$ and Theorem 1′ for $n = 4,5$). The proof of Theorem 1′ is computer-aided.

In the identification of L_n for $n \leqq 5$, we first get messy expressions of L_n in terms of the derivatives $f^{(2+m)}(x)$ *for* $m \geqq 0$ (see Propositions 1 and 1′). These expressions are simplified by using the derivatives of a function $p = p(v)$ defined by

$$p(v) = f''(x) \quad \text{with} \quad v = f'(x).$$

Note that $x \mapsto v$ is a one-dimensional hodograph transformation and that $1/p$ is the second derivative of the Legendre transform of f.

We have in particular $L_3 = -(p^2 p^{(4)})''/4!$. As a corollary of this fact, we show that the global condition $\psi|_{\partial\Omega} = 0$ (i.e. $L_3 = 0$) characterizes the ball. It should be noted that this characterization is not a consequence of the invariant theory which asserts that Ω is locally biholomorphic to a ball in a neighborhood of a boundary point if and only if $L_3=L_4=0$ there (cf. Graham [4]). In order to show that Ω is biholomophic to a ball, we solve the ordinary differential equation

$$\{p(v)^2p^{(4)}(v)\}''=0 \quad \text{for } -\infty<v<0$$

under boundary conditions which come from the strict pseudoconvexity of $\partial\Omega\cap\{z_1z_2=0\}$.

The idea of using the function p is again inspired by the paper of Boichu and Coeuré [1]. In fact, our function p is a modification of theirs defined by

$$p_{\mathrm{BC}}(v)=\log[-f''(x)] \quad \text{with} \quad v=f'(x).$$

They wrote the condition $\psi|_{\partial\Omega}=0$ as a boundary value problem for a differential equation satisfied by p_{BC}, and tried to show that the solution corresponds to a domain which is biholomorphic to a ball [1, Théorème 4]. However, the proof is incorrect—the proof of [1, Lemme 4] involves a wrong use of Taylor's formula and the statement of [1, Lemme 4] is false. It seems to the present author that to prove the above mentioned characterization of a ball by using the function p_{BC} is extremely difficult.

This paper consists of five sections. The first section gives the statement of the results. We first state the identification of the coefficients L_n for $n\leqq 5$ (Theorems 1 and 1′), and then gives the characterization of the ball as Corollary of Theorem 1. We next give a formula by which one can compute, in principle, all the coefficients L_n (Theorem 2). Finally, we add some explanation to that formula.

In Section 2, we assume Theorem 1 and prove its Corollary. In Section 3, Theorems 1 and 1′ are proved, assuming the validity of Theorem 2. We prove Theorem 2 in Section 4. The proof of Theorem 2 requires several technical lemmas, which are proved in the final section.

Let us emphasize that all our results except Corollary of Theorem 1 are those for the real domain $\log|\Omega|$ rather than those for the Reinhardt domain Ω. In Theorems 1, 1′ and 2, the boundedness of Ω and the strict pseudoconvexity of the boundary points at which $z_1z_2=0$ are imposed in order to simplify the description, and these restrictions can be eliminated without changing the proof—even the smoothness of the boundary $\partial\Omega$ up to the portion $z_1z_2=0$ is not necessary to assume. In Subsection 3.5, we consider a family of complete Reinhardt domains which are locally biholomorphic to a ball. The boundary of each domain is strictly pseudoconvex except at the portion $z_2=0$; some domains are unbounded and/or weakly pseudoconvex. This family is used in order to simplify calculations in the proof of Theorem 1.

ACKNOWLEDGMENT. The author would like to thank Professor Gen Komatsu for bringing the subject to his attention and for much helpful advice and encouragement.

Note added in proof. In a recent joint work with K. Hirachi and G. Komatsu, Theorem 1′ of the present paper was used in order to identify a universal constant appearing in Graham's invariant theory [4] for ψ in (0.2) with $N=2$. See Remarks 1, 5 and 7 of K. Hirachi, G. Komatsu and N. Nakazawa, Two method of determining local invariants in the Szegö kernel, in Complex Geometry (G. Komatsu and Y. Sakane, eds.), Lect. Notes in Pure Appl. Math. 143, pp. 77–96, Marcel Dekker, 1992.

1. Statement of the results

1.1. Coefficients of the asymptotic expansion. Let Ω be a bounded strictly pseudoconvex domain in $\boldsymbol{C}^2$ with C^∞ boundary. We assume that Ω is a complete Reinhardt domain, that is,

$$\{(z_1,\ z_2)\in \boldsymbol{C}^2;\ |z_1|\leqq|w_1|,\ |z_2|\leqq|w_2|\}\subset\Omega$$

for every $(w_1,\ w_2)\in\Omega$. In order to express the asymptotic expansion of the Bergman kernel, we consider the logarithmic real representation domain $\log|\Omega|$, which is the image of $\Omega\cap\{z_1z_2\neq0\}$ under the mapping $(z_1,\ z_2)\mapsto(\log|z_1|,\ \log|z_2|)$. Let us set

$$f(x):=\sup\{y\in\boldsymbol{R};(x,y)\in\log|\Omega|\}\qquad\text{for } x\in I:=(-\infty,\ x_+),$$

$$\lambda(z):=\log|z_2|-f(\log|z_1|)\qquad\text{for } z=(z_1,\ z_2)\in\boldsymbol{C}^2 \text{ with } z_1z_2\neq0,$$

where $x_+:=\sup\{x\in\boldsymbol{R};(x,\ y)\in\log|\Omega| \text{ with some } y\in\boldsymbol{R}\}$ Then,

$$\log|\Omega|=\{(x,\ y)\in I\times\boldsymbol{R};\ \lambda:=y-f(x)<0\},$$

and $\lambda=\lambda(z)$ is a defining function of $\partial\Omega\cap\{z_1z_2\neq0\}$. Note that the strict pseudoconvexity of $\partial\Omega\cap\{z_1z_2\neq0\}$ implies $f''<0$. We now make a change of variables $x\mapsto f'(x)$ and introduce a function $p\in C^\infty(f'(I))$ defined by

$$p(v)=f''(x)\qquad\text{with } v=f'(x) \text{ for } x\in I.$$

Let $K(z)$ for $z\in\Omega$ denote the Bergman kernel of Ω restricted to the diagonal of $\Omega\times\Omega$. Recalling that $K(z)$ is independent of $(z_1/|z_1|,z_2/|z_2|)$, we write

$$L(\lambda,\ x)=L(\lambda;v)=(2\pi)^2|z_1z_2|^2K(z). \tag{1.1}$$

Using the notation in Fefferman's expansion (0.2), we have

$$L=\frac{p}{2}\left(\frac{\varphi}{\lambda^3}+\psi\log[-\lambda]\right).$$

We state a preliminary result as follows.

Theorem 0. *Under the conditions and the notation as above, the singularity of* $L(\lambda; v)$ *in* (1.1) *takes the form*

$$L(\lambda; v)\equiv\frac{p(v)}{4}\left\{\frac{2L_0(v)}{\lambda^3}-\frac{L_1(v)}{\lambda^2}+\frac{L_2(v)}{\lambda}+\sum_{n=3}^{l+3}\frac{L_n(v)\lambda^{n-3}}{(n-3)!}\log[-\lambda]\right\}$$

modulo $C^l(\bar{\Omega}\cap\{z_1z_2\neq 0\})$ *for any prescribed* $l\in N_0:=\{0, 1, 2, \cdots\}$, *where the coefficients* $L_n(v)$ *are determined by* p *and its derivatives at* v.

Several coefficients can be explicitly determined. In fact, we have:

Theorem 1. *In Theorem* 0 *above, the coefficients* $L_n=L_n(v)$ *for* $0\leqq n\leqq 3$ *are given by*

$$L_0=1,\quad L_1=-\frac{1}{2}p'',\quad L_2=\frac{1}{3!}(pp^{(3)})',\quad L_3=-\frac{1}{4!}(p^2p^{(4)})''.$$

Theorem 1′. *Two more coefficients are given by*

$$5!L_4=\big(p(p^2p^{(4)})'''\big)'-\frac{1}{2}p^{(2)}(p^2p^{(4)})''+\frac{1}{2}(pp^{(4)})^2,$$

$$\begin{aligned}-6!L_5=&\left(p\big(p(p^2p^{(4)})'''\big)''\right)'+4\big(pp^{(3)}(p^2p^{(4)})''\big)'\\&+3(p^2p^{(4)})''((pp^{(3)})'-(p'')^2/2)\\&+6p^{(6)}p^{(4)}p^3+9(p^{(5)})^2p^3+50p^{(5)}p^{(4)}p'p^2\\&+10(p^{(4)})^2p''p^2+32(p^{(4)})^2(p')^2p.\end{aligned}$$

Let us conclude this subsection by giving a consequence of Theorem 1. By solving the differential equation $(p^2p^{(4)})''=0$ under boundary conditions coming from the strict pseudoconvexity assumption on $\partial\Omega\cap\{z_1z_2=0\}$, we get the following:

Corollary of Theorem 1. *Under the assumption of Theorem* 1, *if the coefficient* L_3 *vanishes identically, then* Ω *is globally equivalent to a* *ball. In other words, if $\psi=0$ on $\partial\Omega$ in Fefferman's expansion* (0.2), *then* $\Omega=\{a|z_1|^2+b|z_2|^2<1\}$ *with* a, $b>0$.

The proof of this corollary is given in Section 2.

It should be emphasized that the boundedness of Ω and the strict pseudoconvexity of $\partial\Omega\cap\{z_1z_2=0\}$ are crucial in the Corollary above.

1.2. Formal integral representation of the full expansion. We shall present, in Theorem 2 below, a formula of giving the coefficients L_n of the asymptotic expansion in Theorem 0. That formula takes the form of an integral representation, which gives rise to Theorems 1 and 1′. The integrand is a formal power series of a parameter, where the coefficients are polynomials multiplied by an exponential function to use the standard notation $\boldsymbol{R}[[\cdot]]$ and $\boldsymbol{R}[\cdot]$. That is, $\boldsymbol{R}[[\tau]]$ denotes the totality of formal power series in τ with real coefficients, and $\boldsymbol{R}[\mu]$ stands for the set of all real polynomials in μ.

In order to motivate the formulation of Theorem 2, let us first write the asymptotic expansion in Theorem 0 as

$$L(\lambda;\, v) \sim \frac{p(v)}{4}\sum_{n=0}^{\infty} L_n(v)\,\partial_\lambda^{2-n}\frac{1}{\lambda}, \tag{1.2}$$

where ∂_λ and ∂_λ^{-1} denote differentiation and indefinite integration, respectively. Since we are concerned only with the singularity at $\lambda=-0$, the ambiguity of indefinite integration is irrelevant. One may naturally express the right side of (1.2) by using a pseudodifferential operator, that is,

$$L^{\#}_{\text{aux}}(\partial_\lambda;\, v)\,\partial_\lambda^2\frac{-1}{2\lambda}=\frac{p(v)}{4}\sum_{n=0}^{\infty} L_n(v)\,\partial_\lambda^{2-n}\frac{1}{\lambda},$$

where $L^{\#}_{\text{aux}}(\hat{\tau};v)\in\boldsymbol{R}[[\hat{\tau}^{-1}]]$.

Let us next recall another implication of Theorem 0 concerning the dependence of $L_n(v)$ on the derivatives of $p(v)$. Noting an elementary relation

$$f^{(2+m)}(x)=(p(v)\partial_v)^m p(v) \text{ with } v=f'(x) \text{ for } m\in\boldsymbol{N}_0, \tag{1.3}$$

we see by Theorem 0 that $L_n(v)$ is determined by $f^{(2+m)}(x)$ for $m\geqq 0$. We shall state Theorem 2 in terms of $f^{(2+m)}(x)$ in place of $p^{(m)}(v)$. It is then natural to consider the formal Taylor expansion $M^{\#}_x(\xi)\in\boldsymbol{R}[[\xi]]$ of

$$M_x(\xi)=\frac{f(x+\xi)-f(x)-f'(x)\xi}{\xi^2}=\int_0^1(1-\sigma)f''(x+\sigma\xi)\,d\sigma, \tag{1.4}$$

and write $L_n(v)=L_n[M_x^\#]$, etc. when we wish to emphasize the dependence on $M_x^\#$.

For technical reasons that will become obvious later, we write $L_{\text{aux}}^\#(\hat{\tau}; v)=L^\#[M_x^\#](\tau)$ with $\hat{\tau}^{-1}=\tau^2$. Then, the problem is reduced to writing down explicitly $L^\#[M_x^\#](\tau)\in \boldsymbol{R}[[\tau^2]]$ such that

$$(1.5)\qquad L(\lambda, x)\sim L^\#[M_x^\#](\partial_\lambda^{-1/2})\ \partial_\lambda^2 \frac{-1}{2\lambda}.$$

Our result is described as follows.

Theorem 2. *Under the assumption of Theorem* 0, *the asymptotic relation* (1.5) *is valid with*

$$(1.6)\qquad L^\#[M_x^\#](\tau):=\frac{1}{2}\int_{\boldsymbol{R}}\frac{d\mu}{H^\#[M_x^\#](\mu,\tau)}\in \boldsymbol{R}[[\tau^2]],$$

where

$$(1.7)\qquad H^\#[M_x^\#](\mu,\tau):=\int_{\boldsymbol{R}}\exp[\mu\xi+M_x^\#(\tau\xi)\xi^2]\,d\xi\in \boldsymbol{R}[[\tau]].$$

In the next subsection, we shall explain the statement above.

