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LEFT CELLS IN THE AFFINE WEYL GROUP $W_a(\tilde{D}_4)$

Dedicated to Professor R.W. Carter on his sixtieth birthday

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The cells of affine Weyl groups have been studied for more than one decade. They have been described explicitly in cases of type $\tilde{A}_n (n \geq 1)$ [13], [9] and of rank ≤ 3 [1], [4], [10]. But there are only some partial results for an arbitrary irreducible affine Weyl group [2], [7], [8], [16], [17]. In [18], we constructed an algorithm to find a representative set of left cells of a certain crystallographic group W in a given two-sided cell. This provides us a practicable way to describe the cell of more groups. In the present paper, we shall apply it to the case when W is the affine Weyl group $W_a(\tilde{D}_4)$ (or denoted by W_a for brevity) of type $\tilde{D}_4$. We shall give an explicit description for all the left cells of W_a by finding a representative set of left cells of W_a. Before this paper, Du Jie gave an explicit description for all the two-sided cells of W_a, but he was unable to find the left cells of this group [5]. Chen Chengdong recently described all the left cells of W_a in terms of certain special reduced expressions of elements [3]. Comparing with their results, our description on the cells of W_a is neater and easier expressable in nature. Moreover, by doing the above work, we develop some technical skill in performing the mentioned algorithm In particular, we could avoid any computation of non-trivial Kazhdan-Lusztig polynomials throughout this work.

The content of the present paper is organized as below. Section 1 is the preliminaries. Some basic concepts and results concerning our algorithm are stated there. In section 2, we introduce the alcove forms of elements of W_a and also state some properties of elements of W_a in terms of alcove forms, which are quite useful in the subsequent sections Then in sections 3–5, we apply our algorithm to find a representative set Σ of left cells of W_a. Finally, in section 6, we describe all the left cells of W_a by making use of the set Σ.

1. Preliminaries

1.1 Let $W=(W, S)$ be a Coxeter group with S its Coxeter generator set. Let

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$\leq$ be the Bruhat order on W. For $w\in W$, we denote by $l(w)$ the length of w. Let $A=\boldsymbol{Z}[u]$ be the ring of polynomials in an indeterminate u with integer coefficients. For each ordered pair $y, w\in W$, there exists a unique polynomial $P_{y,w}\in A$, called a Kazhdan-Lusztig polynomial, which satisfies the conditions: $P_{y,w}=0$ if $y\nleq w$, $P_{w,w}=1$, and $\deg P_{y,w}\leq(1/2)(l(w)-l(y)-1)$ if $y<w$. These polynomials satisfy the following recurrence formula. Let $y, w\in W$ and assume $sw<w$ for some $s\in W$. Then we have

$$(1.1.1)\qquad P_{y,w}=u^{c}P_{sy,sw}+u^{1-c}P_{y,sw}-\sum_{\substack{y\leq z<sw\\ sz<z}}\mu(z, sw)u^{(1/2)(l(w)-l(z))}P_{y,z}$$

where $\mu(z, sw)$ denotes the coefficient of $u^{(1/2)(l(sw)-l(z)-1)}$ in $P_{z,sw}$; $c=1$ if $sy>y$ and $c=0$ if $sy<y$ (see [6]). We denote $y-w$ if either $\deg P_{y,w}$ or $\deg P_{w,y}$ reaches $(1/2)(|l(w)-l(y)|-1)$.

From formula (1.1.1), we see that checking the relation $y-w$ for $y, w\in W$ usually involves very complicated computation of Kazhdan-Lusztig polynomials. But the following fact is simple and useful: if $x, y\in W$ satisfy $y<x$ and $l(y)=l(x)-1$, then we have $y-x$. Another result concerning this relation will be stated in Proposition 1.14.

1.2 The preorders $\underset{L}{\leq}$, $\underset{R}{\leq}$, $\underset{LR}{\leq}$ on W and the associated equivalence relations $\underset{L}{\sim}$, $\underset{R}{\sim}$, $\underset{LR}{\sim}$ on W are defined in [6]. The equivalence classes for $\underset{L}{\sim}$ (resp. $\underset{R}{\sim}$, $\underset{LR}{\sim}$) on W are called left cells (resp. right cells, two-sided cells).

1.3 Now we take $W=W_a$ to be an irreducible affine Weyl group. Lusztig defined a function $a\colon W_a\to\boldsymbol{N}$ which satisfies the following properties:

(1) $a(z)\leq|\Phi|/2$, for any $z\in W_a$, where Φ is the root system determined by W_a;

(2) $x\underset{LR}{\leq}y\Rightarrow a(x)\geq a(y)$. In particular, $x\underset{LR}{\sim}y\Rightarrow a(x)=a(y)$. So we may define the a-value $a(\Gamma)$ on a (left, right or two-sided) cell Γ of W_a by $a(x)$ for any $x\in\Gamma$.

(3) $a(x)=a(y)$ and $x\underset{L}{\leq}y$ (resp. $x\underset{R}{\leq}y$)$\Rightarrow x\underset{L}{\sim}y$ (resp. $x\underset{R}{\sim}y$).

(4) Let $\delta(z)=\deg P_{e,z}$ for $z\in W_a$, where e is the identity of the group W_a. Then the inequality

$$(1.3.1)\qquad l(z)-2\delta(z)-a(z)\geq 0$$

holds for any $z\in W_a$. The set

$$(1.3.2)\qquad \mathscr{D}=\{w\in W_a\,|\,l(w)-2\delta(w)-a(w)=0\}$$

is a finite set of involutions. Each left (resp. right) cells of W_a contains a unique element of $\mathscr{D}$ [11].

(5) For any proper subset I of S, let w_I be the longest element in the subgroup

W_I generated by I. Then $w_I \in \mathscr{D}$ and $a(w_I)=l(w_I)$.

The above properties of function a were shown by Lusztig in his paper [10], [11]. Now we state two more properties of this function which are simple consequences of properties (2), (3) and (5).

Let $W_{(i)}=\{w\in W_a \mid a(w)=i\}$ for any non-negative integer i. Then by (2), $W_{(i)}$ is a union of some two-sided cells of W_a.

To each element $x\in W_a$, we associate two subsets of S as below.

(1.3.3) $\quad \mathscr{L}(x) = \{s\in S \mid sx<x\}$ and $\mathscr{R}(x) = \{s\in S \mid xs<x\}$.

(6) If $W_{(i)}$ contains an element of the form w_I for some $I\subset S$, then $\{w\in W_{(i)} \mid \mathscr{R}(w)=I\}$ forms a single left cell of W_a.

(7) Let $x=yz$ with $l(x)=l(y)+l(z)$ for $x, y, z\in W_a$. Then $x\underset{L}{\leq} z$, $x\underset{R}{\leq} y$ and hence $a(x)\geq a(y), a(z)$. In particular, if $I=\mathscr{R}(x)$ (resp. $I=\mathscr{L}(x)$), then $a(x)\geq l(w_I)$.

1.4 Let G be the connected reductive algebraic group over $\boldsymbol{C}$ whose type is dual to the type of Φ (see 1.3(1)). Then the following result is due to Lusztig [12].

Theorem. *There exists a bijection $\boldsymbol{u}\mapsto \boldsymbol{c}(\boldsymbol{u})$ from the set of unipotent conjugacy classes in G to the set of two-sided cells in W_a. This bjiection satisfies the equation $a(\boldsymbol{c}(\boldsymbol{u}))=\dim \mathscr{B}_u$, where u is any element in $\boldsymbol{u}$, and $\dim \mathscr{B}_u$ is the dimension of the variety of Borel subgroups of G containing u.*

1.5 To each element $x\in W_a$, we associate a set $\Sigma(x)$ of all left cells Γ of W_a satisfying the condition that there is some element $y\in\Gamma$ with $y-x$, $\mathscr{R}(y) \not\subseteq \mathscr{R}(x)$ and $a(y)=a(x)$

Then the following result is known.

Theorem ([18]). *If $x\underset{L}{\sim}y$ in W_a, then $\mathscr{R}(x)=\mathscr{R}(y)$ and $\Sigma(x)=\Sigma(y)$.*

1.6 A subset $K\subset W_a$ is called a representative set of left cells of W_a (resp. of W_a in a two-sided cell Ω), if $|K\cap\Gamma|=1$ for any left cell Γ of W_a (resp. of W_a in Ω), where the notation $|X|$ stands for the cardinality of the set X.

The main purpose of the present paper is to describe the left cells of the affine Weyl group W_a of type $\tilde{D}_4$ by finding a representative set of left cells of W_a. By 1.3(4), we know that the set $\mathscr{D}$ forms such a set. But finding the set $\mathscr{D}$ should involve very complicated computation of Kazhdan-Lusztig polynomials. Thus instead, the author formulated an algorithm to find a representative set of left cells of a certain crystallographic group in a given two-sided cell (see [18]). We shall state the algorithm in the case of W_a right now.

The algorithm is based on the following result which is a consequence of Theorem 1.5.

Theorem ([18]). *Let Ω be a two-sided cell of W_a. Assume that a non-empty*

subset $M\subset\Omega$ satisfies the following conditions.
(1) $x \underset{L}{\not\sim} y$ *for any* $x\neq y$ *in* M*;*
(2) *If for a given element* $y\in W_a$*, there is some element* $x\in M$ *satisfying conditions* $y-x$, $\mathscr{R}(y)\not\subseteq\mathscr{R}(x)$ *and* $a(y)=a(x)$*, then there is some* $z\in M$ *with* $y\underset{L}{\sim}z$*.*
Then M *is a representative set of left cells of* W_a *in* Ω*.*

1.7 To each element $x\in W_a$, we define a set $M(x)$ of all elements y for each of which there is a sequence of elements $x_0=x, x_1, \cdots, x_r=y$ in W_a with some $r\geq 0$, where for every i, $1\leq i\leq r$, the conditions $x_{i-1}^{-1}x_i\in S$ and $\mathscr{R}(x_{i-1})\underset{\not\subseteq}{\not\supseteq}\mathscr{R}(x_i)$ are satisfied.

The following result is well-known.