1.3. Explanation of Theorem 2. Let us begin by explaining (1.6) with (1.7). We define

$$F^\#[M_x^\#](\xi,\tau)=\sum_{m=0}^{\infty}a_m(\xi)\tau^m\in \boldsymbol{R}[[\tau]]$$

by the formal Taylor expansion of $\exp[M_x^\#(\tau\xi)\xi^2-f''(x)\xi^2/2]$. Then, we can write (1.7) more explicitly as follows:

$$\begin{aligned} H^\#[M_x^\#](\mu,\tau)&=\int_{\boldsymbol{R}}F^\#[M_x^\#](\xi,\tau)\ \exp\left[\mu\xi+\frac{f''(x)}{2}\xi^2\right]d\xi\\ &=\sum_{m=0}^{\infty}A_m(\mu)\ \tau^m\ \exp\left[\frac{-\mu^2}{2f''(x)}\right],\end{aligned}$$

where $A_m(\mu)=A_m[M_x^\#](\mu)$ are defined by

$$A_m(\mu):=\int_{\boldsymbol{R}}a_m(\xi)\ \exp\left[\frac{f''(x)}{2}\left\{\xi+\frac{\mu}{f''(x)}\right\}^2\right]d\xi.$$

Using the fact that $a_0(\xi)=1$, we see that

$$A_0(\mu)=\frac{1}{B_0}>0, \qquad \text{where } B_0:=\sqrt{\frac{-f''(x)}{2\pi}}. \tag{1.8}$$

Thus $1/H^{\#}[M_x^{\#}](\mu,\tau)$ makes sense as a formal power series in τ, so that we may write

$$\frac{1}{H^{\#}[M_x^{\#}](\mu,\tau)}=\sum_{m=0}^{\infty} B_m(\mu)\ \tau^m \exp\left[\frac{\mu^2}{2f''(x)}\right]\in \boldsymbol{R}[[\tau]],$$

with functions of μ, $B_m(\mu)=B_m[M_x^{\#}](\mu)$, such that $B_0(\mu)=B_0$. For $m\in \boldsymbol{N}_0$, we set

$$L_{m/2}[M_x^{\#}]:=\frac{-1}{f''(x)}\int_{\boldsymbol{R}} B_m(\mu)\ \exp\left[\frac{\mu^2}{2f''(x)}\right]d\mu, \tag{1.9}$$

which are well-defined by virtue of the following lemma.

Lemma 0. *$A_m(\mu)$ and $B_m(\mu)$ for $m\in \boldsymbol{N}_0$ are polynomials of μ such that*

$$A_m(-\mu)=(-1)^m A_m(\mu),\ B_m(-\mu)=(-1)^m B_m(\mu). \tag{1.10}$$

In particular, the integral in (1.9) *converges and*

$$L_{n+1/2}[M_x^{\#}]=0 \text{ for } n\in \boldsymbol{N}_0,$$

so that (1.6) *makes sense and*

$$L^{\#}[M_x^{\#}](\tau)=-\frac{f''(x)}{2}\sum_{n=0}^{\infty} L_n[M_x^{\#}]\ \tau^{2n}. \tag{1.11}$$

Proof of Lemma 0. Recalling a well-known formula for the Hermite polynomials $H_m(\xi)$:

$$\int_{\boldsymbol{R}} H_m(\xi)\ \exp\left[-\frac{1}{2}(\mu-\xi)^2\right]\frac{d\xi}{\sqrt{2\pi}}=\mu^m,$$

we see that $A_m(\mu)\in \boldsymbol{R}[\mu]$ and thus $B_m(\mu)\in \boldsymbol{R}[\mu]$. By definition, we have $F^{\#}[M_x^{\#}](-\xi,-\tau)=F^{\#}[M_x^{\#}](\xi,\tau)$, and thus $H^{\#}[M_x^{\#}](-\mu,-\tau)=H^{\#}[M_x^{\#}](\mu,\tau)$. This implies (1.10).

Let us restate Theorem 2 as a refinement of Theorem 0.

Restatement of Theorem 2. *Under the same assumption as that of Theorem* 0, *the singularity of $L(\lambda,x)$ in* (1.1) *takes the form*

$$(1.12)\qquad L(\lambda, x) \equiv -\frac{f''(x)}{2}\sum_{n=0}^{l_0+3} L_n[M_x^{\sharp}]\ \partial_\lambda^{2-n}\frac{-1}{2\lambda}$$

modulo $C^{l_0}(\{-\infty<\lambda\leqq 0,\ x\in I\})$ *for* $l_0\in \boldsymbol{N}_0$ *arbitrarily prescribed, where the coefficients* $L_n[M_x^{\sharp}]$ *are defined by* (1.9), *and* ∂_λ^{-1} *stands for integration over* (λ_0, λ) *with* $\lambda_0\in(-\infty, 0)$ *arbitrarily fixed.*

2. Proof of Corollary of Theorem 1

Since $\psi|_{\partial\Omega}=0$ and $p<0$, it follows from Theorem 1 that

$$(2.1)\qquad \frac{d^2}{dv^2}\{p(v)^2 p^{(4)}(v)\}=0 \qquad \text{for } v\in f'(I).$$

We solve this differential equation under boundary conditions which are given by the following lemma.

Lemma 1 (cf. Lemme 6 of [1], page 150). *Under the same assumption as that of Corollary of Theorem* 1,

$$p\in C^\infty((-\infty, 0]),\quad \lim_{v\uparrow 0}\frac{p(v)}{v}=2,\quad \lim_{v\downarrow -\infty}\frac{p(v)}{v^2}=-2.$$

We assume for a moment the validity of Lemma 1 above and prove Corollary of Theorem 1. Integrating both sides of (2.1) twice, we use Lemma 1 and get $p^{(4)}=0$. Using Lemma 1 again, we obtain

$$p(v)=2v-2v^2, \qquad \text{that is, } f''(x)=2f'(x)-2f'(x)^2.$$

It is elementary to solve this differential equation. Using the strict pseudoconvexity of Ω, we get

$$ae^{2x}+be^{2f(x)}=1 \qquad \text{with constants } a, b>0,$$

which implies the conclusion of Corollary of Theorem 1. Thus, we are done if we prove Lemma 1.

Proof of Lemma 1. Recall that Ω is a strictly pseudoconvex complete Reinhardt domain in $\boldsymbol{C}^2$. Since, in addition, Ω is bounded, it follows that the set

$$|\partial\Omega|=\{(|z_1|, |z_2|);\ z=(z_1, z_2)\in\partial\Omega\}\subset\boldsymbol{R}^2$$

is a C^∞ curve which has endpoints $(0, r_2)$, $(r_1, 0)\in\partial\Omega$ with some constants

r_1, $r_2>0$. Using the strict pseudoconvexity of these points, we see that the boundary $\partial\Omega$ is given, locally near these points, by

$$|z_2|=r_2-l_1|z_1|^2+|z_1|^4\mathscr{E}(|z_1|) \quad \text{near } z=(0,r_2), \tag{2.2}$$

$$|z_1|=r_1-l_2|z_2|^2+|z_2|^4\mathscr{E}(|z_2|) \quad \text{near } z=(r_1,0), \tag{2.3}$$

where l_1, $l_2>0$ are constants. Here, and also in what follows, we use the notation $\mathscr{E}(\cdot)$ to denote a C^∞ even function near $0\in \boldsymbol{R}$ which changes from instance to instance.

It is clear that $f'(I)=(-\infty,0)$, which is the domain of definition of the function p. We need to study the behavior of $p(v)$ as $v\uparrow 0$ and that as $v\downarrow -\infty$. This amounts to considering $\partial\Omega$ near the points $(0, r_2)$ and $(r_1, 0)$.

Let us first consider $\partial\Omega$ near $(0, r_2)$. Taking logarithm of both sides of (2.2), we have

$$f(x)=\log r_2-le^{2x}+e^{4x}\mathscr{E}(e^x), \quad \text{where } l:=l_1/r_2>0.$$

We differentiate both sides and get

$$f'(x)=-2le^{2x}+e^{4x}\mathscr{E}(e^x), \quad f''(x)=-4le^{2x}+e^{4x}\mathscr{E}(e^x) \tag{2.4}$$

as $x\downarrow -\infty$. Applying the inverse function theorem to the first equation of (2.4), we obtain $e^x=\sqrt{-v/(2l)}\,\{1+v\mathscr{E}(\sqrt{-v})\}$ as $v\uparrow 0$. This, together with the second equation of (2.4), yields

$$p(v)=2v+v^2\mathscr{E}(\sqrt{-v}) \quad \text{as } v\uparrow 0. \tag{2.5}$$

Therefore, we obtain the first two assertions of Lemma 1.

It remains to verify the last assertion, which follows from

$$p(v)=-2v^2+v\mathscr{E}(1/\sqrt{-v}) \quad \text{as } v\downarrow -\infty. \tag{2.6}$$

The proof of (2.6) is similar to that of (2.5). Starting from (2.3) in place of (2.2) and setting $l':=l_2/r_1>0$, we argue as before. Then we get

$$-1/f'(x)=2l'e^{2y}+e^{4y}\mathscr{E}(e^y), \quad f''(x)=-2f'(x)^2\{1+e^{2y}\mathscr{E}(e^y)\} \tag{2.4$'$}$$

as $y:=f(x)\downarrow -\infty$. The first equation of (2.4)$'$ yields

$$e^y=\sqrt{-w/(2l')}\,\{1+w\mathscr{E}\,(\sqrt{(-w)}\} \quad \text{as } w:=1/v\uparrow 0,$$

which, together with the second equation of (2.4)$'$, implies (2.6) as desired. Therefore, the proof of Lemma 1 is complete.

3. Proofs of Theorems 1 and 1′

Assuming the validity of Theorem 2, we prove Theorems 1 and 1′.

3.1. Preliminary observation on the dependence of $L_n(v)$ on $p^{(k)}(v)$. Recall Theorem 0 which constitutes a part of the assertion of Theorem 2. By Theorem 0, we see that each $L_n(v)$ is determined by the derivatives $p^{(k)}(v)$ for $k \geqq 0$. It is not difficult to see, by recalling the explanation in Subsection 1.3, that each L_n depends only on $p^{(k)}(v)$ with $k \leqq 2n$. In order to prove Theorems 1 and 1′, it is useful to give a refinement of this result. First, some notation.

Multi-index notation. For a multi-index $\gamma = (\gamma_1, \cdots, \gamma_j) \in \boldsymbol{N}_0^j$, we define its length as usual by $|\gamma| := \gamma_1 + \cdots + \gamma_j$. Given another integer $m \geqq j$, we set

$$\boldsymbol{N}^{j,m} := \{\gamma \in \boldsymbol{N}^j;\ |\gamma| = m,\ \gamma_1 \geqq \cdots \geqq \gamma_j > 0\},$$

where $\boldsymbol{N} = \{1, 2, 3, \cdots\}$. In case $j = m$, we also define

$$\boldsymbol{N}_0^{m,m} := \{\eta \in \boldsymbol{N}_0^m;\ |\eta| = m,\ \eta_1 \geqq \cdots \geqq \eta_m \geqq 0\}.$$

Assuming the validity of Theorem 2, we shall prove the following:

Proposition 0. *Each coefficient $L_n(v) = L_n[M_x^\#]$ in Theorem 0 is a linear combination of $f^{(2+\gamma_1)}(x) \cdots f^{(2+\gamma_j)}(x)/f''(x)^{j+n}$ with $\gamma \in \boldsymbol{N}^{j,2n}$. In terms of p and its derivatives, each L_n in Theorem 0 is a linear combination of*

$$e_\eta[p] := p^{(\eta_1)} \cdots p^{(\eta_{2n})}/p^n \quad \text{with } \eta \in \boldsymbol{N}_0^{2n,2n}.$$

Before proving Proposition 0, we do a normalization, which will be useful also in the remaining subsections. Setting

$$\hat{M}_x^\#(\xi) = \frac{M_x^\#(\xi)}{-f''(x)} = \sum_{m=0}^{\infty} \frac{\hat{f}_{2+m}^x}{(2+m)!} \xi^m \in \boldsymbol{R}[[\xi]]$$

so that $\hat{f}_2^x = -1$, we consider $F^\#$, $H^\#$, and $L^\#$ with $\hat{M}_x^\#$ in place of $M_x^\#$, that is,

$$F^\#[\hat{M}_x^\#](\xi, \tau) = \exp[\frac{1}{2}\xi^2 + \hat{M}_x^\#(\tau\xi)\xi^2] = \exp\left[\sum_{m=1}^{\infty} \frac{\hat{f}_{2+m}^x}{(2+m)!} \xi^{2+m}\tau^m\right],$$

$$(3.1) \quad H^\#[\hat{M}_x^\#](\mu, \tau) = \int_{\boldsymbol{R}} F^\#[\hat{M}_x^\#](\xi, \tau) \exp\left[\mu\xi - \frac{1}{2}\xi^2\right] d\xi,$$

$$L^\#[\hat{M}_x^\#](\tau) = \frac{1}{2}\int_{\boldsymbol{R}} \frac{d\mu}{H^\#[\hat{M}_x^\#](\mu, \tau)}.$$

Noting that $L^{\#}[\hat{M}_x^{\#}](C\tau)=C^2L^{\#}[M_x^{\#}/C^2](\tau)$ for an arbitrary constant $C>0$, we have from (1.11) that

$$(3.2)\qquad L^{\#}[\hat{M}_x^{\#}](\tau)=\frac{1}{2}\sum_{n=0}^{\infty}\{-f''(x)\}^n L_n[M_x^{\#}]\ \tau^{2n}\in \boldsymbol{R}[[\tau^2]].$$

Proof of Proposition 0. We prove the former statement—the latter follows immediately from the former via (1.3). It suffices to show that each $L_n[M_x^{\#}]$ is a linear combination of $\hat{f}_{2+\gamma_1}^x\cdots\hat{f}_{2+\gamma_j}^x/f''(x)^n$ with $\gamma\in \boldsymbol{N}^{j,2n}$. In order to prove this, we set

$$\text{weight }(\hat{f}_{2+\gamma_1}^x\cdots\hat{f}_{2+\gamma_j}^x\tau^k)=|\gamma|-k;$$

we say that a polynomial in $\hat{f}_{2+m}^x$ and τ is of homogeneous weight w if it is the sum of monomials with the same weight w. Then we are reduced to showing that each term in the right side of (3.2) is of homogeneous weight 0. But this is clear from the expression of $L^{\#}[\hat{M}_x^{\#}](\tau)$ in terms of $F^{\#}[\hat{M}_x^{\#}](\xi,\tau)$.

3.2. Proof of Theorem 1. We have given the method of computation in Subsection 1.3. It is only necessary to evaluate quantities appearing there and then express the results in terms of the derivatives of p. Observe by (3.2) that $L_0[M_x^{\#}]=2L^{\#}[\hat{M}_x^{\#}](0)=1$. Thus the first task is to represent $L_n[M_x^{\#}]$ for $n=1,2,3$ in terms of the derivatives of f. This is done by the following:

Proposition 1. *The coefficients $L_n=L_n[M_x^{\#}]$ for $n=1,2,3$ are given by*

$$2f''(x)L_1=\hat{f}_4^x+\cdots,$$
$$-3!f''(x)^2L_2=\hat{f}_6^x+6\hat{f}_5^x\hat{f}_3^x+\cdots,$$
$$4!f''(x)^3L_3=\hat{f}_8^x+12\hat{f}_7^x\hat{f}_3^x+24\hat{f}_6^x\hat{f}_4^x+82\hat{f}_6^x(\hat{f}_3^x)^2+\cdots,$$

where $\cdots$ in the expression of each L_n is a linear combination of $\hat{f}_{2+\gamma_1}^x\cdots\hat{f}_{2+\gamma_j}^x$ with $\gamma=(\gamma_1,\cdots,\gamma_j)\in \boldsymbol{N}^{j,2n}$ satisfying $\gamma_1<n+1$.

It is not necessary to write down the abbreviated parts in Proposition 1 above. This is because of the following observation, which refines Proposition 0 in the cases $n=1,2,3$.

Proposition 2. *If $n\leqq 3$ then L_n is a linear combination of $e_\eta[p]$ with $\eta=(\eta_1,\cdots,\eta_{2n})\in \boldsymbol{N}_0^{2n,2n}$ satisfying $\eta_1\geqq n+1$. More explicitly, L_n for $n=1,2,3$ are linear combinations of*

$$p''\quad \textit{for } n=1;\quad p^{(4)}p,\quad p^{(3)}p'\quad \textit{for } n=2;$$

$$\text{and } p^{(6)}p^2, \quad p^{(5)}p'p, \quad p^{(4)}p''p, \quad p^{(4)}(p')^2 \quad \text{for } n=3.$$

Propositions 1 and 2 are proved in Subsection 3.4 and 3.5.

Proof of Theorem 1. We are almost done by virtue of Proposition 1. It remains only to express the right sides in terms of the derivatives of p. This is done by using (1.3); specifically, we need

$$f''=p, \quad -\hat{f}_3^x=p', \quad -\hat{f}_4^x=p^{(2)}p+(p')^2,$$

and

$$-\hat{f}_5^x=p^{(3)}p^2+4p''p'p+(p')^3,$$
$$-\hat{f}_6^x=p^{(4)}p^3+7p^{(3)}p'p^2+4(p'')^2p^2+11p''(p')^2p+(p')^4,$$
$$-\hat{f}_7^x=p^{(5)}p^4+11p^{(4)}p'p^3+15p^{(3)}p''p^3+32p^{(3)}(p')^2p^2+\cdots,$$
$$-\hat{f}_8^x=p^{(6)}p^5+16p^{(5)}p'p^4+26p^{(4)}p''p^4+76p^{(4)}(p')^2p^3+\cdots,$$

where $\cdots$ in the expression of $-\hat{f}_{2+m}^x$ for $m=5,6$ is a linear combination of $p^{(\eta_1)}\cdots p^{(\eta_m)}$ with $\eta=(\eta_1,\cdots,\eta_m)\in N_0^{m,m}$ satisfying $\eta_1<m-2$. By virtue of Proposition 2, the abbreviated parts are irrelevant both in the expressions above and in Proposition 1. After some algebraic manipulations, we get the desired conclusion.

3.3. Proof of Theorem 1′. We follow the argument in the previous subsection. Let us first state substitutes for Propositions 1 and 2.

Proposition 1′. *If $n=4$ or 5, then $\hat{L}_n:=L_n[\hat{M}_x^\sharp]=\{-f''(x)\}^nL_n[M_x^\sharp]$ is given by $(n+1)!\hat{L}_n=Q_n+\cdots$, where $\cdots$ is a linear combination of*

$$\hat{f}_{2+\gamma_1,\cdots,2+\gamma_j}:=\hat{f}_{2+\gamma_1}^x\cdots\hat{f}_{2+\gamma_j}^x \quad \text{with } \gamma=(\gamma_1,\cdots,\gamma_j)\in N^{j,2n}$$

satisfying $\gamma_1<4$ or $j>n$; Q_4 and Q_5 are given by

$$-Q_4=\hat{f}_{10}+20\hat{f}_{9,3}+\frac{105}{2}\hat{f}_{8,4}+\frac{445}{2}\hat{f}_{8,3,3}+90\hat{f}_{7,5}+1030\hat{f}_{7,4,3}$$
$$+1740\hat{f}_{7,3,3,3}+\frac{107}{2}\hat{f}_{6,6}+1525\hat{f}_{6,5,3}+1030\hat{f}_{6,4,4}+10440\hat{f}_{6,4,3,3},$$

$$-Q_5=\hat{f}_{12}+30\hat{f}_{11,3}+97\hat{f}_{10,4}+492\hat{f}_{10,3,3}+210\hat{f}_{9,5}+2880\hat{f}_{9,4,3}$$
$$+5670\hat{f}_{9,3,3,3}+327\hat{f}_{8,6}+5580\hat{f}_{8,5,3}+\frac{7545}{2}\hat{f}_{8,4,4}$$
$$+44550\hat{f}_{8,4,3,3}+\frac{100095}{2}\hat{f}_{8,3,3,3,3}+189\hat{f}_{7,7}+7668\hat{f}_{7,6,3}$$

$$+12900\hat{f}_{7,5,4}+76140\hat{f}_{7,5,3,3}+102930\hat{f}_{7,4,4,3}+462360\hat{f}_{7,4,3,3,3}$$
$$+7668\hat{f}_{6,6,4}+45261\hat{f}_{6,6,3,3}+9540\hat{f}_{6,5,5}+304380\hat{f}_{6,5,4,3}$$
$$+683490\hat{f}_{6,5,3,3,3}+68560\hat{f}_{6,4,4,4}+1385505\hat{f}_{6,4,4,3,3}.$$

Proposition 2′. *If $n=4$ or 5, then $L_n=L_n[M_x^\sharp]$ is a linear combination of $e_\eta[p]$ with $\eta=(\eta_1,\cdots,\eta_{2n})\in N_0^{2n,2n}$ satisfying $\eta_1\geqq 4$ and $\eta_{n+1}=0$.*

Proof of Theorem 1′. The procedure is the same as that of Theorem 1; we use Propositions 1′ and 2′ in place of Propositions 1 and 2. It is necessary to express $\hat{f}^x_{2+m}$ with $m\leqq 10$ in terms of the derivatives of p; we need

$$-\hat{f}^x_{2+m}=R_m+\cdots \quad \text{for } m\leqq 10$$

where $\cdots$ is a linear combination of $p^{(\eta_1)}\cdots p^{(\eta_m)}$ with $\eta=(\eta_1,\cdots,\eta_m)\in N_0^{m,m}$ satisfying the following conditions: for $m\leqq 6$, $\eta_5>0$; for $m=7,8,9$, $\eta_1<4$ or $\eta_5>0$; for $m=10$, $\eta_1<4$ or $\eta_6>0$. In order to shorten the description of R_m for $5\leqq m\leqq 10$, we set for a moment $p_k=p^{(k)}$ for $k\in N_0$; then

$$R_5=p_5p_0^4+11p_4p_1p_0^3+15p_3p_2p_0^3+32p_3p_1^2p_0^2$$
$$+34p_2^2p_1p_0^2+26p_2p_1^3p_0,$$
$$R_6=p_6p_0^5+16p_5p_1p_0^4+26p_4p_2p_0^4+76p_4p_1^2p_0^3+15p_3^2p_0^4$$
$$+192p_3p_2p_1p_0^3+122p_3p_1^3p_0^2+34p_2^3p_0^3+180p_2^2p_1^2p_0^2,$$
$$R_7=p_7p_0^6+22p_6p_1p_0^5+42p_5p_2p_0^5+156p_5p_1^2p_0^4+56p_4p_3p_0^5$$
$$+474p_4p_2p_1p_0^4+426p_4p_1^3p_0^3,$$
$$R_8=p_8p_0^7+29p_7p_1p_0^6+64p_6p_2p_0^6+288p_6p_1^2p_0^5+98p_5p_3p_0^6$$
$$+1038p_5p_2p_1p_0^5+1206p_5p_1^3p_0^4+56p_4^2p_0^6+1344p_4p_3p_1p_0^5$$
$$+768p_4p_2^2p_0^5+5142p_4p_2p_1^2p_0^4,$$
$$R_9=p_9p_0^8+37p_8p_1p_0^7+93p_7p_2p_0^7+491p_7p_1^2p_0^6+162p_6p_3p_0^7$$
$$+2062p_6p_2p_1p_0^6+2934p_6p_1^3p_0^5+210p_5p_4p_0^7+3068p_5p_3p_1p_0^6$$
$$+1806p_5p_2^2p_0^6+14988p_5p_2p_1^2p_0^5+1736p_4^2p_1p_0^6$$
$$+4590p_4p_3p_2p_0^6+18864p_4p_3p_1^2p_0^5+20838p_4p_2^2p_1p_0^5,$$
$$R_{10}=p_{10}p_0^9+46p_9p_1p_0^8+130p_8p_2p_0^8+787p_8p_1^2p_0^7+255p_7p_3p_0^8$$

$$+3788p_7p_2p_1p_0{}^7+6371p_7p_1{}^3p_0{}^6+372p_6p_4p_0{}^8+6426p_6p_3p_1p_0{}^7$$
$$+3868p_6p_2{}^2p_0{}^7+38224p_6p_2p_1{}^2p_0{}^6+25761p_6p_1{}^4p_0{}^5$$
$$+210p_5{}^2p_0{}^8+8220p_5p_4p_1p_0{}^7+11270p_5p_3p_2p_0{}^7$$
$$+55328p_5p_3p_1{}^2p_0{}^6+63456p_5p_2{}^2p_1p_0{}^6+165978p_5p_2p_1{}^3p_0{}^5$$
$$+6326p_4{}^2p_2p_0{}^7+31016p_4{}^2p_1{}^2p_0{}^6+7155p_4p_3{}^2p_0{}^7$$
$$+156894p_4p_3p_2p_1p_0{}^6+203304p_4p_3p_1{}^3p_0{}^5$$
$$+28768p_4p_2{}^3p_0{}^6+325500p_4p_2{}^2p_1{}^2p_0{}^5.$$

After more algebraic manipulations than those in the proof of Theorem 1, we reach the desired result.

Unfortunately, we only have computer-aided proofs of Propositions 1′ and 2′ as well as the expression of R_m for $5 \leqq m \leqq 10$. A more acceptable proof is desired for Theorem 1′, or at least for Proposition 2′.

3.4. Proof of Proposition 1. We need to carry out some explicit computation which corresponds to the explanation in Subsection 1.3. It is then convenient to restate Proposition 0 in terms of $L^\#[\hat{M}_x^\#]$, $H^\#[\hat{M}_x^\#]$ and $F^\#[\hat{M}_x^\#]$—normalized expression in (3.1). Recall first that

$$H^\#[\hat{M}_x^\#](\mu, \tau) = \sqrt{2\pi}\, e^{\mu^2/2} \sum_{m=0}^{\infty} \hat{A}_m(\mu)\, \tau^m \in \boldsymbol{R}[[\tau]],$$

$$\frac{1}{H^\#[\hat{M}_x^\#](\mu, \tau)} = \frac{1}{\sqrt{2\pi}} e^{-\mu^2/2} \sum_{m=0}^{\infty} \hat{B}_m(\mu)\, \tau^m \in \boldsymbol{R}[[\tau]],$$

where $\hat{A}_m(\mu)$ and $\hat{B}_m(\mu)$ are polynomials of μ with $\hat{A}_0(\mu) = \hat{B}_0(\mu) = 1$; furthermore,

$$L^\#[\hat{M}_x^\#](\tau) = \frac{1}{2} \sum_{n=0}^{\infty} \hat{L}_n \tau^{2n} \in \boldsymbol{R}[[\tau^2]],$$

where $\hat{L}_n = (2\pi)^{-1/2} \int_{\boldsymbol{R}} \hat{B}_{2n}(\mu) \exp[-\mu^2/2]\, d\mu$ are the same as those in (3.2), that is, $\hat{L}_n = \{-f''(x)\}^n L_n[M_x^\#]$. Then Proposition 0 is restated as

$$\hat{L}_n = \sum_{\gamma \in \boldsymbol{N}^{j,2n}} c_n(\gamma) \frac{\hat{f}^x_{2+\gamma_1}}{(2+\gamma_1)!} \cdots \frac{\hat{f}^x_{2+\gamma_j}}{(2+\gamma_j)!},$$

where $c_n(\gamma) \in \boldsymbol{R}$ for $\gamma = (\gamma_1, \cdots, \gamma_j) \in \boldsymbol{N}^{j,2n}$ are universal constants independent of the domain Ω.

We wish to evaluate $c_n(\gamma)$ explicitly. This is done by recalling that

$$F^{\#}[\hat{M}_x^{\#}](\xi,\tau)=\sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{m=1}^{\infty}\frac{\hat{f}_{2+m}^{x}}{(2+m)!}\xi^{2+m}\tau^{m}\right)^{k}\in \boldsymbol{R}[[\tau]].$$

In fact, writing $F^{\#}[\hat{M}_x^{\#}](\xi,\tau)=\sum_{m=0}^{\infty}\hat{a}_m(\xi)\tau^m\in \boldsymbol{R}[[\tau]]$ with $\hat{a}_m(\xi)\in\boldsymbol{R}[\xi]$, we have

$$\hat{A}_m(\mu)=\int_{\boldsymbol{R}}\hat{a}_m(\xi)\ \exp\left[-\frac{1}{2}(\mu-\xi)^2\right]\frac{d\xi}{\sqrt{2\pi}},$$

while $\hat{B}_m(\mu)$ are determined by

$$\sum_{k=0}^{m}\hat{A}_{m-k}(\mu)\hat{B}_k(\mu)=0\quad\text{for } m\geqq 1.$$

In order to prove Proposition 1, it suffices to verify the following three statements:

(1°) $\dfrac{c_n(2n)}{(2+2n)!}=\dfrac{-1}{(n+1)!}$ for $n\geqq 1$;

(2°) $\dfrac{c_n(2n-1,1)}{(1+2n)!\ 3!}=\dfrac{-1}{(n-1)!}$ for $n\geqq 2$, and $\dfrac{c_3(4,2)}{6!\ 4!}=-1$;

(3°) $\dfrac{c_3(4,1,1)}{6!\ (3!)^2}=-\dfrac{41}{12}\cdot$

Before proving (1°)–(3°) above, let us prepare an elementary Calculus Lemma. Given $\gamma=(\gamma_1,\cdots,\gamma_j)\in\boldsymbol{N}^{j,2n}$, we set

$$I_n(\gamma)=\int_{\boldsymbol{R}}G_{\gamma_1}(\mu)\cdots G_{\gamma_j}(\mu)\ e^{-\mu^2/2}\frac{d\mu}{\sqrt{2\pi}},$$

where $G_m(\mu)\in\boldsymbol{R}[\mu]$ for $m\in\boldsymbol{N}_0$ is defined by

$$G_m(\mu)=\int_{\boldsymbol{R}}\xi^m\exp\left[-\frac{1}{2}(\mu-\xi)^2\right]\frac{d\xi}{\sqrt{2\pi}}.$$

Calculus Lemma. *Let $I_n(\gamma)$ for $\gamma=(\gamma_1,\cdots,\gamma_j)\in\boldsymbol{N}^{j,2n}$ be defined as above. Then, the following five statements hold*:

(i) $I_n(2n)=\frac{(2n)!}{n!}$ *for* $n\geqq 1$;

(ii) $I_n(2n-3,3)=(n+4)\frac{(2n-3)!}{(n-2)!}$ *for* $n\geqq 3$;

(iii) $I_5(6,4)=6480=\frac{3}{8}6!\ 4!$;

(iv) $I_6(6,6)=101520=\frac{47}{12}6!\ (3!)^2$;

(v) $I_6(6,3,3)=53280=\frac{37}{18}6!\ (3!)^2$.