Proposition ([18]). *Given* $x, x'\in W_a$*. If there are elements* $y, z\in M(x)$ *and* $y', z'\in M(x')$ *such that* $y-y'$, $z-z'$, $\mathscr{R}(y)\not\subseteq\mathscr{R}(y')$ *and* $\mathscr{R}(z')\not\subseteq\mathscr{R}(z)$*, then* $x\underset{R}{\sim}x'$*. In particular, we have* $a(x)=a(x')$*.*

1.8 A subset $P\subset W_a$ is said to be distinguished if $P\neq\emptyset$ and $x\underset{L}{\not\sim}y$ for any $x\neq y$ in P.

Given a subset P of W_a. The following are two processes on P.
(A) Find a largest possible subset Q from the set $\bigcup_{x\in P} M(x)$ with Q distinguished.
(B) For each $x\in P$, find elements $y\in W_a$ such that $y-x$, $\mathscr{R}(y)\supsetneqq\mathscr{R}(x)$ and $a(y)=a(x)$, add these elements y on the set P to form a set P' and then take a largest possible subset Q from P' with Q distinguished.
1.9 A subset P of W_a is called **A**-saturated (resp. **B**-saturated) if Process (**A**) (resp. Process (**B**)) can't produce any element z satisfying $z\underset{L}{\not\sim}x$ for all $x\in P$.

Clearly, a set of the form $\bigcup_{x\in K} M(x)$ for any $K\subset W_a$ is always **A**-saturated.

It follows from Theorem 1.6 that a representative set of left cells of W_a in a two-sided cell Ω is exactly a distinguished subset of Ω which is both **A**- and **B**-saturated. So to get such a set, we may use the following
1.10 ALGORITHM ([18]) (1) Find a non-empty subset P of Ω (Usually we take P to be distinguished for avoiding unnecessary complication if possible); (2) Perform Processes (**A**) and (**B**) alternately on P until the resulting distinguished set can't be further enlarged by both processes.
1.11 We define a graph $\mathscr{M}(x)$ associated to each $x\in W_a$ as follows. Its vertex set is $M(x)$. Its edge set consists of all two-elements subsets $\{y, z\}\subset M(x)$ with $y^{-1}z\in S$ and $\mathscr{R}(y)\underset{\not\subseteq}{\not\supseteq}\mathscr{R}(z)$, To each vertex $y\in M(x)$, we are given a subset $\mathscr{R}(y)$ of S. To each edge $\{y, z\}$ of $\mathscr{M}(x)$, we are given an element $s\in S$ with $s=y^{-1}z$.
1.12 Two graphs $\mathscr{M}(x)$ and $\mathscr{M}(x')$ are called quasi-isomorphic if there exists

a bijection ϕ from the set $M(x)$ to the set $M(x')$ satisfying the following conditions.

(1) $\mathcal{R}(w)=\mathcal{R}(\phi(w))$ for $w\in M(x)$.

(2) For $y, z\in M(x)$, $\{y, z\}$ is an edge of $\mathcal{M}(x)$ if and only if $\{\phi(y), \phi(z)\}$ is an edge of $\mathcal{M}(x')$.

1.13 By a path in graph $\mathcal{M}(x)$, we mean a sequence of vertices $z_0, z_1, \cdots, z_t$ in $M(x)$ such that $\{z_{i-1}, z_i\}$ is an edge of $\mathcal{M}(x)$ for any i, $1\leq i\leq t$. Two elements $x, x'\in W_a$ are said to have the same generalized τ-invariant if for any path $z_0=x, z_1, \cdots, z_t$ in graph $\mathcal{M}(x)$, there is a path $z_0'=x', z_1', \cdots, z_t'$ in $\mathcal{M}(x')$ with $\mathcal{R}(z_i')=\mathcal{R}(z_i)$ for every i, $0\leq i\leq t$, and if the same condition holds when interchanging the roles of x with x'.

The following result is known.

Proposition ([18]). *The elements in the same left cell of W_a have the same generalized τ-invariant.*

1.14 Suppose that the product st of two generators $s, t\in S$ has order 3. We call an ordered pair of the form (ys, yst) or (yt, yts) an $\{s, t\}$-string if $y\in W_a$ satisfies $\mathcal{R}(y)\cap\{s, t\}=\emptyset$.

Now we are given two $\{s, t\}$-strings (x_1, x_2) and (y_1, y_2). Then we have the following known result.

Proposition ([18]). (1) $x_1 — y_1 \Leftrightarrow x_2 — y_2$;

(2) $x_1 — y_2 \Leftrightarrow x_2 — y_1$;

(3) $x_1 \underset{L}{\sim} y_1 \Leftrightarrow x_2 \underset{L}{\sim} y_2$;

(4) $x_1 \underset{L}{\sim} y_2 \Leftrightarrow x_2 \underset{L}{\sim} y_1$.

1.15 Say a set Σ of left cells of W_a to be represented by a set M of elements of W_a if Σ is the set of all left cells Γ of W_a with $\Gamma\cap M\neq\emptyset$.

As an easy consequence of Theorem 1.5, we have

Proposition. *If $x\underset{L}{\sim}y$ in W_a, then $M(x)$ and $M(y)$ represent the same set of left cells of W_a.*

1.16 We state some results of a Coxeter group (W, S) which will be useful in performing Processes (**A**) and (**B**) on a set.

(1) If $x, y\in W$ satisfy $x — y$ and $\mathcal{R}(x)\underset{\not\subseteq}{\not\supseteq}\mathcal{R}(y)$, then $x^{-1}y\in S$. More precisely, we have $x^{-1}y\in\mathcal{R}(x)\vee\mathcal{R}(y)$, where the notation $X\vee Y$ stands for the symmetric difference of two sets X and Y.

(2) If $x, y\in W$ satisfy $y — x$, $\mathcal{R}(y)\supsetneqq\mathcal{R}(x)$ and $a(x)=a(y)$, then we have either $y^{-1}x\in S$ or $y<x$ with $l(x)-l(y)$ odd, and we also have $\mathcal{L}(y)=\mathcal{L}(x)$.

The following known result are concerning the Bruhat order on elements of (W, S).

(a) Let $y \leq w$ in W. Then for any reduced form $w = s_1 s_2 \cdots s_r$ with $s_i \in S$, there is a subsequence $i_1, i_2, \cdots, i_t$ of $1, 2, \cdots, r$ such that $y = s_{i_1} s_{i_2} \cdots s_{i_t}$ is a reduced expression of y.

(b) Suppose $J = \mathcal{L}(w)$ (resp. $J = \mathcal{R}(w)$) for $w \in W$. Then there is some $x \in W$ with $w = w_J \cdot x$ (resp. $w = x \cdot w_J$) and $l(w) = l(w_J) + l(x)$.

Now let $w \in W$ with $J = \mathcal{L}(w)$. By (b), we can find a reduced expression

$$w = s_1 s_2 \cdots s_r, \qquad s_i \in S$$

with $w_J = s_1 s_2 \cdots s_t$, where $t = l(w_J)$. Denote $w_j = s_1 s_2 \cdots s_j$ for $t \leq j \leq r$. Let P_j be the set of all elements y with $y \leq w_j$ and $\mathcal{L}(y) \supseteq J$. Then $P_t = \{w_J\}$. Suppose that the set P_k has been found for $t \leq k < r$. Then by (a), we have

$$P_{k+1} = P_k \cup \{x s_{k+1} \mid x \in P_k, \quad s_{k+1} \notin \mathcal{R}(x)\}.$$

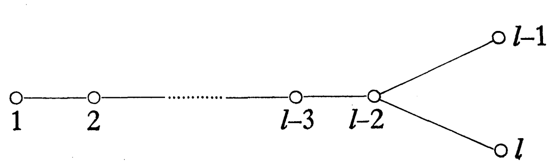
This provides an inductive procedure to find all the elements y with $y \leq w$ and $\mathcal{L}(y) \supseteq \mathcal{L}(w)$ for any given $w \in W$.

2. Aclove forms of elements of $W_a(\tilde{D}_l)$.

Although any element of W_a can be expressed as a product of generators in S, there are some disadvantages for such an expression in practical usage. For example, it is not easy to tell whether such an expression is reduced or not, and it is also difficult to determine the sets $\mathcal{L}(w)$ and $\mathcal{R}(w)$ directly from such an expression of an element $w \in W_a$. In the present section, we shall introduce the alcove forms of elements of W_a by which one can overcome the above obscurities.

2.1 Let E be the euclidean space spanned by the root system Φ of type D_l, $l \geq 4$. Let $\langle\, ,\, \rangle$ be an inner product in E. The affine Weyl group W_a of type $\tilde{D}_l$ can be regarded as a group of right isometric transformations on E. More precisely, let W be the Weyl group of Φ generated by the reflections s_α on E for $\alpha \in \Phi$: s_α sends $x \in E$ to $x - \langle x, \alpha^\vee \rangle \alpha$, where $\alpha^\vee = 2\alpha / \langle \alpha, \alpha \rangle$. We denote by N the group of all translations T_λ on E: T_λ sends x to $x + \lambda$, where λ ranges over the root lattice $\boldsymbol{Z}\Phi$. Then W_a can be regarded as the semi-direct product $N \rtimes W$. There is a canonical homomorphism from W_a to W: $w \mapsto \bar{w}$.

Let Φ^+ be a positive root system of Φ with $\Delta = \{\alpha_1, \cdots, \alpha_l\}$ its simple root system, where the indices of simple roots are compatible with the following Dynkin diagram:



Let $-\alpha_0$ be the highest root in Φ^+. We define $s_0=s_{\alpha_0}T_{-\alpha_0}$ and $s_i=s_{\alpha_i}$, $1\leq i\leq l$. Then the generator set S of W_a can be taken as $S=\{s_0, s_1, \cdots, s_l\}$.

2.2 For $\alpha\in\Phi^+$ and $m, k\in \boldsymbol{Z}$ with $m>0$, we define a stripe of E as below.

$$H^m_{\alpha;k} = H^m_{-\alpha;-k} = \{v\in E \mid k<\langle v, \alpha^\vee\rangle<k+m\}.$$

By an alcove, we mean a non-empty set of E of the form

$$\bigcap_{\alpha\in\Phi} H^1_{\alpha;k_\alpha}$$

with all $k_\alpha\in \boldsymbol{Z}$. The action of W_a on E induces an action on the set of all alcoves of E which is simply transitive. This enables us to identify an element $w\in W_a$ with the corresponding alcove

$$A_w = \bigcap_{\alpha\in\Phi} H^1_{\alpha;k(w,\alpha)}$$

for some set of integers $k(w, \alpha)$. This correspondence is determined uniquely by the following properties.