Proof of (1°). We are concerned with the case $\gamma=2n\in \boldsymbol{N}^{1,2n}$. This case is easy, because the linear term $\hat{f}^x_{2+2n}$ appears only as the lowest order term of $\hat{a}_{2n}(\xi)\in \boldsymbol{R}[\xi]$. We get

$$\hat{A}_{2n}(\mu)=\frac{\hat{f}^x_{2+2n}}{(2+2n)!}G_{2+2n}(\mu)+\cdots,$$

where $\cdots$ is the part which is irrelevant to our purpose. Since $\hat{B}_{2n}(\mu)=-\hat{A}_{2n}(\mu)+\cdots$, it follows from (i) of the Calculus Lemma above that

$$c_n(2n)=-I_{n+1}(2n+2)=-\frac{(2n+2)!}{(n+1)!}\cdot$$

Proof of (2°). Let us consider the case $\gamma=(\gamma_1,\gamma_2)\in \boldsymbol{N}^{2,2n}$ with $\gamma_1>\gamma_2$. We wish to extract the quadratic term $\hat{f}^x_{2+\gamma_1}\hat{f}^x_{2+\gamma_2}$ from $\hat{L}_n$. Inspecting the expansion of $F^\#[\hat{M}^\#_x]$, we see that

$$\hat{a}_{2n}(\xi)=\frac{\hat{f}^x_{2+\gamma_1}}{(2+\gamma_1)!}\frac{\hat{f}^x_{2+\gamma_2}}{(2+\gamma_2)!}\xi^{4+2n}+\cdots,$$

$$\hat{a}_{\gamma_1}(\xi)=\frac{\hat{f}^x_{2+\gamma_1}}{(2+\gamma_1)!}\xi^{2+\gamma_1}+\cdots,\quad \hat{a}_{\gamma_2}(\xi)=\frac{\hat{f}^x_{2+\gamma_2}}{(2+\gamma_2)!}\xi^{2+\gamma_2}+\cdots,$$

and $\hat{B}_{2n}(\mu)=-\hat{A}_{2n}(\mu)+2\hat{A}_{\gamma_1}(\mu)\hat{A}_{\gamma_2}(\mu)+\cdots$. Consequently,

$$\begin{aligned}c_n(\gamma)&=-I_{n+2}(2n+4)+2I_{n+2}(2+\gamma_1,2+\gamma_2)\\&=-\frac{(2n+4)!}{(n+2)!}+2I_{n+2}(2+\gamma_1,2+\gamma_2),\end{aligned}$$

where we have used (i) of the Calculus Lemma. In order to get the desired results, we also use (ii) and (iii) of the Calculus Lemma. Restricting ourselves to the case $\gamma=(2n-1,1)$, we have

$$I_{n+2}(2+\gamma_1, 2+\gamma_2)=(n+6)\frac{(2n+1)!}{n!},$$

which implies the first assertion of (2°). It remains to consider the case $\gamma=(4,2)$. We have

$$c_3(4,2)=-I_5(10)+2I_5(6,4)=-\frac{10!}{5!}+\frac{3}{4}6!\ 4!=-6!\ 4!.$$

Thus the proof of (2°) is complete.

Proof of (3°). It remains to study the case $\gamma=(4,1,1)\in \boldsymbol{N}^{3,6}$—the cubic term $\hat{f}_6^x(\hat{f}_3^x)^2$ in $\hat{L}_3$ is concerned. We argue as before. It is sufficient to look at

$$\hat{a}_6(\xi)=\frac{1}{2}\frac{\hat{f}_6^x}{6!}\left(\frac{\hat{f}_3^x}{3!}\right)^2\xi^{12}+\cdots,\qquad \hat{a}_5(\xi)=\frac{\hat{f}_6^x\hat{f}_3^x}{6!\,3!}\xi^9+\cdots,$$

$$\hat{a}_4(\xi)=\frac{\hat{f}_6^x}{6!}\xi^6+\cdots,\qquad \hat{a}_2(\xi)=\frac{1}{2}\left(\frac{\hat{f}_3^x}{3!}\right)^2\xi^6+\cdots,\qquad \hat{a}_1(\xi)=\frac{\hat{f}_3^x}{3!}\xi^3+\cdots,$$

and $\hat{B}_6(\mu)=-\hat{A}_6(\mu)+2\hat{A}_5(\mu)\hat{A}_1(\mu)+\hat{A}_4(\mu)\{2\hat{A}_2(\mu)-3\hat{A}_1(\mu)^2\}+\cdots$. From these,

$$c_3(4,1,1)=-\frac{1}{2}I_6(12)+2I_6(9,3)+I_6(6,6)-3I_6(6,6,3).$$

Again, we use the Calculus Lemma—(i), (ii) and (iv), (v); then

$$-\frac{1}{2}I_6(12)+2I_6(9,3)=-\frac{7}{6}6!(3!)^2,$$

$$I_6(6,6)-3I_6(6,3,3)=-\frac{9}{4}6!(3!)^2,$$

from which the conclusion follows.

Proof of the Calculus Lemma. The first assertion (i) is a consequence of the well-known formula

$$\int_{\boldsymbol{R}}\exp\left[-\frac{1}{2}(\xi-\mu)^2\right]\exp\left[-\frac{1}{2}\mu^2\right]\frac{d\mu}{\sqrt{2\pi}}=\frac{1}{\sqrt{2}}\exp\left[-\frac{1}{4}\xi^2\right].$$

In fact, changing the order of integration

$$I_n(2n)=\frac{1}{\sqrt{2}}\int_{\boldsymbol{R}}\xi^{2n}\exp\left[-\frac{1}{4}\xi^2\right]\frac{d\xi}{\sqrt{2\pi}}=\frac{(2n)!}{n!}.$$

In order to get (ii)–(v), we note that $G_m(\mu)$ are modified Hermite polynomials in the sense that $G_m(\mu)=i^{-m}H_m(i\mu)$. It is well-known that

$$\exp\left[\mu s-\frac{1}{2}s^2\right]=\sum_{m=0}^{\infty}H_m(\mu)\frac{s^m}{m!},$$

which implies

$$\frac{1}{\gamma!}H_{\gamma_1}(\mu)H_{\gamma_2}(\mu)=\sum_{0\leqq k\leqq\gamma_2}\frac{1}{k!}\frac{H_{m-2k}(\mu)}{(\gamma_1-k)!\,(\gamma_2-k)!}$$

for $\gamma=(\gamma_1,\gamma_2)\in\boldsymbol{N}^{2,m}$, where $\gamma!=\gamma_1!\,\gamma_2!$. Setting $m=2n$, we get

$$\text{(3.3)}\qquad \frac{1}{\gamma!}G_{\gamma_1}(\mu)G_{\gamma_2}(\mu)=\sum_{0\leqq k\leqq\gamma_2}\frac{(-1)^k}{k!}\frac{G_{2n-2k}(\mu)}{(\gamma_1-k)!(\gamma_2-k)!}$$

for $\gamma=(\gamma_1,\gamma_2)\in\boldsymbol{N}^{2,2n}$. Multiplying both sides of (3.3) by $\exp[-\mu^2/2]$, we integrate the results with respect to μ. Using the conclusion of (i), we obtain

$$\text{(3.4)}\qquad \frac{n!}{\gamma!}I_n(\gamma)=\sum_{0\leqq k\leqq\gamma_2}(-1)^k\binom{n}{k}\frac{(2n-2k)!}{(\gamma_1-k)!\,(\gamma_2-k)!}$$

for $\gamma=(\gamma_1,\gamma_2)\in\boldsymbol{N}^{2,2n}$. Therefore, the proofs of (ii)–(iv) are done by evaluating the right side of (3.4) in these cases. Also, (v) is proved by a repeated use of (3.3) and (3.4).

Let us first prove (ii)–(iv). We write $\gamma_2=l$, and thus $\gamma_1=2n-l$. In order to do efficient computation, we note that the right side of (3.4) is the coefficient of s^l in the expansion of $\{(1+s)^2-s\}^n$. Since $(1+s)^2-s=1+s+s^2=(1-s^3)/(1-s)$, it follows that

$$\{(1+s)^2-s\}^n=1+(n)_1s+\frac{(n)_2}{2}s^2+\left(\frac{(n)_3}{3!}+(-n)_1\right)s^3$$

$$(3.5)\qquad +\left(\frac{(n)_4}{4!}+(-n)_1(n)_1\right)s^4+\left(\frac{(n)_5}{5!}+(-n)_1\frac{(n)_2}{2}\right)s^5$$

$$+\left(\frac{(n)_6}{6!}+(-n)_1\frac{(n)_3}{3!}+\frac{(-n)_2}{2}\right)s^6+O(s^7),$$

where $(n)_l=n(n+1)\cdots(n+l-1)$. By using this formula (3.5), we obtain (ii)–(iv).

It remains to consider the case $\gamma=(6,3,3)\in \boldsymbol{N}^{3,12}$. Using (3.3), we have

$$\frac{1}{6!3!}I_6(6,3,3)=\sum_{k=0}^{3}\frac{(-1)^k}{k!}\frac{I_{6-k}(9-2k;3)}{(6-k)!(3-k)!}.$$

We evaluate $I_{6-k}(9-2k,3)$ by using (3.4). This is not difficult, because computation is simplified with the aid of (3.5). We thus obtain the desired result (v).

3.5. Proof of Proposition 2. It is possible to verify the desired conclusion by direct computation which, however, is lengthy. In order to shorten the proof, we require a lemma.

Lemma 2. *For* $a, b\in 2\boldsymbol{N}_0$ *with* $b>0$ *arbitrarily fixed, consider a complete Reinhardt domain in* $\boldsymbol{C}^2$ *given by* $\Omega_{a,b}:=\{|z_1|^2|z_2|^a+|z_2|^b<1\}$. *Then the following statements hold*:

(i) $p(v)=\frac{1}{2}(2+av)v(2+(a-b)v)=2v+(2a-b)v^2+\frac{1}{2}a(a-b)v^3$, *and the domain of definition of* p *is the interval* $-2/a<v<0$.

(ii) *The Bergman kernel* $K_{a,b}$ *does not have log term in the sense that* $\psi=0$ *in* (0.2).

(iii) *If* $a=0$ *then* $L_0=1$, $L_1=b$ *and* $L_n=0$ *for all* $n\geqq 2$.

Here is the only place in the present paper where we encounter unbounded domains or the breakdown of the strict pseudoconvexity at boundary points $z=(z_1,z_2)$ satisfying $z_1z_2=0$. Note that if $a>0$ then $\Omega_{a,b}$ is an unbounded domain such that the boundary is everywhere strictly pseudoconvex. If $b>2$ then $\Omega_{0,b}$ is a bounded domain such that the boundary is strictly pseudoconvex except at the portion $z_2=0$. The domain $\Omega_{0,2}$ is the unit ball. As we mentioned at the end of Introduction, Theorem 2 remains valid for the domains $\Omega_{a,b}$ in Lemma 2.

Postponing the proof of Lemma 2 for a moment, let us first prove Proposition 2 with the aid of Lemma 2.

Proof of Proposition 2. Regarding $L_n = L_n[M_x^\#]$ as a function of the variable v, we write, according to Proposition 0,

$$L_n = L_n(v) = \sum_\eta C_n(\eta)\ e_\eta[p] \quad \text{with } e_\eta[p] = \frac{p^{(\eta_1)} \cdots p^{(\eta_{2n})}}{p^n} \tag{3.6}$$

and coefficients $C_n(\eta) = C_n(\eta_1, \cdots, \eta_{2n}) \in \boldsymbol{R}$, where summation is taken over multi-indices $\eta = (\eta_1, \cdots, \eta_{2n}) \in \boldsymbol{N}_0^{2n,2n}$.

Let us first observe that the desired conclusion will be obtained if we prove the following three statements:

(1°) $C_n(\eta) = 0$ if $\eta_1 \leqq 2$ and $\eta \neq \eta^0 := (2, \cdots, 2, 0, \cdots, 0) \in \boldsymbol{N}_0^{2n,2n}$;
(2°) $C_n(\eta^0) = 0$ for $n \geqq 2$;
(3°) $C_3(\eta) = 0$ in case $\eta_1 = 3$.

In fact, if $n = 1$, then the possible choice of η is (2,0) and (1,1), but (1°) yields $C_1(1, 1) = 0$. In case $n = 2$, what we want to show is that

$$C_2(\eta) = 0 \text{ for } \eta = (1, 1, 1, 1),\ (2, 1, 1, 0),\ (2, 2, 0, 0),$$

a fact which is a consequence of (1°) and (2°). Similarly, if $n = 3$ then the desired conclusion is

$$C_3(\eta) = 0 \text{ in case } \eta_1 \leqq 3,$$

which follows from (3°) if $\eta_1 = 3$, and from (1°) and (2°) if $\eta_1 \leqq 2$.

We shall prove (1°), (2°) and (3°) by using Lemma 2. The idea is simple. That is, for the domains $\Omega_{a,b}$ in Lemma 2, we compute $e_\eta[p]$ by using (i), and specify L_n with the aid of (ii) and (iii). Substituting the results to the both sides of (3.6), we see that certain coefficients $C_n(\eta)$ must vanish and obtain the desired conclusions (1°), (2°) and (3°). More precisely, we regard $e_\eta[p]$ as functions of v and consider the behavior as $v \uparrow 0$. Then, the left hand side of (3.6) is constant for each n fixed, whereas the functions $e_\eta[p]$ appearing in the right hand side behave differently according as the multi-indices η vary. Thus, if $e_\eta[p]$ is non-constant then correspondingly $C_n(\eta) = 0$.

Let us proceed to the proofs of (1°), (2°) and (3°). In order to show (1°) and (2°), we take a$=0$ in Lemma 2, and thus $p = 2v - bv^2$, $L_0 = 1$, $L_1 = b$ and $L_n = 0$ for $n \geqq 2$. Hence, setting $n = 1$ in (3.6), we get

$$b = C_1(2,0)p'' + C_1(1,1)\,\frac{(p')^2}{p} = -2bC_1(2,0) + C_1(1,1)\,\frac{(2-2bv)^2}{2v-bv^2}$$

This yields at first $C_1(1,1)=0$ and then $C_1(2,0)=-1/2$. We thus get $L_1=-p''/2$ (and $L_0=1$). Let us next take $n \geqq 2$ in (3.6). If $n=2$ then

$$C_2(1,1,1,1)\frac{(p')^4}{p^2} + C_2(2,1,1,0)\frac{p''(p')^2}{p} + C_2(2,2,0,0)(p'')^2$$
$$+ C_2(3,1,0,0)p^{(3)}p' + C_2(4,0,0,0)p^{(4)}p = 0.$$

Observing the behavior as $v \uparrow 0$, we first get $C_2(1,1,1,1)=C_2(2,1,1,0)=0$ and then $C_2(2,2,0,0)=0$. The same reasoning applies to the general case $n \geqq 2$—we see first (1°) and then (2°).

In order to prove (3°), we need to consider Lemma 2 with $a \neq 0$. Let $n=3$. Recall by (1°) and (2°) that $C_3(\eta)=0$ in case $\eta_1 \leqq 2$. Thus (3.6) takes the form

$$0 = \sum_{j=1}^{3} C_3(\eta^j)\; e_{\eta^j}[p], \tag{3.7}$$

where $\eta^1 := (3,1,1,1,0,0)$, $\eta^2 := (3,2,1,0,0,0)$, $\eta^3 := (3,3,0,0,0,0)$, and thus $e_{\eta^1}[p] = p^{(3)}(p')^3/p$, $e_{\eta^2}[p] = p^{(3)}p''p'$, $e_{\eta^3}[p] = (p^{(3)})^2 p$. As before, we first get $C_3(\eta^1)=0$ from (3.7). Let us next observe that

$$e_{\eta^2}[p] \to 2p^{(3)}(0)p''(0) \text{ and } e_{\eta^3}[p] \to 0 \text{ as } v\uparrow 0.$$

Then (3.7) yields $C_3(\eta^2)=0$, if we choose a and b in such a way that $p^{(3)}(0)p''(0) \neq 0$, that is, $a(a-b)(2a-b) \neq 0$. Noting that $e_{\eta^3}[p] \neq 0$ as a function of v, we again use (3.7) and get $C_3(\eta^3)=0$. Therefore, we obtain (3°), and the proof of Proposition 2 is complete.

Proof of Lemma 2. Direct computation yields (i). Using the fact that a complete orthogonal system of L^2 holomorphic functions is given by monomials

$$z_1^j z_2^k \quad \text{with} \quad k+1-a(j+1)/2 > 0,$$

we can show without difficulty that

$$(2\pi)^2 K_{a,b}(z) = \frac{|z_2|^a}{b\rho_1^{2+2/b}}\left(\frac{8(1+|z_2|^2\rho_1^{-2/b})}{\rho^3} + \frac{4b}{\rho^2}\right), \tag{3.8}$$

where $\rho := 1-|z_2|^2\rho_1^{-2/b}$ with $\rho_1 := 1-|z_1|^2|z_2|^a$. We see that ρ is a smooth defining function of the portion $\partial\Omega\cap\{z_2\neq 0\}$, of which every point is strictly pseudoconvex. Thus (ii) follows from (3.8). If $a=0$, then (3.8) yields

$$(2\pi)^2|z_1z_2|^2 K_{a,b}(z)=\frac{p(v)}{4}\left(\frac{2\cosh\lambda}{\sinh^3\lambda}-\frac{b}{\sinh^2\lambda}\right),$$

which implies (iii).

4. Proof of Theorem 2

4.1. Reduction to the real domain $\log|\Omega|$ and some normalization. Since monomials form a complete orthogonal system for the Hilbert space of L^2 holomorphic functions in Ω, it follows that

$$|z_1z_2|^2K(z)=\sum_{\alpha\in N^2}|z^\alpha|^2\left\{\int_\Omega|\zeta^\alpha|^2dV_4(\zeta)\Big/|\zeta_1\zeta_2|^2\right\}^{-1},$$

where $dV_4(\zeta)$ denotes the volume element of C^2 identified with R^4. Making a change of integration variables $\zeta_j=\exp[\xi_j+\sqrt{-1}\,\theta_j]\mapsto(\xi_j,\theta_j)$ for $j=1,2$, we get

$$\int_\Omega|\zeta^\alpha|^2dV_4(\zeta)\Big/|\zeta_1\zeta_2|^2=(2\pi)^2\int_{\log|\Omega|}e^{2\alpha\cdot\xi}dV_2(\xi),$$

where $dV_2(\xi)$ is the volume element of R^2. Recalling that $\log|z_1|=x$ and $\log|z_2|=f(x)+\lambda$ for $x\in I$, we see that $(2\pi)^2|z_1z_2|^2K(z)$ is a function of (λ,x). Let us denote it by $L(\lambda,x)$ as in (1.1). Then

$$L(\lambda,x)=\sum_{\alpha\in N^2}\frac{2\alpha_2\exp[2\alpha_1x+2\alpha_2\{f(x)+\lambda\}]}{\int_I\exp[2\alpha_1\xi+2\alpha_2f(\xi)]\,d\xi},$$

where we have written ξ in place of ξ_1. We reduce each fraction by the factor $\exp[2\alpha_1x+2\alpha_2f(x)]$ and shift the integration variable ξ. Then we get

$$L(\lambda,x)=\sum_{\alpha\in N^2}\frac{2\alpha_2\exp[2\alpha_2\lambda]}{D(\alpha;x)}\quad\text{for }\lambda<0,\ x\in I,\tag{4.1}$$

where

$$(4.2)\qquad D(\alpha; x) := \int_{I-x} \exp[2\alpha_1\xi + 2\alpha_2\{f(x+\xi) - f(x)\}]\, d\xi$$

with $I - x = \{\xi \in \boldsymbol{R}; \xi + x \in I\}$.

Regarding $x \in I$ as a parameter, we shall investigate the behavior of $L(\lambda, x)$ as $\lambda \uparrow 0$. It is convenient to set

$$E_s(\tau; \lambda) := \tau^{-s}\exp[\lambda/\tau^2] \quad \text{with } \tau := (2\alpha_2)^{-1/2}$$

for $s \in \boldsymbol{Z}$, and write (4.1) as

$$(4.1)_j\qquad L(\lambda, x) = \sum_{\alpha \in \boldsymbol{N}^2} \frac{E_{2+j}(\tau; \lambda)}{D_j(\alpha; x)} \quad \text{for } j = 0, 1, 2,$$

where $D_j(\alpha; x) := \tau^{-j} D(\alpha; x)$. The desired asymptotic expansion will be obtained by using a localized version of $(4.1)_1$, and the proof of localization requires $(4.1)_2$.

Let us look at the dependence of $D(\alpha; x)$ on x more precisely. Recalling by the definition (1.4) that $f(x+\xi) - f(x) = f'(x)\xi + M_x(\xi)\xi^2$, we have

$$2\alpha_1\xi + 2\alpha_2\{f(x+\xi) - f(x)\} = \mu(\alpha; V_x)\xi + 2\alpha_2 M_x(\xi)\xi^2,$$

where $\mu(\alpha; V_x) := 2\alpha_1 + \alpha_2 V_x$ with $V_x := f'(x)$. Setting for $j = 0, 1, 2$

$$(4.3)\qquad \mu_j(\alpha; V_x) := \tau^j \mu(\alpha; V_x) \quad \text{with } \tau = (2\alpha_2)^{-1/2}$$

and $\tau^{-j}(I - x) := \{\xi \in \boldsymbol{R}; \tau^j\xi + x \in I\}$, we get

$$(4.2)_j\qquad D_j(\alpha; x) = \int_{\tau^{-j}(I-x)} \exp[\mu_j(\alpha; V_x)\xi + \tau^{2j-2} M_x(\tau^j\xi)\xi^2]\, d\xi.$$

4.2. Heuristic computation explaining Theorem 2. Before proceeding to the rigorous proof of Theorem 2, we give here two heuristic arguments. The first heuristics is very simple and is given as follows. Setting

$$(4.4)\qquad L_{\text{aux}}(\hat{\tau}) := \sum_{\alpha_1 = 1}^{\infty} \frac{1}{D_2(\alpha; x)} \quad \text{with } \hat{\tau} := 2\alpha_2,$$

we assume that the following Laurent expansion is valid:

$$L_{\text{aux}}(\hat{\tau}) \sim \sum_{n=0}^{\infty} L_n^{\text{aux}}(x)\ \hat{\tau}^{-n} =: L_{\text{aux}}^{\#}(\hat{\tau}) \quad \text{about } \hat{\tau} = \infty.$$

Then, one may expect that the asymptotic expansion of $L(\lambda, x)$ as $\lambda \uparrow 0$ is reduced to that of $L_{\mathrm{aux}}(2\alpha_2)$ as $\alpha_2\uparrow+\infty$. In fact, using $(4.1)_2$ and changing the order of summation, we are led to

$$L(\lambda, x)\sim\sum_{n=0}^{\infty}L_n^{\mathrm{aux}}(x)\left(\sum_{\alpha_2=1}^{\infty}(2\alpha_2)^{2-n}\exp[2\alpha_2\lambda]\right).$$

Noting that

$$\sum_{\alpha_2=1}^{\infty}(2\alpha_2)^{2-n}\exp[2\alpha_2\lambda]\equiv\partial_\lambda^{2-n}\frac{-1}{2\lambda}\quad\text{modulo } C^\infty \text{ at } \lambda=0,$$

we arrive at the asymptotic expansion as in (1.2):

$$L(\lambda, x)\sim L_{\mathrm{aux}}^{\#}(\partial_\lambda)\ \partial_\lambda^2\frac{-1}{2\lambda}\quad\text{as } \lambda\uparrow 0.$$

This explains Theorem 0, which is contained as a part of Theorem 2, except for the fact that the coefficients $L_n^{\mathrm{aux}}(x)$ depend only on $f^{(m+2)}(x)$ for $m\geqq 0$.

The second heuristics is concerned with the remaining part of Theorem 2. This will suggest the integral representation (1.6) of the symbol $L^{\#}[M_x^{\#}](\tau)=L_{\mathrm{aux}}^{\#}(1/\tau^2)$. Let us begin by recalling a theorem of Hörmander (0.1) in Introduction. This together with Fefferman's expansion (0.2) yields

$$(0.2)_0\qquad L(\lambda, x)=\frac{f''(x)}{2}\frac{1}{\lambda^3}+O\left(\frac{1}{\lambda^2}\right),\quad\text{that is, } L_0^{\mathrm{aux}}(x)=-\frac{f''(x)}{2}.$$

Starting from $(4.1)_1$ with $(4.2)_1$, we shall first reproduce $(0.2)_0$ above by giving an integral representation of $L_0^{\mathrm{aux}}(x)$, which can be regarded as an approximate form of Theorem 2. The idea is to regard the definition of $L_{\mathrm{aux}}(2\alpha_2)$ in (4.4) as a Riemann's approximate sum of an integral. Let us explain these more precisely. Setting

$$H_{\mathrm{aux}}(\mu, \tau):=\int_{\tau^{-1}(I-x)}\exp[\mu\xi+M_x(\tau\xi)\xi^2]\ d\xi\qquad\text{with } \tau=(2\alpha_2)^{-1/2},$$

we have $D_1(\alpha; x)=H_{\mathrm{aux}}(\mu_1, \tau)$ with $\mu_1:=\mu_1(\alpha; V_x)$, and thus

$$L_{\mathrm{aux}}(2\alpha_2)=\frac{1}{2}\sum_{\alpha_1=1}^{\infty}\frac{2\tau}{H_{\mathrm{aux}}(\mu_1, \tau)}.$$

Noting how $\mu_1=\mu_1(\alpha; V_x)$ varies with $\alpha_2\in\boldsymbol{N}$, we may regard the series above as a Riemann's approximate sum, and get

$$L_0^{\mathrm{aux}}(x)=\lim_{\alpha_2\uparrow+\infty} L_{\mathrm{aux}}(2\alpha_2)=\frac{1}{2}\int_{\boldsymbol{R}}\frac{d\mu}{H_{\mathrm{aux}}(\mu,+0)}.$$

This implies $(0.2)_0$, because

$$H_{\mathrm{aux}}(\mu,+0)=\int_{\boldsymbol{R}}\exp\left[\mu\xi+\frac{f''(x)}{2}\xi^2\right]d\xi,$$

and thus $\int_{\boldsymbol{R}} d\mu/H_{\mathrm{aux}}(\mu,+0)=-f''(x)$.

We have heuristically gotten the zeroth order approximation of $L_{\mathrm{aux}}(2\alpha_2)$. Higher order asymptotics of $L_{\mathrm{aux}}(2\alpha_2)$ will be obtained if we take into account of the remainder $M_x(\tau\xi)-f''(x)/2$. This idea will be realized by considering the formal power series expansion about $\tau=0$. We then reach the precise form of Theorem 2.