(a) $k(e, \alpha)=0$, $\forall\alpha\in\Phi$, where e is the identity of W_a;

(b) If $w'=ws_i(0\leq i\leq l)$, then

$$k(w', \alpha) = k(w, (\alpha)\bar{s}_i)+\varepsilon(\alpha, i)$$

with

$$\varepsilon(\alpha, i) = \begin{cases} 0 & \text{if } \alpha\neq\pm\alpha_i; \\ -1 & \text{if } \alpha=\alpha_i; \\ 1 & \text{if } \alpha=-\alpha_i, \end{cases}$$

where $\bar{s}_i=s_i$ if $1\leq i\leq l$, and $\bar{s}_0=s_{\alpha_0}$ (see [14]).

2.3 An alcove $\bigcap_{\alpha\in\Phi} H^1_{\alpha;k_\alpha}$ of E is determined completely by a Φ-tuple $(k_\alpha)_{\alpha\in\Phi}$ (resp. a Φ^+-tuple $(k_\alpha)_{\alpha\in\Phi^+}$) over $\boldsymbol{Z}$. So we can simply write $(k_\alpha)_{\alpha\in\Phi}$ (resp. $(k_\alpha)_{\alpha\in\Phi^+}$) for an alcove $\bigcap_{\alpha\in\Phi} H^1_{\alpha;k_\alpha}$. Note that not any Φ-tuple $(k_\alpha)_{\alpha\in\Phi}$ over $\boldsymbol{Z}$ gives rise to an alcove of E in the above way. It is so if and only if the following conditions are satisfied.

(a) $k_{-\alpha}=-k_\alpha$ for any $\alpha\in\Phi$;

(b) for any $\alpha, \beta\in\Phi$ with $\alpha+\beta\in\Phi$, the inequality

$$k_\alpha+k_\beta\leq k_{\alpha+\beta}\leq k_\alpha+k_\beta+1$$

holds (see [14]).

2.4 Property (2.2) (b) actually defines a set of operators $\{s_i \mid 0\leq i\leq l\}$ on the alcoves of E:

$$s_i\colon (k_\alpha)_{\alpha\in\Phi} \mapsto (k_{(\alpha)\bar{s}_i}+\varepsilon(\alpha, i))_{\alpha\in\Phi}\,.$$

These operators could be described graphically. We shall only deal with the case of $l=4$ which is actually needed in the present paper. We denote a root $\alpha=\sum_{i=1}^{4} a_i\alpha_i$ by its coordinate form (a_1, a_2, a_3, a_4). Now we arrange the entries of a Φ^+-tuple $(k_\alpha)_{\alpha\in\Phi^+}$ in the following way.

$$\begin{array}{ccccc} & & k_{(1,1,1,0)} & & \\ & k_{(1,1,0,0)} & & k_{(0,1,1,0)} & \\ k_{(1,0,0,0)} & & k_{(0,1,0,0)} & & k_{(0,0,1,0)} \\ k_{(1,2,1,1)} & & k_{(0,1,1,1)} & & k_{(0,0,0,1)} \\ & k_{(1,1,1,1)} & & k_{(0,1,0,1)} & \\ & & k_{(1,1,0,1)} & & \end{array}$$

Then the effect of the operator s_i on a Φ^+-tuple $w=\begin{array}{ccccc} & & a & & \\ & b & & c & \\ d & & e & & f \\ t & & u & & v \\ & x & & y & \\ & & z & & \end{array}$ are listed as in the following table:

s	s_0	s_1	s_2	s_3	s_4
ws	$\begin{array}{ccccc} & & -y & & \\ & -u & & -z & \\ * & & -x & & * \\ -t+1 & & -b & & * \\ & -e & & -a & \\ & & -c & & \end{array}$	$\begin{array}{ccccc} & & c & & \\ & e & & a & \\ -d-1 & & b & & * \\ * & & x & & * \\ & u & & z & \\ & & y & & \end{array}$	$\begin{array}{ccccc} & & * & & \\ & d & & f & \\ b & & -e-1 & & c \\ x & & * & & y \\ & t & & v & \\ & & * & & \end{array}$	$\begin{array}{ccccc} & & b & & \\ & a & & e & \\ * & & c & & -f-1 \\ * & & y & & * \\ & z & & u & \\ & & x & & \end{array}$	$\begin{array}{ccccc} & & x & & \\ & z & & u & \\ * & & y & & * \\ * & & c & & -v-1 \\ & a & & e & \\ & & b & & \end{array}$

where the entries in the $*$ positions remain unchanged.

2.5 It is known that any permutation on the set $\{s_i \mid i=0,1,3,4\}$ can be extended to a unique automorphism of W_a which fixes s_2. Let $\mathfrak{S}$ be the group of all permutations σ on the set $\{0,1,2,3,4\}$ satisfying $\sigma(2)=2$. Let f_σ be the automorphism of W_a satisfying $f_\sigma(s_i)=s_{\sigma(i)}$ for any $s_i\in S$. We denote $f_{(ij)}$ simply by f_{ij}, where (ij) is the transposition of i and j for $i\neq j$ in $\{0,1,3,4\}$. Then the effect of the f_{ij}'s on an element $w=\begin{array}{ccccc} & & a & & \\ & b & & c & \\ d & & e & & f \\ t & & u & & v \\ & x & & y & \\ & & z & & \end{array}$ are listed as below.

$\{i, j\}$	$\{0, 1\}$	$\{0, 3\}$	$\{0, 4\}$	$\{1, 3\}$	$\{1, 4\}$	$\{3, 4\}$
$f_{ij}(w)$	$-z$	$-u$	$*$	$*$	u	z
	$-x \quad *$	$* \quad -x$	$* \quad *$	$c \quad b$	$y \quad *$	$* \quad y$
	$-t \quad * \quad *$	$* \quad * \quad -t$	$* \quad * \quad *$	$f \quad * \quad d$	$v \quad * \quad *$	$* \quad * \quad v$
	$-d \quad * \quad *$	$-f \quad -a \quad *$	$-v \quad -z \quad -t$	$* \quad z \quad *$	$* \quad a \quad d$	$* \quad * \quad f$
	$-b \quad *$	$-c \quad *$	$-y \quad -x$	$* \quad *$	$* \quad b$	$* \quad c$
	$-a$	$*$	$-u$	u	$*$	a

2.6 For $w, w' \in W_a$, we say that w' is a left extension of w if $l(w')=l(w)+l(w'w^{-1})$. Then the following results on the alcove form $(k(w, \alpha))_{\alpha\in\Phi}$ of an element $w \in W_a$ are known.

Proposition [14], [15]. (1) $l(w)=\sum_{\alpha\in\Phi^+}|k(w, \alpha)|$, *where the notation* $|x|$ *stands for the absolute value of* x:
(2) $\mathcal{R}(w)=\{s_i \mid k(w, \alpha_i)<0\}$.
(3) *Let* $w'=(k(w', \alpha))_{\alpha\in\Phi}\in W_a$. *Then* w' *is a left extension of* w *if and only if the inequalities* $k(w', \alpha)k((w, \alpha)\geq 0$ *and* $|k(w', \alpha)|\geq|k(w, \alpha)|$ *hold for any* $\alpha\in\Phi$.

3. Left cells in $W_{(i)}$, $i\in\{0, 1, 3, 4, 12\}$.

From now on, we always assume that W_a is the affine Weyl group of type $\tilde{D}_4$. We shall apply Algorithm 1.10 to find a representative set of left cells of W_a in each of its two-sided cells Ω.

3.1 Let $W_{(i)}=\{w\in W_a \mid a(w)=i\}$ for $i\geq 0$. Then from the knowledge of unipotent classes of the complex connected reductive algebraic group of type D_4 and from Theorem 1.4, we see that $W_{(i)}=\emptyset$ unless $i\in\{0, 1, 2, 3, 4, 6, 7, 12\}$. $W_{(i)}$ is a single two-sided cell of W_a if $i\in\{0, 1, 3, 4, 7, 12\}$. On the other hand, $W_{(i)}$ is a union of three two-sided cells of W_a if $i\in\{2, 6\}$.

3.2 The case $W_{(0)}=\{e\}$ is trivial. The two-sided cell $W_{(1)}$ consists of all non-identity elements y of W_a each of which has a unique reduced expression. The set S forms a representative set of left cells of W_a in $W_{(1)}$ (see [8]). The set $W_{(12)}$ can be described as follows

$$\begin{aligned} W_{(12)} &= \{w\in W_a \mid k(w, \alpha)\neq 0\ \forall\alpha\in\Phi\} \\ &= \{w\in W_a \mid w = x\cdot w_J\cdot y \text{ for some } J\subset S \text{ and } x, y\in W_a \text{ with} \\ &\qquad l(w_J) = 12 \text{ and } l(w) = l(x)+l(w_J)+l(y)\} \end{aligned} \tag{3.2.1}$$

It is known that the set

$$N = \{w\in W_{(12)} \mid \mathcal{L}(w) = J \text{ satisfies } l(w_J) = 12 \text{ and } sw\notin W_{(12)} \text{ for any } s\in J\} \tag{3.2.2}$$

forms a representative set of left cells of W_a in $W_{(12)}$ which has cardinality

192 (see [16][17]).

For the sake of brevity, we shall denote each generator s_i of W_a by $\boldsymbol{i}$ (bold-faced) in the remaining part of this paper. Let $T=\{0, 1, 3, 4\}$.