4.3. Localization. In this subsection, we basically follow Boichu-Coeuré [1, Lemme 1]. Let a point $x_0\in I$ be arbitrarily fixed, and thus $(x_0,f(x_0))\in\partial\log|\Omega|$. In (4.1) with (4.2), we replace f by a function $\tilde f\in C^\infty(\tilde I)$ satisfying

(4.5) $$\tilde f(x)=f(x)\quad\text{for } x\in\tilde I_0,\quad \tilde f''(x)<0\ \text{for } x\in\tilde I,$$

where $\tilde I$ and $\tilde I_0$ are open intervals such that $x_0\in\tilde I_0\subset\tilde I$. Then, localization about x_0 is possible. Furthermore, we may restrict the index set of summation from $\boldsymbol{N}^2$ to $\Lambda\subset\boldsymbol{N}^2$ satisfying

(4.6) $$\Lambda\supset\Lambda_\varepsilon:=\{\alpha\in\boldsymbol{N}^2;\,|\mu_{20}(\alpha)|<\varepsilon\}\quad\text{with some }\varepsilon>0,$$

where $\mu_{20}(\alpha)=\mu_2(\alpha;f'(x_0))=\alpha_1/\alpha_2+f'(x_0)$. In order to state these more precisely, we first set, corresponding to $(4.1)_j$,

$$L_\Lambda(\lambda,x)=\sum_{\alpha\in\Lambda}\frac{E_{2+j}(\tau;\lambda)}{D_j(\alpha;x)}\quad\text{with }\tau=(2\alpha_2)^{-1/2}\quad\text{for } j=0,1,2.$$

We then set

$$\tilde L_\Lambda(\lambda,x)=\sum_{\alpha\in\Lambda}\frac{E_{2+j}(\tau;\lambda)}{\tilde D_j(\alpha;x)}\quad\text{with }\tau=(2\alpha_2)^{-1/2}\quad\text{for } j=0,1,2,$$

where $\tilde D_j(\alpha;x)$ are defined as in $(4.2)_j$ with $\tilde f$ in place of f, and the variable

x is restricted the interval $\tilde{I}_0$. That is,

$$\tilde{D}_j(\alpha; x) = \int_{\tau^{-j}(\tilde{I}-x)} \exp[\mu_j(\alpha; V_x)\xi + \tau^{2j-2}\tilde{M}_x(\tau^j\xi)\xi^2]\, d\xi,$$

where

$$\tilde{M}_x(\xi) := \frac{\tilde{f}(x+\xi) - \tilde{f}(x) - \tilde{f}'(x)\xi}{\xi^2} = \frac{\tilde{f}(x+\xi) - f(x) - f'(x)\xi}{\xi^2} \tag{4.7}$$

for $x \in \tilde{I}_0$ and $\xi \in \tilde{I} - x$. Using the expression for $j=2$, we shall show in Subsection 5.1 below that:

Lemma 3 (Boichu-Coeuré [1, Lemme 1]). *If $\tilde{f} \in C^\infty(\tilde{I})$ and $\Lambda \subset \boldsymbol{N}^2$ satisfy* (4.5) *and* (4.6), *respectively, then $L(\lambda, x) \equiv \tilde{L}_\Lambda(\lambda, x)$ modulo C^∞ at* $(\lambda, x) = (0, x_0)$.

4.4. One-parameter family of domains. Let $\tilde{f} \in C^\infty(\tilde{I})$ satisfy (4.5). We assume that $\tilde{I} = \boldsymbol{R}$. Then the set $\{(x, y) \in \boldsymbol{R}^2;\ y < \tilde{f}(x)\}$ is the logarithmic real representation domain of an unbounded (and possibly incomplete) Reinhardt domain $\tilde{\Omega} \subset \boldsymbol{C}^2$. In this subsection, we construct a one-parameter family of Reinhardt domains $\tilde{\Omega}_x^t \subset \boldsymbol{C}^2$ $(0 \leqq t \leqq 1)$ for each $x \in \tilde{I}_0$ such that $\tilde{\Omega}_x^1 = \tilde{\Omega}$ and that $\tilde{\Omega}_x^0$ is a quadratic model which is locally biholomorphic to a ball. The logarithmic real representation domain of $\tilde{\Omega}_x^t$ takes the form

$$\{(x+\xi, y) \in \boldsymbol{R}^2; y < \tilde{f}_x^t(x+\xi)\} \qquad \text{with } \tilde{f}_x^t \in C^\infty(\boldsymbol{R}).$$

Such a family $\tilde{f}_x^t$ $(0 \leqq t \leqq 1)$ for $x \in \tilde{I}_0$ has been constructed by Boichu and Coeuré [1]. In order to get Theorem 2, we need a more careful construction as follows, because we have to show in particular that desired asymptotic expansion of $L(\lambda, x)$ depends only on $M_x^\sharp(\xi) \in \boldsymbol{R}[[\xi]]$.

Let us begin by observing that the assumption $\tilde{I} = \boldsymbol{R}$ enables us to simplify the expression of $\tilde{D}_j(\alpha; x)$—we only use the case $j=1$. That is, we have, corresponding to $(4.2)_1$,

$$\tilde{D}_1(\alpha; x) = H[\tilde{M}_x](\mu_1(\alpha; V_x), \tau) \quad \text{with } \tau = (2\alpha_2)^{-1/2} \tag{4.8}$$

for $x \in \tilde{I}_0$, where $\mu_1(\alpha; V_x)$ was defined by (4.3) and

$$H[\tilde{M}_x](\mu, \tau) := \int_{\boldsymbol{R}} \exp[\mu\xi + \tilde{M}_x(\tau\xi)\xi^2]\, d\xi \qquad \text{for } \mu, \tau \in \boldsymbol{R},$$

with $\tilde{M}_x$ given by (4.7). Consequently,

$$(4.9)\qquad \tilde{L}_\Lambda(\lambda, x)=\sum_{\alpha\in\Lambda}\frac{E_3(\tau;\lambda)}{H[\tilde{M}_x](\mu_1(\alpha; V_x),\tau)}\quad \text{with } \tau=(2\alpha_2)^{-1/2}.$$

We are now in a position to define a family $\tilde{f}_x^t$ together with the associated $\tilde{L}_\Lambda(\lambda, x)$. Given $x\in\tilde{I}_0$ and $0\leqq t\leqq 1$, we define $\tilde{f}_x^t\in C^\infty(\boldsymbol{R})$ by

$$\tilde{f}_x^t(x+\xi):=f(x)+f'(x)\xi+\tilde{M}_x(t\xi)\xi^2.$$

Then $\tilde{f}_x^1(x+\xi)=\tilde{f}(x+\xi)$ and $\tilde{f}_x^0(x+\xi)=f(x)+f'(x)\xi+f''(x)\xi^2/2$. Thus $\tilde{\Omega}_x^1=\tilde{\Omega}$ and $\tilde{\Omega}_x^0$ is locally biholomorphic to a ball. Note by the expressions of (1.4) that

$$\partial_\xi^2\tilde{f}_x^t(x+\xi)=\tilde{f}''(x+t\xi)\ \text{and}\ \tilde{M}_x^t(\xi)=\tilde{M}_x(t\xi),$$

where $\tilde{M}_x^t(\xi):=\{\tilde{f}_x^t(x+\xi)-\tilde{f}_x^t(x)-(\tilde{f}_x^t)'(x)\xi\}/\xi^2$. We now define $\tilde{L}_\Lambda^t(\lambda, x)$ and $\tilde{D}_1^t(\alpha; x)$ by (4.9) and (4.8) with $\tilde{M}_x^t$ in place of $\tilde{M}_x$, and thus $\tilde{L}_\Lambda^1(\lambda, x)=\tilde{L}_\Lambda(\lambda, x)$ and $\tilde{D}_1^1(\alpha; x)=\tilde{D}_1(\alpha; x)$. That is,

$$(4.9)^t\qquad \tilde{L}_\Lambda^t(\lambda, x)=\sum_{\alpha\in\Lambda}\frac{E_3(\tau;\lambda)}{H[\tilde{M}_x^t](\mu_1(\alpha; V_x),\tau)}=\sum_{\alpha\in\Lambda}\frac{E_3(\tau;\lambda)}{\tilde{D}_1^t(\alpha; x)}$$

with $\tau=(2\alpha_2)^{-1/2}$. Then, an asymptotic expansion of $L(\lambda, x)$ will be obtained as the Taylor expansion of $\tilde{L}_\Lambda^t(\lambda, x)$ about $t=0$ evaluated at $t=1$. We shall justify this in the next subsection with additional assumptions of $\tilde{f}\in C^\infty(\boldsymbol{R})$.

In [1], Boichu and Coeuré considered a family $\tilde{f}_x^t$ defined by

$$\tilde{f}_x^t(x+\xi)=f(x)+f'(x)\xi+f''(x)\xi^2/2+t(\tilde{M}_x(\xi)-f''(x)/2)\xi^2.$$

We have modified it in order to clarify the dependence of $\tilde{L}_\Lambda^t(\lambda, x)$ on $f^{(2+m)}(x)$ for $m\in\boldsymbol{N}_0$ and on $V_x=f'(x)$.

4.5. Asymptotic expansion via boundary variations. In order to get the asymptotic expansion via boundary variations, we require more conditions of $\tilde{f}$. In addition to (4.5), we assume $\tilde{I}=\boldsymbol{R}$ and that

$$(4.10)\quad \sup|\tilde{f}^{(k)}|<+\infty\quad \text{for } k\geqq 2,\quad C_-\leqq -\tilde{f}''(x)\leqq C_+\quad \text{for } x\in\boldsymbol{R}$$

with some constants $C_+>C_->0$. It is easily seen that the conditions (4.5) and (4.10) are realized by a function $\tilde{f}\in C^\infty(\boldsymbol{R})$ such that the support of the third order derivative $\tilde{f}^{(3)}$ is compact and that

$$|\tilde{f}''(x)-f''(x_0)|<\varepsilon\quad \text{for } x\in\boldsymbol{R},$$

where $\varepsilon>0$ is arbitrarily prescribed.

We are now in a position to state the asymptotic expansion via boundary variations.

Lemma 4 (cf. Boichu-Coeuré [1, Théorème 1]). *Assume that* $\tilde{f} \in C^{\infty}(\boldsymbol{R})$ *and* $\Lambda \subset \boldsymbol{N}^2$ *satisfy respectively* (4.10) *and*

$$(4.11) \qquad \Lambda \subset \Lambda_a^+ := \{\alpha \in \boldsymbol{N}^2; \alpha_1/\alpha_2 \leqq a\} \quad \textit{with some } 0 < a \in \boldsymbol{R}.$$

Let k, l, m be non-negative integers. Then:

(i) If $m + l < C_-/(C_+ - C_-)$, *then* $\partial_\lambda^k \partial_x^l \partial_t^m \tilde{L}_\Lambda^t(\lambda, x)$ *exists as a continuous function of* (λ, x, t) *in* $(-\infty, 0) \times \tilde{I}_0 \times [0, 1]$.

(ii) *In addition to the assumption of* (i), *if* $2k + l < m - 7$, *then*

$$\partial_\lambda^k \partial_x^l \left(\tilde{L}_\Lambda - \sum_{j=0}^{m-1} \frac{1}{j!} \partial_t^j \tilde{L}_\Lambda^t \bigg|_{t=0} \right) \quad \textit{for } (\lambda, x) \in (-\infty, 0) \times \tilde{I}_0$$

extends continuously to $\lambda \leqq 0$.

Roughly speaking, we have by Lemmas 3 and 4 that

$$(4.12) \qquad L(\lambda, x) \sim \sum_{m=0}^{\infty} \frac{1}{m!} \partial_t^m \tilde{L}_\Lambda^t(\lambda, x) \bigg|_{t=0}$$

up to $\lambda = 0$, but we have to be careful because of the conditions of Lemma 4. In fact, if k and l are large, then m must be also large, in which case $C_+ - C_-$ should be small. However, $\tilde{f}$ is chosen when C_- and C_+ are specified. In order to make $C_+ - C_-$ small, we must replace $\tilde{f}$ by a new one. Also, we have to shrink the neighborhood of x_0 on which $\tilde{f} = f$. Keeping these remarks in mind, one may say that (4.12) is valid after an obvious modification.

4.6. Asymptotic expansion in Theorem 2. In this subsection, we prove Theorem 2. It is necessary to extract the singularity from each term of the right side of the asymptotic expansion (4.12) via boundary variations. The conclusion is very simple; we shall have

$$(4.13) \qquad \partial_t^m \tilde{L}_\Lambda^t(\lambda, x)|_{t=0} \equiv \partial_\tau^m L^\#[M_x^\#](\tau)|_{\tau=0} \, \partial_\lambda^{2-m/2} \frac{-1}{2\lambda}$$

modulo C^∞ at $(\lambda, x) = (0, x_0)$, where $L^\#[M_x^\#](\tau) \in \boldsymbol{R}[[\tau^2]]$ is given by (1.6) in Theorem 2. This is because the homotopy parameter t has been introduced in Subsection 4.4 in such a way that (4.13) holds.

Recall by (1.9) that the right side of (4.13) is written as

$$(4.14)\qquad \left(\frac{m!}{2}\int_{\boldsymbol{R}} B_m(\mu)\exp\left[\frac{\mu^2}{2P_x}\right]d\mu\right)\partial_\lambda^{2-m/2}\frac{-1}{2\lambda}\quad \text{with } P_x=f''(x).$$

Corresponding to this, we first show that:

Lemma 5. *Under the assumption of* (i) *in Lemma* 4, *the left side of* (4.13) *for* $\lambda<0$, $x\in\tilde{I}_0$ *is given by*

$$(4.15)\qquad m!\sum_{\alpha\in A} E_{3-m}(\tau;\lambda)\ B_m(\mu_1)\exp\left[\frac{\mu_1^2}{2P_x}\right]$$

evaluated at $\tau=(2\alpha_2)^{-1/2}$ *and* $\mu_1=\mu_1(\alpha;V_x)$ *with* $V_x=f'(x)$.

We shall prove Lemma 5 at the end of Subsection 5.2.

Our next task is to compute (4.15) in Lemma 5 with monomials μ_1^k in place of the polynomials $B_m(\mu_1)$. We thus set

$$L_{m,k}^{\mathrm{sum}}(\lambda,x)=\sum_{\alpha\in A}E_{3-m}(\tau;\lambda)\ \mu_1^k\exp\left[\frac{\mu_1^2}{2P_x}\right],$$

$$L_{m,k}^{\mathrm{int}}(\lambda,x)=\left(\frac{1}{2}\int_{\boldsymbol{R}}\mu^k\exp\left[\frac{\mu^2}{2P_x}\right]d\mu\right)\ \partial_\lambda^{2-m/2}\frac{-1}{2\lambda},$$

for $m,k\in\boldsymbol{N}_0$ such that $m+k$ is even. Note that $L_{m,k}^{\mathrm{int}}(\lambda,x)$ depends only P_x, whereas $L_{m,k}^{\mathrm{sum}}(\lambda,x)$ depends on P_x and V_x. Recall by Lemma 0 that $B_m(\mu_1)$ is a linear combination of μ_1^k such that $m+k$ is even. In this case, if m is odd then k is odd and thus $L_{m,k}^{\mathrm{int}}(\lambda,x)=0$. Now (4.13) is established if we show that:

Proposition 3. $L_{m,k}^{\mathrm{sum}}\equiv L_{m,k}^{\mathrm{int}}$ *modulo* C^∞ *at* $(\lambda,x)=(0,x_0)$, *provided* $\Lambda=\Lambda_a^+$ *with some* $a\in\boldsymbol{N}$ *satisfying* $a>-V_{x_0}$, *where* Λ_a^+ *is given in* (4.11).

Postponing the proof of Proposition 3 to the next subsection, let us prove Theorem 2. It is only necessary to summarize the argument we have had in this section.

Proof of Theorem 2. For $x_0\in I$ and $l_0\in\boldsymbol{N}_0$ arbitrarily fixed, we shall show that there exists a neighborhood $I_0\subset I$ of x_0 such that the relation (1.12) is valid modulo $C^{l_0}(\{-\infty<\lambda\leqq 0, x\in I_0\})$. This statement implies Theorem 2 in a form stated precisely in Subsection 1.3. Setting $m_0=2l_0+8$, we choose two constants $C_\pm>0$ satisfying $C_-<-f''(x_0)<C_+$

and $m_0 + l_0 < C_-/(C_+ - C_-)$. We take $a \in \boldsymbol{N}$ satisfying $a > -f'(x_0)$ and set $\Lambda = \Lambda_a^+$. We then take $\tilde{f} \in C^\infty(\tilde{I})$ with $\tilde{I} = \boldsymbol{R}$ satisfying (4.5) and (4.10). It then follows from Lemmas 3 and 4 that

$$L(\lambda, x) \equiv \sum_{m=0}^{m_0-1} \frac{1}{m!} \partial_t^m \tilde{L}_\Lambda^t(\lambda, x)\Big|_{t=0}$$

modulo $C^{l_0}(\{-\infty < \lambda \leqq 0, x \in I_0\})$, where $I_0 \subset I$ is a neighborhood of x_0. Shrinking I_0 if necessary, we see by (4.13) that

$$L(\lambda, x) \equiv \sum_{m=0}^{m_0-1} \frac{1}{m!} \partial_\tau^m L^\# [M_x^\#](\tau)\Big|_{\tau=0} \partial_\lambda^{2-m/2} \frac{-1}{2\lambda}$$

modulo $C^{l_0}(\{-\infty < \lambda \leqq 0, x \in I_0\})$. Here, fractional integration does not appear by virtue of Lemma 0. We thus get the desired result.

4.7. Proof of Proposition 3. What we have to do is to replace the series $L_{m,k}^{\mathrm{sum}}(\lambda, x)$ by an integral $L_{m,k}^{\mathrm{int}}(\lambda, x)$ up to a C^∞ error. The case $m = k = 0$ was done by Boichu and Coeuré [1]. In order to extend their argument to the general case, we begin by stating a lemma, which is essentially due to Boichu and Coeuré.

Given $a \in \boldsymbol{N}$ and $b > 0$, we set

$$U_{a,b} := \{(\rho, V, P) \in \boldsymbol{R}^3;\ 0 \leqq \rho < b,\ -a < V < 0,\ P < 0\},$$

and consider an integral

$$J(\rho, V, P) := \int_{\boldsymbol{R}} S(\xi; \rho, V, P)\, d\xi \quad \textit{for } (\rho, V, P) \in U_{a,b}$$

with the integrand defined by

$$S(\xi; \rho, V, P) := \sum_{\alpha \in \Lambda_a^+} Z_1(\xi)^{\alpha_1} Z_2(\xi; \rho, V, P)^{\alpha_2},$$

where $Z_1(\xi) := \exp[4i\xi]$ and $Z_2(\xi; \rho, V, P) := \exp[-\rho + 4iV\xi + 4P\xi^2]$. Then,

Lemma 6. *If $b > 0$ is small enough, then*

$$J(\rho, V, P) \equiv \frac{\pi}{2\rho} \quad \text{modulo } C^\omega(U_{a,b}).$$

We shall prove Lemma 6 in Subsection 5.3.

With the aid of Lemma 6, it is fairly easy to prove Proposition 3 as follows.

The case $m=k=0$ (cf. Boichu and Coeuré [1]). We write

$$L_{0,0}^{\text{sum}}(\lambda, x)=\partial_\lambda^2 \sum_{\alpha\in\Lambda_a^+} \exp\left[\frac{\lambda}{\tau^2}\right]\left(\tau \exp\left[\frac{\mu_1^2}{2P_x}\right]\right)$$

with $\tau=(2\alpha_2)^{-1/2}$, $\mu_1=\mu_1(\alpha; V_x)$, and use an elementary equality

$$2B_0\int_{\mathbf{R}} \exp\left[2i\mu_0\xi+\frac{2P_x}{\tau^2}\xi^2\right]d\xi=\tau\exp\left[\frac{\mu_1^2}{2P_x}\right] \tag{4.16}$$

with $\mu_0=\mu_0(\alpha; V_x)$ where $B_0=\sqrt{-P_x/(2\pi)}$ as in (1.8). Then,

$$L_{0,0}^{\text{sum}}(\lambda, x)=2B_0\, \partial_\lambda^2\, J(-2\lambda, V_x, P_x),$$

so that Lemma 6 implies $L_{0,0}^{\text{sum}}\equiv L_{0,0}^{\text{int}}$ modulo C^∞ at $(\lambda, x)=(0, x_0)$, as required.

General case. Recalling that $m+k$ is even, we first write, as in the simplest case $m=k=0$,

$$L_{m,k}^{\text{sum}}(\lambda, x)=\partial_\lambda^{2-(m+k)/2} \sum_{\alpha\in\Lambda_a^+} \exp\left[\frac{\lambda}{\tau^2}\right]\left(\tau\,\mu_0^k \exp\left[\frac{\mu_1^2}{2P_x}\right]\right),$$

where ∂_λ^{-1} stands for integration over $(-\infty, \lambda)$. Noting next that $\mu_0^k=\partial_s^k \exp[\mu_0 s]|_{s=0}$, we get

$$L_{m,k}^{\text{sum}}(\lambda, x)=2B_0\partial_\lambda^{2-(m+k)/2}\, \partial_s^k \tilde{J}(s; \lambda, x)\Big|_{s=0},$$

where

$$\tilde{J}(s; \lambda, x):=\sum_{\alpha\in\Lambda_a^+} \exp\left[\frac{\lambda}{\tau^2}\right]\left(\frac{\tau}{2B_0}\exp\left[\mu_0 s+\frac{\mu_1^2}{2P_x}\right]\right).$$

By using an analogue of (4.16), we see that

$$\tilde{J}(s; \lambda, x)=J(-2\lambda+P_x s^2, V_x+P_x s, P_x),$$

so that Lemma 6 yields

$$L_{m,k}^{\text{sum}}(\lambda, x)\equiv B_0\partial_\lambda^{2-(m+k)/2}\, \partial_s^k \frac{\pi}{-2\lambda+P_x s^2}\Big|_{s=0}, \tag{4.17}$$

where integration ∂_λ^{-1} is restricted to an interval $(-\tilde{b}, \lambda)$ with $\tilde{b}$ satisfying

$0<\tilde{b}<b$. It is immediately seen from (4.17) that if k is odd then $L^{\text{sum}}_{m,k}(\lambda, x)\equiv 0$. If k and m are even, then the right side of (4.17) equals

$$\partial_\lambda^{2-m/2}\left(\partial_\lambda^{-k/2}\,\partial_s^k\frac{\pi B_0}{-2\lambda+P_x s^2}\bigg|_{s=0}\right)\equiv\left\{\pi B_0\frac{k!}{(k/2)!}\left(\frac{-P_x}{2}\right)^{k/2}\right\}\partial_\lambda^{2-m/2}\frac{-1}{2\lambda},$$

a fact which implies the desired result. Therefore, the proof of Proposition 3 is complete.

5. Proofs of Lemmas 3–6

5.1. Proof of Lemma 3. We shall show that

(I) $\tilde{L}_{N^2}-\tilde{L}_\Lambda\equiv 0$ provided $\Lambda\supset\Lambda_\varepsilon$ with some $\varepsilon>0$;

(II) $L_{\Lambda_\varepsilon}-\tilde{L}_{\Lambda_\varepsilon}\equiv 0$ provided $\varepsilon>0$ is small enough.

Recalling the assumptions (4.5) and (4.6), we see that the desired result is obtained by using (I) and (II) as follows:

$$L=L_{N^2}\equiv L_{\Lambda_\varepsilon}\equiv\tilde{L}_{\Lambda_\varepsilon}\equiv\tilde{L}_{N^2}\equiv\tilde{L}_\Lambda.$$

In order to prove (I) and (II), let us note that each term of the series defining $\tilde{L}_\Lambda$ is a monomial of the form $C_\alpha X^{2\alpha_1}\tilde{Y}^{2\alpha_2}$ with $X=e^x$ and $\tilde{Y}=e^{\tilde{f}(x)+\lambda}$. If $x\in\tilde{I}_0$, then $\tilde{f}(x)=f(x)$, so that the left sides of (I) and (II) are both power series of X and $Y=e^{f(x)+\lambda}$. We shall prove the convergence of these series in a neighborhood of $(X, Y)=(e^{x_0}, e^{f(x_0)})$. By Abel's lemma, it suffices to estimate uniformly

$$T_{\rm I}(\alpha):=\sup_{U_I}\frac{E_4(\tau;\lambda)}{\tilde{D}_2(\alpha;x)},$$

$$T_{\rm II}(\alpha):=\sup_{U_{\rm II}}E_4(\tau;\lambda)\frac{|D_2(\alpha;x)-\tilde{D}_2(\alpha;x)|}{D_2(\alpha;x)\,\tilde{D}_2(\alpha;x)},$$

where $U_{\rm I}$ and $U_{\rm II}$ are appropriate neighborhoods of $(\lambda, x)=(0, x_0)$. More precisely,

Sublemma. (i) *For any* $\varepsilon>0$, *there exists a neighborhood* $U_{\rm I}\subset\boldsymbol{R}^2$ *of* $(\lambda, x)=(0, x_0)$ *such that* $\sup\{T_{\rm I}(\alpha);\alpha\in\boldsymbol{N}^2\setminus\Lambda_\varepsilon\}<+\infty$.

(ii) *If* $\varepsilon>0$ *is small enough, then there exists a neighborhood* $U_{\rm II}\subset\boldsymbol{R}^2$ *of* $(\lambda, x)=(0, x_0)$ *such that* $\sup\{T_{\rm II}(\alpha);\ \alpha\in\Lambda_\varepsilon\}<+\infty$.

Proof of Sublemma. Setting $\tilde{R}_x(\xi):=[\tilde{f}(x+\xi)-f(x)]/\xi-f'(x_0)$ for $x\in\tilde{I}_0$, we have

$$(5.1)\qquad \tilde{D}_2(\alpha; x) = \int_{2\alpha_2(\tilde{I}-x)} \exp\,[\mu_{20}(\alpha)\xi + \tilde{R}_x(\tau^2\xi)\xi]\, d\xi.$$

For $\varepsilon_{\mathrm{I}}>0$, we choose δ_{I}, $\delta'_{\mathrm{I}}>0$ so small that $|\tilde{R}_x(\xi)| \leqq \varepsilon_{\mathrm{I}}$ for $|\xi| \leqq \delta_{\mathrm{I}}$ and $|x-x_0|<\delta'_{\mathrm{I}}$. Then

$$(5.2)\qquad \tilde{D}_2(\alpha; x) \geqq \int_0^{2\alpha_2\delta_{\mathrm{I}}} \exp\,[(|\mu_{20}(\alpha)| - \varepsilon_{\mathrm{I}})\xi]\, d\xi \qquad \text{for } |x-x_0|<\delta'_{\mathrm{I}}.$$

Given $\varepsilon>0$, we require $\varepsilon_{\mathrm{I}}<\varepsilon$ and set $\varepsilon'_{\mathrm{I}}=\varepsilon-\varepsilon_{\mathrm{I}}$. If $\alpha \notin \Lambda_\varepsilon$ then, by (5.2),

$$\frac{\exp[2\alpha_2\delta_{\mathrm{I}}\varepsilon'_{\mathrm{I}}]}{\tilde{D}_2(\alpha; x)} \leqq \frac{\varepsilon'_{\mathrm{I}}}{C(\alpha_2)} \leqq \frac{\varepsilon'_{\mathrm{I}}}{C(1)} \qquad \text{with} \qquad C(\alpha_2) := 1 - \exp[-2\alpha_2\delta_{\mathrm{I}}\varepsilon'_{\mathrm{I}}],$$

which implies (i). We note that (5.2) also yields

$$(5.3)\qquad \tilde{D}_2(\alpha; x) \geqq \int_0^{2\delta_{\mathrm{I}}} \exp[-\varepsilon_{\mathrm{I}}\xi]\; d\xi \qquad \text{for } |x-x_0|<\delta'_{\mathrm{I}}$$

without any restriction on $\alpha \in \boldsymbol{N}^2$.