3.3 Now we consider $W_{(3)}$. The set of elements of $W_{(3)}$ of the form w_J with $J \subset S$ is

$$P = \{\mathbf{020}, \mathbf{121}, \mathbf{323}, \mathbf{424}, \mathbf{013}, \mathbf{014}, \mathbf{034}, \mathbf{134}\}. \tag{3.3.1}$$

Graph $\mathcal{M}(\boldsymbol{ijk})$ with distinct $i, j, k, m \in T$ are

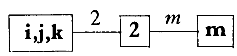


Figure 1. $\mathcal{M}(\boldsymbol{ijk})$

where the vertices x are represented by boxes, inside which we describe the corresponding subset $\mathcal{R}(x)$ of S, the vertex x with $\mathcal{R}(x)=\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ is the element $\boldsymbol{ijk}$. The graphs $\mathcal{M}(\boldsymbol{i2i})$ with $i \in T$ are all infinite and are quasi-isomorphic to each other. By 1.3(6) and Proposition 1.13, we can find a subgraph $\mathcal{M}$ of graph $\mathcal{M}(\mathbf{121})$ such that its vertex set M is a largest distinguished subset in the set $\bigcup_{i \in T} M(\boldsymbol{i2i})$

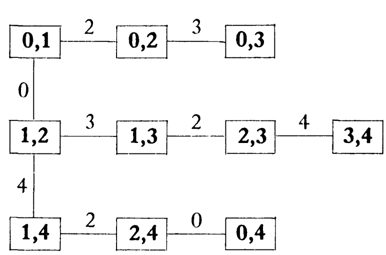


Figure 2. $\mathcal{M}$

where the vertex x with $\mathcal{R}(x)=\{\mathbf{1}, \mathbf{2}\}$ is the element $\mathbf{121}$.

Let

$$I = \{\{i, j, k\} \subset T \mid i, j, k \text{ are distinct}\}. \tag{3.3.2}$$

Then the **A**-saturated set

$$M \cup \Big(\bigcup_{\{i,j,k\} \in I} M(\boldsymbol{ijk}) \Big) \tag{3.3.3}$$

is distinguished by Proposition 1.13. It is easily checked that this set is also **B**-saturated. In fact, by 1.3(6) and by symmetry, one need only show that if $y \in W_a$ satisfies $y - \mathbf{01324}$, $\mathcal{R}(y) \supsetneqq \{\mathbf{4}\}$ and $a(y)=3$, then there exists some element z of the set in (3.3.3) with $y \underset{L}{\sim} z$. This could be done by using 1.16(2). Hence the set in (3.3.3) forms a representative set of left cells of W_a in $W_{(3)}$

by Theorem 1.6.

3.4 Next we consider $W_{(4)}$. There exists only one element in $W_{(4)}$, i.e. $\mathbf{0134}$, which has the form w_J. The graph $\mathcal{M}(\mathbf{0134})$ is as below.

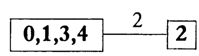


Figure 3. $\mathcal{M}(\mathbf{0134})$

The set $M(\mathbf{0134})$ is distinguished and **A**-saturated. But it is not **B**-saturated. In fact, let $y=\mathbf{01342}$ and $y_i=y\boldsymbol{i}$, $i\in T$. Then $y_i - y$, $\mathcal{R}(y_i)=\{\mathbf{2},\boldsymbol{i}\}\supsetneqq\{\mathbf{2}\}=\mathcal{R}(y)$ and $a(y_i)=4$, where the assertion $a(y_i)=4$ can be shown by Propositions 1.7 and 1.14 from the graph

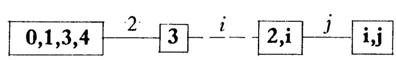


Figure 4.

where $i, j\in T$ are distinct, and the vertex x with $\mathcal{R}(x)=\{\mathbf{2},\boldsymbol{i}\}$ is the element y_i. The graphs $\mathcal{M}(y_i)$, $i\in T$, are finite which are all the same, i.e.

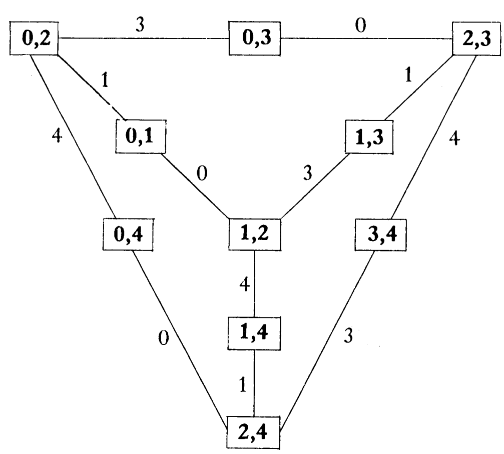


Figure 5. $\mathcal{M}(y_0)$

where the vertex x with $\mathcal{R}(x)=\{\mathbf{2},\boldsymbol{i}\}$ is the element y_i for any $i\in T$. Note that the above graph could be drawn tetrahedrally which looks more symmetric. The union set $M(\mathbf{0134})\cup M(y_0)$ is distinguished and **A**-saturated. But it is still not **B**-saturated. Let $y_{ij}=y_i\cdot\boldsymbol{j}$ for distinct $i, j\in T$. Then $\mathcal{R}(y_{ij})=\{\boldsymbol{i},\boldsymbol{j}\}$, and the y_{ij}'s are vertices of graph $\mathcal{M}(y_0)$. Let k, m, i, j be four numbers with $\{k, m, i, j\}=T$ and let $z_{ijk}=y_{ij}\cdot\boldsymbol{k}$. Then $z_{ijk}-y_{ij}$ and $\mathcal{R}(z_{ijk})=\{\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}\}\supsetneqq\{\boldsymbol{i},\boldsymbol{j}\}=\mathcal{R}(y_{ij})$. We have graphs $\mathcal{M}(z_{ijk})$ as below.

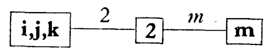


Figure 6. $\mathcal{M}(z_{ijk})$

where the vertex x with $\mathcal{R}(x)=\{i,j,k\}$ is the element z_{ijk}. By Propositions 1.7 and 1.14, we see from Figures 5 and 6 that $a(z_{ijk})=4=a(y_{ij})$. We have $z_{ijk}=z_{i'j'k'}$ if and only if i',j',k' is a permutation of i,j,k. Thus we get four distinct graphs: $\mathcal{M}(z_{ijk})$, $\{i,j,k\}\in I$ (see (3.3.1)). It is easily checked that the set

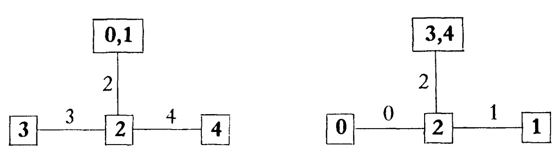
$$M(\mathbf{0134})\cup M(y_0)\cup\Big(\bigcup_{\{i,j,k\}\in I} z_{ijk}\Big)$$

is distinguished which is both **A**- and **B**-saturated. Thus by Theorem 1.6, this forms a representative set of left cells of W_a in $W_{(4)}$.

4. Left cells in $W_{(2)}$ and $W_{(6)}$

Since neither $W_{(2)}$ nor $W_{(6)}$ is a single two-sided cell of W_a, we shall deal with these two sets in a different way. As a starting set in the algorithm, P couldn't be chosen the set of all the elements of $W_{(i)}(i=2,6)$ of the form w_J. This is because the latter set in $W_{(i)}$ may not be wholely contained in some two-sided cell of W_a.

4.1 Let us first consider the set $W_{(2)}$. It contains six elements of the form w_J: $\mathbf{01},\mathbf{03},\mathbf{04},\mathbf{13},\mathbf{14}$ and $\mathbf{34}$. We start with the set $P=\{\mathbf{01}\}$ and consider the two-sided cell Ω_1 of W_a containing $\mathbf{01}$. Graph $\mathcal{M}(\mathbf{01})$ is the left one in Figure 7.



$\mathcal{M}(\mathbf{01})$ $\mathcal{M}(y')$

Figure 7.

Its vertex set $M(\mathbf{01})$ is distinguished and **A**-saturated. But it is not **B**-saturated. In fact, let $y=\mathbf{0123}$ and $y'=y\cdot\mathbf{4}$. Then we have $y'-y$ and $\mathcal{R}(y')=\{\mathbf{3},\mathbf{4}\}\supsetneqq\{\mathbf{3}\}=\mathcal{R}(y)$. By observing graphs $\mathcal{M}(\mathbf{01})$ and $\mathcal{M}(y')$ (see Figure 7), we see from Propositions 1.7 and 1.14 that $y'\underset{R}{\sim}y$ and hence $y'\in\Omega_1$. Now by 1.3(6), we have $\mathbf{34}\underset{L}{\sim}y'$. Thus by Proposition 1.15, the set $M(\mathbf{34})$ represents the same set of left cells of W_a as the set $M(y')$ does. We see by Proposition 1.13 that the union $M(\mathbf{01})\cup M(\mathbf{34})$ is distinguished and **A**-saturated. It is easily checked that this set is also **B**-saturated (By symmetry, we need only check that if $y\in W_a$ satisfies $y-x$, $\mathcal{R}(y)\supsetneqq\mathcal{R}(x)$ and $a(y)=2$ for $x=\mathbf{0123}$, then $\mathcal{R}(y)=\{\mathbf{3},\mathbf{4}\}$. Hence it foms a representative set of left cells of W_a in Ω_1.

4.2 It is known that any S-preserving automorphism of W_a stabilizes the sets $W_{(i)}$, $i\geq 0$, and induces a permutation on the set of two-sided cells of W_a in each $W_{(i)}$. Let $\Omega_2=f_{13}(\Omega_1)$ and $\Omega_3=f_{14}(\Omega_1)$ (see 2.5). Then both Ω_2 and Ω_3 are two-sided cells of W_a in $W_{(2)}$. Ω_1, Ω_2 and Ω_3 are all distinct since each of them contains exactly one of the sets $\{\mathbf{01},\mathbf{34}\}$, $\{\mathbf{03},\mathbf{14}\}$ and $\{\mathbf{04},\mathbf{13}\}$, and no two of these Ω_i's contain the same one. Clearly, the image of the set $M(\mathbf{01})\cup M(\mathbf{34})$ under the map f_{13} (resp. f_{14}), i.e. $M(\mathbf{03})\cup M(\mathbf{14})$ (resp. $M(\mathbf{04})\cup M(\mathbf{13})$), forms a representative set of left cells of W_a in Ω_2 (resp. Ω_3).

4.3 Next we consider the set $W_{(6)}$. There are six elements of the form w_J in $W_{(6)}$. They are $w_{\{i,j,2\}}$ with distinct $i,j\in T$. Let Ω_1' be the two-sided cell of W_a in $W_{(6)}$ containing $w_{\{0,1,2\}}=\mathbf{020120}$. Graph $\mathcal{M}(\mathbf{020120})$ is as in Figure 8.

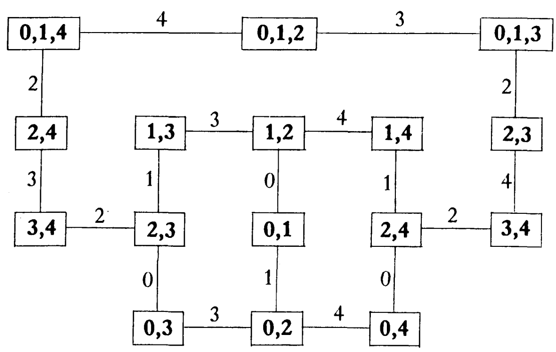
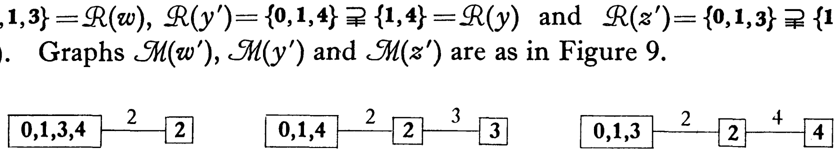


Figure 8. $\mathcal{M}(\mathbf{020120})$

where the vertex x with $\mathcal{R}(x)=\{\mathbf{0},\mathbf{1},\mathbf{2}\}$ is the element $\mathbf{020120}$. The **A**-saturated set $M(\mathbf{020120})$ is distinguished by Proposition 1.13, but it is not **B**-saturated. In fact, take the elements $w=\mathbf{0201203}$, $y=\mathbf{02012032421}$ and $z=\mathbf{02012042321}$ in $M(\mathbf{020120})$. Let $w'=w\cdot\mathbf{4}$, $y'=y\cdot\mathbf{0}$ and $z'=z\cdot\mathbf{0}$. Then $w'—w$, $y'—y$, $z'—z$, $\mathcal{R}(w')=\{\mathbf{0},\mathbf{1},\mathbf{3},\mathbf{4}\}$ $\supsetneqq\{\mathbf{0},\mathbf{1},\mathbf{3}\}=\mathcal{R}(w)$, $\mathcal{R}(y')=\{\mathbf{0},\mathbf{1},\mathbf{4}\}\supsetneqq\{\mathbf{1},\mathbf{4}\}=\mathcal{R}(y)$ and $\mathcal{R}(z')=\{\mathbf{0},\mathbf{1},\mathbf{3}\}\supsetneqq\{\mathbf{1},\mathbf{3}\}=\mathcal{R}(z)$. Graphs $\mathcal{M}(w')$, $\mathcal{M}(y')$ and $\mathcal{M}(z')$ are as in Figure 9.



$\mathcal{M}(w')$ $\mathcal{M}(y')$ $\mathcal{M}(z')$

Figure 9.

where the vertices x with $\mathcal{R}(x)=\{\mathbf{0},\mathbf{1},\mathbf{3},\mathbf{4}\}$, $\{\mathbf{0},\mathbf{1},\mathbf{4}\}$, and $\{\mathbf{0},\mathbf{1},\mathbf{3}\}$ are w', y' and z', respectively. Thus by Propositions 1.7 and 1.14, we get $w'\underset{R}{\sim}w$, $y'\underset{R}{\sim}y$ and $z'\underset{R}{\sim}z$. In particular, we have $w',y',z'\in\Omega_1'$.

4.4 By Proposition 1.13, we see that the **A**-saturated set

(4.4.1) $$\tilde{M}(\mathbf{020120}) = M(\mathbf{020120}) \cup M(w') \cup M(y') \cup M(z')$$

is distinguished. But it is still not **B**-saturated. In fact, let $v=z\cdot\mathbf{4}$. Then $v-z$ and $\mathcal{R}(v)=\{\mathbf{1},\mathbf{3},\mathbf{4}\} \supsetneq \{\mathbf{1},\mathbf{3}\}=\mathcal{R}(z)$. Graph $\mathcal{M}(v)$ is displayed in Figure 10.

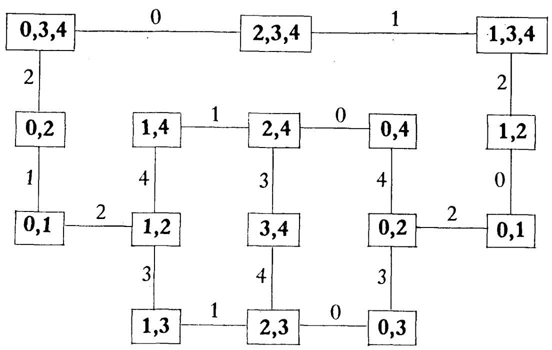


Figure 10. $\mathcal{M}(v)$

where the vertex x with $\mathcal{R}(x)=\{\mathbf{1},\mathbf{3},\mathbf{4}\}$ is the element v. By Propositions 1.7 and 1.14, we see that $v\underset{R}{\sim}z$ and hence the set $M(v)$ is contained in Ω_1'. Now $\mathcal{R}(v\cdot\mathbf{1})=\mathcal{R}(w_{\{2,3,4\}})=\{\mathbf{2},\mathbf{3},\mathbf{4}\}$. By 1.3(6), we have $w_{\{2,3,4\}}\underset{L}{\sim}v\cdot\mathbf{1}$. In particular, this implies that $w_{\{2,3,4\}}=\mathbf{323423}\in\Omega_1'$. Moreover, by Proposition 1.15, the sets $M(\mathbf{323423})$ and $M(v)$ $(=M(v\cdot\mathbf{1}))$ represent the same set of left cells of W_a.

4.5 The set $M(\mathbf{323423})$ is the image of the set $M(\mathbf{020120})$ under the automorphism $f=f_{(03)(14)}$ of W_a (see 2.5). This implies that the two-sided cell Ω_1' is stable under f. Let $w''=f(w')$, $y''=f(y')$ and $z''=f(z')$. Then w'', y'', $z''\in\Omega_1'$. Graphs $\mathcal{M}(w'')$, $\mathcal{M}(y'')$ and $\mathcal{M}(z'')$ are as in Figure 11.

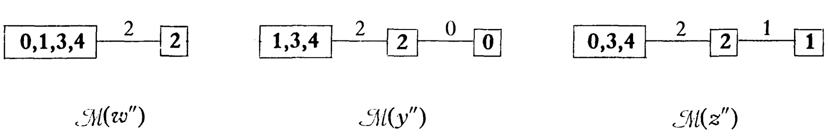


Figure 11.

It is easily seen by Proposition 1.13, that the union set

$$M(\mathbf{020120}) \cup M(\mathbf{323423}) \cup M(y') \cup M(y'') \cup M(z') \cup M(z'') \cup M(x)$$

is distinguished, where $x\in\{w', w''\}$. But since graphs $\mathcal{M}(w')$ and $\mathcal{M}(w'')$ are quasi-isomorphic, it is not clear whether the sets $M(w')$ and $M(w'')$ represent the same set of left cells of W_a or not.

4.6 For $x\in W_a$, we denote by Γ_w the left cell of W_a containing w.

Lemma. *The left cells of W_a represented by the sets $M(w')$ and $M(w'')$ are disjoint.*

Proof. Let $\alpha=w'\cdot 2=\mathbf{020120342}\in M(w')$ and $\beta=w''\cdot 2=\mathbf{323423012}\in M(w'')$. It is enough to show $\alpha\underset{L}{\not\sim}\beta$. By Theorem 1.5, we need only show $\Sigma(\alpha)\neq\Sigma(\beta)$. Observe the graph

$$\boxed{0,1,2}\overset{3}{\text{———}}\boxed{0,1,3}\;-\overset{4}{-}-\;\boxed{0,1,3,4}\overset{2}{\text{———}}\boxed{2}$$

where the vertex x with $\mathscr{R}(x)=\{\mathbf{0,1,2}\}$ (resp. $\mathscr{R}(x)=\{\mathbf{2}\}$) is the element $\mathbf{020120}$ (resp. α). We see from Proposition 1.14 that $\mathbf{020120}-\alpha$ and hence $\Gamma_{\mathbf{020120}}\in\Sigma(\alpha)$. On the other hand, it is easily seen by 1.16(2) that there is no element $x\in W_a$ satisfying both conditions $x-\beta$ and $\mathscr{R}(x)=\{\mathbf{0,1,2}\}$. So $\Gamma_{\mathbf{020120}}\notin\Sigma(\beta)$. Our result follows. ■

4.7 Let $\tilde{M}(\mathbf{323423})=f(\tilde{M}(\mathbf{020120}))$. Then by Lemma 4.6, we see that the union set $\tilde{M}(\mathbf{020120})\cup\tilde{M}(\mathbf{323423})$ is distinguished and **A**-saturated.

Proposition. *$\tilde{M}(\mathbf{020120})\cup\tilde{M}(\mathbf{323423})$ forms a representative set of left cells of W_a in Ω_1'.*

The proposition is amount to assert that the set $\tilde{M}(\mathbf{020120})\cup\tilde{M}(\mathbf{323423})$ is **B**-saturated. We postpone the proof of this assertion to §5.

4.8 Now let us assume Proposition 4.7. Let $\Omega_2'=f_{03}(\Omega_1')$ and $\Omega_3'=f_{04}(\Omega_1')$. Then both Ω_2' and Ω_3' are two-sided cells of W_a in $W_{(6)}$. We assert that Ω_1', Ω_2' and Ω_3' are all distinct since each of them contains exactly one of the sets $\{\mathbf{020120}, \mathbf{323423}\}$, $\{\mathbf{323123}, \mathbf{020420}\}$ and $\{\mathbf{424124}, \mathbf{323023}\}$, and no two of these Ω_i''s contains the same one. Clearly, the set $f_{03}(\tilde{M}(\mathbf{020120})\cup\tilde{M}(\mathbf{323423}))$ forms a representative set of left cells of W_a in Ω_2'. A similar result holds for Ω_3'.

5. Left cells in $W_{(7)}$.

Unfortunately, there is no element of the form w_J in $W_{(7)}$. So the previous method can't be carried on to the case of $W_{(7)}$. We must find some suitable starting set of our algorithm.