Let us prove (ii). We first take δ_{II}, $\delta'_{\mathrm{II}}>0$ small satisfying $f(x+\xi)=\tilde{f}(x+\xi)$ for $|\xi|<\delta_{\mathrm{II}}$ and $|x-x_0|<\delta'_{\mathrm{II}}$. Then

$$|D_2(\alpha; x) - \tilde{D}_2(\alpha; x)| \leqq D_2^{\delta_{\mathrm{II}}}(\alpha; x) + \tilde{D}_2^{\delta_{\mathrm{II}}}(\alpha; x) \quad \text{for } |x-x_0|<\delta'_{\mathrm{II}},$$

where $\tilde{D}_2^{\delta_{\mathrm{II}}}(\alpha; x)$ is defined by the right side of (5.1) with the interval of integration $2\alpha_2\{(\tilde{I}-x)\setminus[-\delta_{\mathrm{II}}, \delta_{\mathrm{II}}]\}$ in place of $2\alpha_2(\tilde{I}-x)$. If δ'_{II} is sufficiently small, then there exists a constant $\varepsilon_{\mathrm{II}}>0$ such that $\tilde{R}_x(\xi)\xi \leqq -\varepsilon_{\mathrm{II}}|\xi|$ for $|\xi| \geqq \delta_{\mathrm{II}}$ and $|x-x_0|<\delta'_{\mathrm{II}}$, where the second assumption of (4.5) was used. Consequently,

$$\tilde{D}_2^{\delta_{\mathrm{II}}}(\alpha; x) \leqq 2\int_{2\alpha_2\delta_{\mathrm{II}}}^{+\infty} \exp[(|\mu_{20}(\alpha)| - \varepsilon_{\mathrm{II}})\xi]\, d\xi \quad \text{for } |x-x_0|<\delta'_{\mathrm{II}}.$$

If $\varepsilon>0$ is chosen so small that $\varepsilon<\varepsilon_{\mathrm{II}}$, we get (ii). In fact, the uniform boundedness of $1\Big/\Big(D_2(\alpha; x)\,\tilde{D}_2(\alpha; x)\Big)$ is clear from (5.3).

5.2. Proofs of Lemmas 4 and 5. Recall that $\tilde{L}^t_A = \Sigma_{\alpha\in A} E_3/\tilde{D}^t_1$ and that $E_3/\tilde{D}^t_1$ is C^∞ smooth for (λ, x, t) in $(-\infty, 0] \times \tilde{I}_0 \times [0,1]$. Setting $U = \tilde{I}_0 \times [0,1]$, we shall show that:

Sublemma (cf. Boichu-Coeuré [1, Lemme 2]). *If* $m+l<C_-/$

$(C_+ - C_-)$, *then there exists a constant* $C_{m,l} > 0$ *independent of* $\alpha \in \boldsymbol{N}^2$ *such that*

$$\sup_U |\partial_x^l \partial_t^m (1/\tilde{D}_1^t)| \leqq C_{m,l} \tau^{m-l} \quad \textit{with} \quad \tau = (2\alpha_2)^{-1/2}.$$

Assuming for a moment the validity of Sublemma above, we first prove (i) and (ii).

Let us begin by observing that Sublemma implies

$$\text{(5.4)} \qquad \sup_{x,t} |\partial_\lambda^k \partial_x^l \partial_t^m (E_3/\tilde{D}_1^t)| \leqq C_{m,l} E_{3-m+l+2k}.$$

We next note by the assumption (4.11) that

$$\text{(5.5)} \qquad \sum_{\alpha \in A} \tau^N \leqq \frac{a}{2} \sum_{\alpha_2 = 1}^{\infty} (2\alpha_2)^{1-N/2} < +\infty \quad \text{if } 4 < N \in \boldsymbol{R}.$$

Then (i) follows from (5.4) and (5.5), because the supremum of $E_{N'}(\tau; \lambda)$ over $\alpha_2 \in \boldsymbol{N}$ and $\lambda < -\varepsilon$ is bounded for any $\varepsilon > 0$ and $N' \in \boldsymbol{R}$ prescribed.

In order to prove (ii), we note that (i) permits us to use Taylor's formula which yields

$$\tilde{L}_A - \sum_{j=0}^{m-1} \frac{1}{j!} \partial_t^j \tilde{L}_A^t \bigg|_{t=0} = \frac{1}{(m-1)!} \int_0^1 (1-t)^{m-1} \partial_t^m \tilde{L}_A^t \, dt.$$

It then suffices to estimate $|\partial_\lambda^k \partial_x^l \partial_t^m (E_3/\tilde{D}_1^t)|$ uniformly up to $\lambda = 0$. Since $E_0(\tau; \lambda) \leqq 1$, it follows from (5.4) that

$$\sup_{\lambda < 0} \sup_{x,t} |\partial_\lambda^k \partial_x^l \partial_t^m (E_3/\tilde{D}_1^t)| \leqq C_{m,l} \tau^{-3+m-l-2k}.$$

Noting that $-3+m-l-2k > 4$ is guaranteed by assumption, we see that (5.5) yields the desired estimate.

Therefore, the proof of Lemma 4 is complete if we justify Sublemma.

Proof of Sublemma. For $b \in \boldsymbol{N}_0$, $\mu \in \boldsymbol{R}$ and $\tau = (2\alpha_2)^{-1/2}$ with $\alpha_2 \in \boldsymbol{N}$, we set

$$H_b[\tilde{M}_x](\mu, \tau) := \int_{\boldsymbol{R}} |\xi|^b \exp[\mu\xi + \tilde{M}_x(\tau\xi)\xi^2] \, d\xi,$$

and thus $H_0[\tilde{M}_x] = H[\tilde{M}_x]$. Recalling that $\tilde{D}_1^t(\alpha; x) = H[\tilde{M}_x](\mu_1, t\tau)$ with $\mu_1 = \mu_1(\alpha; f'(x))$, we see by (4.10) that $|\partial_x^l \partial_t^m \tilde{D}_1^t|$ is dominated by a constant multiple of $\tau^{m-l} \max H_b[\tilde{M}_x](\mu_1, t\tau)$, where the maximum is taken over $b \leqq 3m+2l$. Given $n \in \boldsymbol{N}$ and $\beta = (\beta_1, \cdots, \beta_n) \in \boldsymbol{N}_0^n$, we set

$$H_{\beta,n}[\tilde{M}_x](\mu,\tau):=H[\tilde{M}_x](\mu,\tau)^{-n-1}\prod_{j=1}^{n}H_{\beta_j}[\tilde{M}_x](\mu,\tau).$$

Then $|\partial_x^l\partial_t^m(1/\tilde{D}_1^t)|$ is dominated by a constant multiple of $\tau^{m-l}\max H_{\beta,n}[\tilde{M}_x](\mu_1,t\tau)$, where the maximum is taken over $n=m+l$ and $|\beta|\leqq 3m+2l$. Therefore, it suffices to show that

$$\sup\{H_{\beta,n}[\tilde{M}_x](\mu,\sigma);\mu\in\boldsymbol{R},\sigma\geqq 0,x\in\tilde{I}_0\}<+\infty \tag{5.6}$$

with $n\in\boldsymbol{N}$, $\beta\in\boldsymbol{N}_0^n$ fixed.

Let us prove (5.6). Noting by (4.10) that $C_-\leqq -2\tilde{M}_x(\xi)\leqq C_+$, we have

$$H[\tilde{M}_x](\mu,\sigma)\geqq\sqrt{\frac{2\pi}{C_+}}\exp\left[\frac{\mu^2}{2C_+}\right],\quad H_b[\tilde{M}_x](\mu,\sigma)\leqq P_b(\mu)\exp\left[\frac{\mu^2}{2C_-}\right],$$

where $P_b(\mu):=\int_{\boldsymbol{R}}|\xi+\mu/C_-|^b\exp[-C_-\xi^2/2]\,d\xi$. Setting

$$P_{\beta,n}(\mu):=\left(\frac{C_+}{2\pi}\right)^{(n+1)/2}\prod_{j=1}^{n}P_{\beta_j}(\mu),$$

we have $H_{\beta,n}[\tilde{M}_x](\mu,\sigma)\leqq P_{\beta,n}(\mu)\exp[C\mu^2]$, where

$$C:=\frac{n}{2C_-}-\frac{n+1}{2C_+}=\frac{n(C_+-C_-)-C_-}{2C_-C_+}.$$

Note by assumption that $C<0$. Since $P_{\beta,n}(\mu)$ increases at most in polynomial order as $|\mu|\to\infty$, it follows that

$$\sup_{\mu\in\boldsymbol{R}}P_{\beta,n}(\mu)\exp[C\mu^2]<+\infty,$$

which implies (5.6) as desired. Therefore, the proof of Sublemma is finished.

Proof of Lemma 5. We have observed that the derivative $\partial_t^m\tilde{L}_\Lambda^t(\lambda,x)|_{t=0}$ can be computed by termwise differentiation in $(4.9)^t$. Let us compare the result with $\partial_\tau^m L^\#[M_x^\#](\tau)|_{\tau=0}$. Then, in view of (4.14) and (4.15), we see that the conclusion follows from the obvious equalities

$$\partial_t^m H[\tilde{M}_x^t](\mu,\tau)|_{t=0}=\tau^m(\partial_\tau^m H^\#[M_x^\#](\mu,\tau)|_{\tau=0}).$$

5.3. Proof of Lemma 6. We follow the argument of Boichu and Coeuré [1, Lemme 4 and Proposition].

Regarding V and P as parameters, we write $J(\rho)=J(\rho, V, P)$ and $S(\xi,\rho)=S(\xi;\rho,V,P)$. Then,

$$S(\xi,\rho)=\frac{1-Z_1^a}{1-Z_1}\frac{Z_1Z_2}{(1-Z_2)(1-Z_1^aZ_2)},$$

which is a meromorphic function of $\xi \in \boldsymbol{C}$ having no poles on the real line $\boldsymbol{R}$, where $Z_1=Z_1(\xi)$ and $Z_2=Z_2(\xi,\rho)=Z_2(\xi,\rho,V,P)$. Observe that $S(\xi,\rho)$ has exactly two poles $\xi=i\eta_\pm$ which approach to the real axis as $\rho\downarrow 0$. More precisely, $i\eta_+$ and $i\eta_-$ come from the zeros of $1-Z_2$ and $1-Z_1^aZ_2$, respectively, and

$$0\lessgtr\eta_\pm=\frac{V_\pm}{2P}\pm\sqrt{\frac{V_\pm^2}{4P^2}-\frac{\rho}{4P}}\to 0 \quad \text{as } \rho\downarrow 0,$$

where $V_+=-V>0$ and $V_-=-V-a<0$. Let us take a contour Γ on the upper half-plane $\{\operatorname{Im}\xi>0\}$ in such a way that

$$J(\rho)=2\pi i \operatorname*{Res}_{\xi=i\eta_+} S(\xi,\rho)+J_+(\rho) \quad \text{with} \quad J_+(\rho):=\int_\Gamma S(\xi,\rho)\,d\xi$$

and that $J_+(\rho)$ remains analytic when $\rho\downarrow 0$ as a function of (ρ, V, P), where $\operatorname{Res}_{\xi=i\eta_+}S(\xi,\rho)$ stands for the residue of $S(\xi,\rho)$ at $\xi=i\eta_+$. It was shown by Boichu and Coeuré [1], pp. 142–143, that

$$\operatorname*{Res}_{\xi=i\eta_+} S(\xi,\rho)=\operatorname*{Res}_{\xi=i\eta_+}\frac{Z_1}{(1-Z_1)(1-Z_2)}\equiv\frac{1}{4i\rho}, \tag{5.7}$$

which implies the desired result. Thus the proof of Lemma 6 is complete.

We would like to add that the residue calculus (5.7) can be replaced by simpler calculation as follows. Let us begin by observing that

$$J(\rho)\equiv J^\delta(\rho)=\int_{-\delta}^{\delta} S(\xi,\rho)d\xi,$$

where $\delta>0$ is arbitrarily fixed. We next set

$$S_1(\xi,\rho):=\frac{Z_1}{1-Z_1}\frac{Z_2}{1-Z_2},\quad S_2(\xi,\rho):=\frac{Z_1}{1-Z_1}\frac{Z_1^aZ_2}{1-Z_1^aZ_2},$$

so that $S(\xi,\rho)=S_1(\xi,\rho)-S_2(\xi,\rho)$. If $\delta<\pi/2$, then the origin $\xi=0$ is the only pole of $S_1(\xi,\rho)$ and $S_2(\xi,\rho)$ on the interval $-\delta<\xi<\delta$—this simple pole comes from the zero of $1-Z_1$. We then write $J^\delta(\rho)=J_1^\delta(\rho)-J_2^\delta(\rho)$, where

$$J_j^\delta(\rho):=\mathrm{PV}\int_{-\delta}^{\delta} S_j(\xi,\rho)\,d\xi \quad \text{for} \quad j=1,\ 2.$$

Let us first consider $J_1^\delta(\rho)$. Note that $\xi=i\eta_+$ is the unique pole of $S_1(\xi,\rho)$ which approaches to the real axis as $\rho\downarrow 0$. Therefore, by considering a contour on the lower half-plane $\{\mathrm{Im}\ \xi<0\}$, we see that

$$J_1^\delta(\rho)\equiv -\pi i \operatorname*{Res}_{\xi=0} S_1(\xi,\rho)=\frac{\pi}{4}\frac{e^{-\rho}}{1-e^{-\rho}}\equiv\frac{\pi}{4\rho}.$$

Similarly for $J_2^\delta(\rho)$. That is, considering the pole $\xi=i\eta_-$ of $S_2(\xi,\rho)$, we get

$$J_2^\delta(\rho)\equiv \pi i \operatorname*{Res}_{\xi=0} S_2(\xi,\rho)=\frac{-\pi}{4\rho}.$$

We thus obtain $J^\delta(\rho)\equiv\pi/(2\rho)$ as desired.

References

[1] D. Boichu and G. Coeuré: *Sur le noyau de Bergman des domaines de Reinhardt*, Invent. Math. **72** (1983), 131–152.

[2] C. Fefferman: *The Bergman kernel and biholomorphic mappings of pseudoconvex domains*, Invent. Math. **26** (1974), 1–65.

[3] C. Fefferman: *Parabolic invariant theory in complex analysis*, Adv. in Math. **31** (1979), 131–262.

[4] C. R. Graham: *Scalar boundary invariants and the Bergman kernel*, in "Complex Analysis II" (C.A. Berenstein, ed.), Lect. Notes in Math. 1276, pp.108–135, Springer, 1987.

[5] L. Hörmander: *L^2 estimates and existence theorems for the $\bar{\partial}$ operator*, Acta Math. **113** (1965), 89–152.

Mathematical Institute
Faculty of Science
Tôhoku University
Sendai 980, Japan