5.1 Let us consider the element $w=\mathbf{020120321}$. We know $a(w)=7$ by a result of Du [5, Lemma 2.9]. Graph $\mathscr{M}(w)$ is as in Figure 12, where the vertex x with $\mathscr{R}(x)=\{\mathbf{1,2,3}\}$ is the element w. Note that this graph could be drawn tetrahedrally which looks more symmetric. The **A**-saturated set $M(w)$ is distinguished by Proposition 1.13. But it is not **B**-saturated. Take $\alpha=w\cdot\mathbf{0}$, $\beta=w\cdot\mathbf{42021}\in M(w)$. Let $\alpha'=\alpha\cdot\mathbf{4}$ and $\beta'=\beta\cdot\mathbf{3}$ Then $\alpha'-\alpha$, $\beta'-\beta$, $\mathscr{R}(\alpha')=\{\mathbf{0,1,3,4}\}\supsetneqq\{\mathbf{0,1,3}\}=\mathscr{R}(\alpha)$ and $\mathscr{R}(\beta')=\{\mathbf{0,1,3}\}\supsetneqq\{\mathbf{0,1}\}=\mathscr{R}(\beta)$. Graphs $\mathscr{M}(\alpha')$ and $\mathscr{M}(\beta')$ are as in Figure 13.

By Propositions 1.7 and 1.14, we see from Figures 12 and 13 that $\alpha'\underset{R}{\sim}\alpha$ and $\beta'\underset{R}{\sim}\beta$. This implies

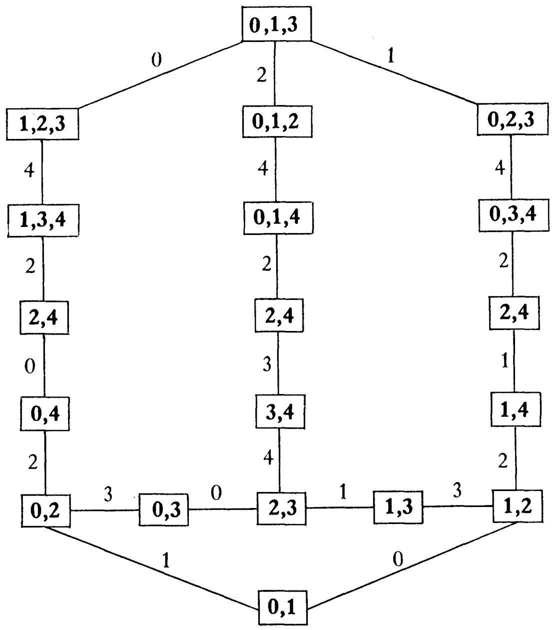


Figure 12. $\mathcal{M}(w)$

$\mathcal{M}(\alpha')$ $\mathcal{M}(\beta')$

Figure 13.

(5.1.1) $$\bar{M}(w) = M(w) \cup M(\alpha') \cup M(\beta') \subset W_{(7)}\,.$$

The **A**-saturated set $\bar{M}(w)$ is distinguished by Proposition 1.13.

5.2 We have $f_{01}(w)=w\cdot\mathbf{01}\in M(w)$. Moreover, it is easily seen that $f_{01}(\bar{M}(w))=\bar{M}(w)$. Let $\mathfrak{S}'=\{\mathbf{1}, (01)\}$, where $\mathbf{1}$ is the identity of the group $\mathfrak{S}$ (see 2.5). Then $\mathfrak{S}'$ is the stabilizer of $\bar{M}(w)$ in $\mathfrak{S}$. Let $R\subset\mathfrak{S}$ be the set of distinguished left coset representatives of $\mathfrak{S}$ with respect to $\mathfrak{S}'$, i.e. $R=\{\sigma\in\mathfrak{S}\mid\sigma(0)<\sigma(1)\}$. For $\sigma\in R$, we denote the set $f_\sigma(M(x))$ by $M_\sigma(x)$ for $x\in\{w, \alpha', \beta'\}$ and $f_\sigma(\bar{M}(w))$ by $\bar{M}_\sigma(w)$.

5.3 Let us record some facts on elements of $\bar{M}(w)$ which are useful in the proof of the subsequent lemmas.

(1) α is the unique element x in $M(w)$ satisfying the following properties: (i) $|\mathcal{R}(x)|=3$; (ii) If $\{x, y\}$ is an edge of graph $\mathcal{M}(w)$, then $|\mathcal{R}(y)|=3$ and $2\in\mathcal{R}(y)$. These properties are preserved under the action of $\mathfrak{S}$ on α.

(2) $\alpha'=\alpha\cdot\mathbf{4}$ is the unique element x of W_a satisfying the conditions $x—\alpha$ and $\mathcal{R}(x)=\{\mathbf{0,1,3,4}\}$.

(3) $\beta'=\beta\cdot\mathbf{3}$ is the unique element x of W_a satisfying the conditions $x-\beta$, $\mathscr{R}(x)=\{\mathbf{0},\mathbf{1},\mathbf{3}\}$ and $a(x)=7$.
(4) Let $\gamma=\alpha'\cdot\mathbf{2}$. Then $\mathscr{R}(\gamma)=\{\mathbf{2}\}$ and the elements y with $y-\gamma$, $\mathscr{R}(y)\supsetneqq\mathscr{R}(\gamma)$ and $a(y)=7$ are all contained in the set $M(w)$.

5.4. Lemma *Let $\sigma,\sigma'\in R$. Then sets $\bar{M}_\sigma(w)$ and $\bar{M}_{\sigma'}(w)$ represent the same set of left cells of W_a in $W_{(7)}$ if and only if $\mathscr{R}(f_\sigma(\alpha))=\mathscr{R}(f_{\sigma'}(\alpha))$.*

Proof. It is enough to show our result in the case of $\sigma'=\mathbf{1}$. Note that if $\bar{M}_\sigma(w)$ and $\bar{M}(w)$ represent the same set of left cells of W_a in $W_{(7)}$, then graphs $\mathscr{M}(f_\sigma(w))$ and $\mathscr{M}(w)$ must be quasi-isomorphic. Hence the direction "$\Rightarrow$" is obvious since we see from 5.3(1) that the equality $\mathscr{R}(f_\sigma(\alpha))=\mathscr{R}(\alpha)$ is a necessary condition for graphs $\mathscr{M}(w)$ and $\mathscr{M}(f_\sigma(w))$ to be quasi-isomorphic. Now assume $\mathscr{R}(f_\sigma(\alpha))=\mathscr{R}(\alpha)$. Then $\sigma\in\{\mathbf{1},(03),(13)\}$. The case $\sigma=\mathbf{1}$ is trivial. By symmetry, it suffices to show our result in the case of $\sigma=(03)$. By Proposition 2.6, we see from the alcove forms of elements that the element $f_{03}(\alpha\cdot\mathbf{2})$ is a left extension of w (see 2.6). Then we have $f_{03}(\alpha\cdot\mathbf{2})\underset{L}{\sim}w$ by 1.3(3). Hence $M(f_{03}(\alpha\cdot\mathbf{2}))=M_{(03)}(w)$. This implies that the sets $M(w)$ and $M_{(03)}(w)$ represent the same set of left cells of W_a in $W_{(7)}$. Now we can assert by 5.3(2), (3) and Theorem 1.5 that $\bar{M}(w)$ and $\bar{M}_{(03)}(w)$ also represent the same set of left cells of W_a in $W_{(7)}$. ■

5.5 Let $\mathfrak{M}_\sigma(x)$ (resp. $\bar{\mathfrak{M}}_\sigma(w)$) be the set of left cells of W_a represented by the set $M_\sigma(x)$ (resp. $\bar{M}_\sigma(w)$) for $x\in\{w,\alpha',\beta'\}$ and $\sigma\in R$. We denote $\mathfrak{M}_1(x)$ simply by $\mathfrak{M}(x)$.

Lemma. *Let $\sigma,\sigma'\in R$. If $\bar{\mathfrak{M}}_\sigma(w)\neq\bar{\mathfrak{M}}_{\sigma'}(w)$, then $\bar{\mathfrak{M}}_\sigma(w)\cap\bar{\mathfrak{M}}_{\sigma'}(w)=\emptyset$.*

Proof. It suffices to show our assertion in the case of $\sigma'=\mathbf{1}$. Thus by our assumption, we have $\sigma\neq\mathbf{1}$. By Lemma 5.4, we see that $\mathscr{R}(f_\sigma(\alpha))\neq\mathscr{R}(\alpha)$. So by 5.3(1) and Proposition 1.13, we have

$$\mathfrak{M}(w)\cap\mathfrak{M}_\sigma(w)=\emptyset\,. \tag{5.5.1}$$

On the other hand, we have $\mathscr{R}(\beta')=\mathscr{R}(\alpha)$ and hence $\mathscr{R}(f_\sigma(\beta'))=\mathscr{R}(f_\sigma(\alpha))$. Thus $\mathscr{R}(\beta')\neq\mathscr{R}(f_\sigma(\beta'))$ and so we have

$$\mathfrak{M}(\beta')\cap\mathfrak{M}(f_\sigma(\beta'))=\emptyset \tag{5.5.2}$$

by Proposition 1.13 and by observing graph $\mathscr{M}(\beta')$. Finally, by (5.5.1), 5.3(4) and Theorem 1.5, we have $\gamma\underset{L}{\not\sim}f_\sigma(\gamma)$ and hence

$$\mathfrak{M}(\alpha')\cap\mathfrak{M}(f_\sigma(\alpha'))=\emptyset \tag{5.5.3}$$

by observing graph $\mathscr{M}(\alpha')$. Thus our result follows from (5.5.1), (5.5.2), (5.5.3) and Figures 12, 13. ■

5.6 By Lemmas 5.4 and 5.5, we get a largest possible distinguished subset $\bar{M}$ from the set $\bigcup_{\sigma\in R}\bar{M}_\sigma(w)$, which is

$$\bar{M}=\bar{M}(w)\cup\bar{M}_{(14)}\cup\bar{M}_{(34)}(w)\cup\bar{M}_{(014)}(w) \tag{5.6.1}$$

Proposition. *The set $\bar{M}$ forms a representative set of left cells of W_a in $W_{(7)}$.*

Before showing this proposition, we first consider the following

Lemma 5.7. *Given any $x\in\bar{M}(w)$. If $y\in W_a$ satisfies $y-x$, $\mathscr{R}(y)\supsetneqq\mathscr{R}(x)$ and $a(y)=7$, then there is some $z\in\bar{M}$ with $z\underset{L}{\sim}y$.*

Let K be the set of all elements of W_a of the form $y=xs$ for some $x\in\bar{M}(w)$ and $s\in S-\mathscr{R}(x)$ (set difference) with $\mathscr{R}(y)\supset\mathscr{R}(x)$ and $y\notin W_{(12)}$ (i.e. $k(y,\alpha)=0$ for some $\alpha\in\Phi$ by (3.2.1)). Let K' be the set of all elements y of W_a such that $y<x$, $y-x$, $\mathscr{R}(y)\supsetneqq\mathscr{R}(x)$ and $y\notin W_{(12)}$ for some $x\in\bar{M}(w)$. Then by 3.1 and 1.16(2), we see that Lemma 5.7 is equivalent to

Lemma 5.8. *For any $y\in K\cup K'$, there is some $x\in\bar{M}$ with $x\underset{L}{\sim}y$.*

Proof. First assume $y\in K$. Then $y\in W_{(7)}$ by 1.3(7) and 3.1. By Proposition 2.6(3), we can see from the alcove form of y that y is a left extension of some $z'\in\bigcup_{\sigma\in R}\bar{M}_\sigma(w)$, i.e. $y\underset{L}{\sim}z'$ by 1.3(3), (7). This implies $y\underset{L}{\sim}z$ for some $z\in\bar{M}$ by the choice of the set $\bar{M}$.

Next assume $y\in K'$. Note that there is a unique maximal element in $\bar{M}(w)$ with respect to the Bruhat order. This maximal elment is $d=\mathbf{020120321420021324}\in M(\beta')$. Consider the set H of all elements z of W_a such that $z<d$, $\mathscr{L}(z)=\{0,1,2\}$, $z\notin W_{(12)}$ and $|\mathscr{R}(z)|\geq 2$. Then $H\cap W_{(7)}\supseteq K'$. The set H can be found by the inductive procedure given in 1.16 and by expressing elements in alcove forms. By direct checking, we see that each element z of H satisfies one of the following conditions.

(1) z is a left extension of some element in $\bigcup_{\sigma\in R}\bar{M}_\sigma(w)$;

(2) z belongs to the set described in (4.4.1);

(3) $z\underset{R}{\sim}h$ with $h=\mathbf{020120342102}$;

(4) $z\underset{R}{\sim}k$ with $k=\mathbf{02012042324}$.

By a result of Du (see [5, the proof of Lemma 3.7], and note that there Du showed $f_{03}(h), f_{03}(k)\in W_{(6)}$ in our notations), we see that the elements $z\in H$ satisfying condition (3) or (4) are in $W_{(6)}$. Also, the elements $z\in H$ satisfying condition (2) are in $W_{(6)}$ since the set in (4.4.1) is in $W_{(6)}$. This implies that $y\in K'\subseteq$

$H\cap W_{(7)}$ satisfies condition (1). So $y \underset{L}{\sim} z$ for some $z\in \bar{M}$ by the argument given in the first paragraph of this proof. ■

5.9 *The proof of Proposition 5.6.* We know that the set $\bar{M}$ is both distinguished and **A**-saturated. Thus it remains to show that $\bar{M}$ is also **B**-saturated. Since $\bar{M}$ is a largest possible distinguished subset of $\bigcup_{\sigma\in R} \bar{M}_\sigma(w)$ and the latter set is $\mathfrak{S}$-stable, it is enough to show Lemma 5.7. But Lemma 5.7 is equivalent to Lemma 5.8 which has been shown. So our result follows. ■

5.10 From Lemma 5.4 and Proposition 5.6, the discription of the set $W_{(7)}$ by Du (see [5, the proof of Theorem 4.6]) could be restated in more explicit way when elements of W_a are expressed in alcove forms.

Proposition. *The set $W_{(7)}$ consists of all elements y of W_a such that y is a left extension of some element in $\bigcup_{\sigma\in R} \bar{M}_\sigma(w)$ and satisfies $k(y, \alpha)=0$ for some $\alpha\in\Phi$.*

5.11 Now we shall show Proposition 4.7.

Proof of Proposition 4.7. Let us denote $M_1=M(\mathbf{020120})$, $M_2=M(\mathbf{323423})$, $\tilde{M}_1=\tilde{M}(\mathbf{020120})$, $\tilde{M}_2=\tilde{M}(\mathbf{323423})$ and $\tilde{M}=\tilde{M}_1\cup\tilde{M}_2$.

We say that a set $Q\subset W_a$ has property (L), if the left cells represented by Q are contained in the set of left cells represented by $\tilde{M}$.

Clearly, if Q has property (L), then any subset of Q also has property (L); if both sets Q and P have property (L), then so does their union $Q\cup P$.

Let N be the set of all elements $y\in W_a$ such that there is some $x\in\tilde{M}_1$ with $y-x$, $\mathcal{R}(y)\supsetneqq\mathcal{R}(x)$ and $a(y)=6$. Then Proposition 4.7 is amount to the following statement

(a) The set $N\cup f_{(03)(14)}(N)$ has property (L).

Since the set $\tilde{M}$ is stable under the automorphism $f_{(03)(14)}$, statement (a) is equivalent to the statement

(b) The set N has property (L).

Let N_1 be the set of all elements of W_a of the form $y=xs$ with $\mathcal{R}(y)\supset\mathcal{R}(x)$ and $a(y)=6$ for some $x\in\tilde{M}_1$ and $s\in S-\mathcal{R}(x)$. Let N_2 be the set of all elements y of W_a such that $y<x, y-x$, $a(y)=6$ and $\mathcal{R}(y)\supsetneqq\mathcal{R}(x)$ for some $x\in\tilde{M}_1$. Then $N=N_1\cup N_2$ by 1.16(2). So statement (b) is equivalent to the statement

(c) Both N_1 and N_2 have property (L).

Note that if we remove the restriction $a(y)=6$ in the definitions of the sets N_i, $i=1, 2$, then by 1.3(2), we have the inequality $a(y)\geq 6$ for $y\in N_1\cup N_2$. Thus by 3.1, (3.2.1) and Proposition 5.10, the requirement $a(y)=6$ is amount to that $k(y, \alpha)=0$ for some $\alpha\in\Phi$ and that y is not a left extension of any element of $\bigcup_{\sigma\in R} \bar{M}_\sigma(w)$. This can be checked by the alcove form of y quite easily. So the set N_1 can be found easily. But finding the set N_2 is somewhat difficult since checking the condition $y-x$ on y for a given x involves very complicated computation of Kazhdan-Lusztig polynomials. So instead to find N_2 and to check the property (L) of N_2, we shall find a larger set, say Q, containing N_2 and check Q to have property (L), by which we deduce that N_2 has property (L) immediately. Finding the set Q will be easier and will not involve any computation of Kazhdan-Lusztig polynomials. Note that such a trick has already been used in the proof of Lemma 5.8. The sets Q, N_2 here play the same roles as the sets H, K' there.

Let us first show N_1 to have property (L). By a direct computation, we get the inclusion

$$N_1 \subset M(k) \cup M(k\cdot\mathbf{01}) \cup M(h) \cup \tilde{M}_1 . \tag{5.10.1}$$

where elements k, h are as defined in the proof of Lemma 5.8. Since $\mathcal{R}(k)=\{\mathbf{2},\mathbf{3},\mathbf{4}\}$, $\mathcal{R}(h)=\{\mathbf{0},\mathbf{1},\mathbf{2}\}$ and $k, h \in W_{(6)}$, this implies by 1.3(6) that $k \underset{L}{\sim} \mathbf{323423}$ and $h \underset{L}{\sim} \mathbf{020120}$. So by Proposition 1.15, the sets $M(k)$ and M_2 (resp. $M(h)$ and M_1) represent the same set of left cells of W_a. This implies immediately that both sets $M(k)$ and $M(h)$ have property (L). Next note that $k\cdot\mathbf{01}-k\cdot\mathbf{0}$, $\mathcal{R}(k\cdot\mathbf{01})=\{\mathbf{0},\mathbf{1},\mathbf{3},\mathbf{4}\} \supsetneqq \{\mathbf{0},\mathbf{3},\mathbf{4}\}=\mathcal{R}(k\cdot\mathbf{0})$, $k\cdot\mathbf{0}\in M(k)$ and $a(k\cdot\mathbf{01})=6$. Also, note that for $\alpha=\mathbf{323423}\cdot\mathbf{0}\in M_2$, there is a unique element $x\in W_{(6)}$ satisfying $x-\alpha$ and $\mathcal{R}(x)=\{\mathbf{0},\mathbf{1},\mathbf{3},\mathbf{4}\} \supsetneqq \{\mathbf{0},\mathbf{3},\mathbf{4}\}=\mathcal{R}(\alpha)$. Actually, we have $x=\alpha\cdot\mathbf{1}$. Since $\alpha \underset{L}{\sim} k\cdot\mathbf{0}$, this implies $\alpha\cdot\mathbf{1} \underset{L}{\sim} k\cdot\mathbf{01}$ by Theorem 1.5. But $\alpha\cdot\mathbf{1}\in\tilde{M}_2$. So the set $M(k\cdot\mathbf{01})$ has property (L). Thus the set on the right hand side of (5.10.1) has property (L) and hence so does the set N_1.

Now we want to show that N_2 has property (L). There are two maximal elements in the set $\tilde{M}_1$ with respect to the Bruhat order. They are $b_1=\mathbf{02012042320124}$ and $b_2=\mathbf{02012032421023}$. Let $Q_i (i=1, 2)$ be the set of all elements y of W_a such that $y<b_i$, $|\mathcal{R}(y)|\geq 2$, $\mathcal{L}(y)=\{\mathbf{0},\mathbf{1},\mathbf{2}\}$ and $a(y)=6$. Then $Q_1\cup Q_2 \supseteq N_2$. Thus to show N_2 has property (L), it is enough to show that both Q_1 and Q_2 have property (L). Since the automorphism f_{34} of W_a stabilizes the set $\tilde{M}$ and maps Q_1 onto Q_2, we need only show that Q_1 has property (L). By a direct computation, we get the inclusion

$$Q_1 \subset M(k) \cup M(k\cdot\mathbf{01}) \cup M(h) \cup \tilde{M}_1 . \tag{5.10.2}$$

Since we have shown that the set on the right hand side has property (L), this implies that Q_1 has property (L). Hence Proposition 4.7 follows. ■

6. Description of left cells of $W_a(\tilde{D}_4)$.

So far, we have got a representative set of left cells of W_a in each of its two-sided cells. By taking a union of all these sets, we get a representative set of left cells of the whole group W_a which is denoted by Σ. The numbers $n(\Omega)$ of left cells of W_a in the two-sided cells Ω are listed in the following table.

Ω	$W_{(0)}$	$W_{(1)}$	Ω_i ($\subset W_{(2)}$) $i=1,2,3$	$W_{(3)}$	$W_{(4)}$	Ω_i' ($\subset W_{(6)}$) $i=1,2,3$	$W_{(7)}$	$W_{(12)}$
$n(\Omega)$	1	5	8	22	24	48	96	192

So the total number of left cells of W_a is 508.

Now we ask how to use the set Σ to describe left cells of W_a explicitly. In other words, for any given element x of W_a, how can one tell what left cell it belongs to?

6.1 We may assume $x\neq e$ since otherwise it is trivial. If $x\in W_{(12)}$, i.e. $k(x,\alpha)\neq 0$ for all $\alpha\in\Phi$, then by [17, Corollary 1.2], there is a unique element $y\in\Sigma$ which has the same sign type as x does (see [15] for the definition of a sign type). We can conclude $x\in\Gamma_y$ (see 4.6). Now assume $x\notin W_{(12)}$. By Proposition 1.13, there is some element $y\in\Sigma$ such that x and y have the same generalized τ-invariant (This could be checked by comparing graphs $\mathcal{M}(x)$, $\mathcal{M}(y)$ and the positions of x, y in the respective graphs). We can conclude $x\in\Gamma_y$ except for the cases when $\mathcal{M}(x)$ is quasi-isomorphic to one of the following graphs.

(1) $\boxed{\mathbf{0,1,3,4}}\overset{2}{\text{---}}\boxed{\mathbf{2}}$;

(2) $\boxed{\boldsymbol{i,j,k}}\overset{2}{\text{---}}\boxed{\mathbf{2}}\overset{m}{\text{---}}\boxed{\boldsymbol{m}}$, where $\{i,j,k,m\}=T$ (see 3.2).

(3) $\mathcal{M}(\mathbf{121})$ or $\mathcal{M}(\mathbf{013420})$.

6.2 Let $y_{ij}=\boldsymbol{i2ij2ikm}$ and $y_{ijk}=\boldsymbol{i2ij2ik2jim}$ for distinct $i,j,k,m\in T$. The following facts could be deduced from the previous results or by directly checking.

(1) $\mathcal{R}(y_{ij})=\mathcal{R}(y_{ijk})=\{\mathbf{0,1,3,4}\}$, $a(y_{ij})=6$ and $a(y_{ijk})=7$ for any distinct $i,j,k\in T$.

(2) Graphs $\mathcal{M}(y_{ij})$ and $\mathcal{M}(y_{ijk})$ are all quasi-isomorphic to $\boxed{\mathbf{0,1,3,4}}\overset{2}{\text{---}}\boxed{\mathbf{2}}$ for any distinct $i,j,k\in T$.

(3) If $x\notin W_{(12)}$ and $\mathcal{R}(x)=\{\mathbf{0,1,3,4}\}$, then $a(x)\in\{4,6,7\}$.

(4) $y_{ij}\underset{L}{\sim}y_{i'j'}\Leftrightarrow y_{ij}=y_{i'j'}\Leftrightarrow\{i',j'\}=\{i,j\}$.

(5) $y_{ij}\in\Sigma$ for any distinct $i,j\in T$.

(6) y_{ij} is a shortest element in the left cell $\Gamma_{y_{ij}}$. Conversely, if Γ is a left cell of W_a in $W_{(6)}$ with $\mathcal{R}(\Gamma)=\{\mathbf{0,1,3,4}\}$, then any shortest element of Γ has the form y_{ij} for some distinct $i,j\in T$.

(7) $y_{ijk}\underset{L}{\sim}y_{i'j'k'}\Leftrightarrow\{i',j',k'\}=\{i,j,k\}$.

(8) $y_{ijk}\in\Sigma\Leftrightarrow(i,j,k)$ is in the set $\{(0,1,3),(0,4,3),(0,1,4),(4,1,3)\}$.

(9) $y_{ijk}\in W_{(7)}$ is a shortest element in the left cell $\Gamma_{y_{ijk}}$. Conversely, if Γ is a left cell of W_a in $W_{(7)}$ with $\mathcal{R}(\Gamma)=\{\mathbf{0,1,3,4}\}$, then any shortest element of Γ has the form y_{ijk} for some distinct $i,j,k\in T$.

(10) y_{ijk} is a left extension of $y_{i'j'} \Leftrightarrow \{i',j'\} \subset \{i,j,k\}$.

6.3 The following result is a direct consequence of the above facts, which could be used to determine the left cell of W_a containing a given element x in the exceptional case (1).

Proposition. *Let $x \in W_a$ satisfy the conditions $x \notin W_{(12)}$ and $\mathcal{R}(x) = \{\mathbf{0,1,3,4}\}$ with graph $\mathcal{M}(x)$:* $\boxed{\mathbf{0,1,3,4}} \overset{2}{\text{—}} \boxed{\mathbf{2}}$.

(1) *If x is not a left extension of y_{ij} for any distinct $i, j \in T$, then $x \in \Gamma_{0134}$.*

(2) *If x is a left extension of some y_{ij} but is not a left extension of any $y_{i'j'}$ with $\{i',j'\} \neq \{i,j\}$, then $x \in \Gamma_{y_{ij}}$.*

(3) *If x is a left extension of both y_{ij} and $y_{i'j'}$ with $\{i,j\} \neq \{i',j'\}$, then $\{i,j\} \cap \{i',j'\} \neq \emptyset$. We may assume $j = j'$ without loss of generality. Then $x \in \Gamma_{y_{mnp}}$, where m, n, p is a permutation of i, j, i' such that $y_{mnp} \in \Sigma$.*

6.4 For any distinct $i, j, k, m \in T$, we define the following elements:

$$w_{ijk} = \boldsymbol{ijkm2ijk}\,, \qquad x_{ijk} = \boldsymbol{i2ij2ikm2ijk}\,,$$
$$z_{ijk} = \boldsymbol{i2ij2ik2jim2ijk}\,.$$

The following results could be shown by the results in previous section or by directly checking.

(1) $\mathcal{R}(w_{ijk}) = \mathcal{R}(x_{ijk}) = \mathcal{R}(z_{ijk}) = \{\boldsymbol{i,j,k}\}$, $a(w_{ijk}) = 4$, $a(x_{ijk}) = 6$ and $a(z_{ijk}) = 7$ for any distinct $i, j, k \in T$.

(2) Graphs $\mathcal{M}(w_{ijk})$, $\mathcal{M}(x_{ijk})$ and $\mathcal{M}(z_{ijk})$ are all quasi-isomorphic to $\boxed{\boldsymbol{i,j,k}} \overset{2}{\text{—}} \boxed{\mathbf{2}} \overset{m}{\text{—}} \boxed{\boldsymbol{m}}$ for any distinct $i, j, k, m \in T$.

(3) If an element $x \in W_a$ satisfies $x \notin W_{(12)}$ and $\mathcal{R}(x) = \{\boldsymbol{i,j,k}\}$ for some distinct $i, j, k \in T$, then $a(x) \in \{3, 4, 6, 7\}$.

(4) $w_{ijk} \underset{L}{\sim} w_{i'j'k'} \Leftrightarrow w_{ijk} = w_{i'j'k'} \Leftrightarrow \{i,j,k\} = \{i',j',k'\}$.

(5) $w_{ijk} \in \Sigma$ for any distinct $i, j, k \in T$.

(6) w_{ijk} is the shortest element in the left cell $\Gamma_{w_{ijk}}$. Any element of $\Gamma_{w_{ijk}}$ is a left extension of w_{ijk}.

(7) $x_{ijk} \underset{L}{\sim} x_{i'j'k'} \Leftrightarrow x_{ijk} = x_{i'j'k'} \Leftrightarrow \{i,j\} = \{i',j'\}$ and $k = k'$.

(8) $x_{ijk} \in \Sigma$ for any distinct $i, j, k \in T$.

(9) x_{ijk} is the shortest element in the left cell $\Gamma_{x_{ijk}}$. Any element of $\Gamma_{x_{ijk}}$ is a left extension of x_{ijk}.

(10) $z_{ijk} = z_{i'j'k'} \Leftrightarrow \{i,j\} = \{i',j'\}$ and $k = k'$.

(11) $z_{ijk} \underset{L}{\sim} z_{i'j'k'} \Leftrightarrow \{i,j,k\} = \{i',j',k'\}$.

(12) $z_{ijk} \in \Sigma \Leftrightarrow$ either (i,j,k) or (j,i,k) is in the set $\{(0,1,3), (0,1,4), (0,4,3), (1,4,3)\}$.

(13) z_{ijk} is a shortest element in the left cell $\Gamma_{z_{ijk}}$. Any shortest element of

the left cell $\Gamma_{z_{ijk}}$ has the form $z_{i'j'k'}$ for some permutation i',j',k' of i,j,k.
(14) x_{ijk} is a left extension of $w_{i'j'k'} \Leftrightarrow \{i,j,k\}=\{i',j',k'\}$.
(15) z_{ijk} is a left extension of $x_{i'j'k'} \Leftrightarrow \{i,j,k\}=\{i',j',k'\}$ and $k \neq k'$.
6.5 The following proposition is a consequence of the above results, which could be used to determine the left cell of W_a containing a given element x in the exceptional case (2).

Proposition. *Let $x \in W_a$ satisfy $x \notin W_{(12)}$ and $\mathscr{R}(x)=\{i,j,k\}$ with graph $\mathscr{M}(x)$ quasi-isomorphic to $\boxed{i,j,k}\overset{2}{\text{---}}\boxed{2}\overset{m}{\text{---}}\boxed{m}$ for some distinct $i,j,k,m \in T$.*
(1) *If x is not a left extension of w_{ijk}, then $x \in \Gamma_{ijk}$.*
(2) *If x is a left extension of w_{ijk} but is not a left extension of $x_{i'j'k'}$ for any permutation i',j',k' of i,j,k, then $x \in \Gamma_{w_{ijk}}$.*
(3) *If x is a left extension of x_{ijk} but is not a left extension of $x_{i'j'k'}$ for any permutation i',j',k' of i,j,k with $k \neq k'$, then $x \in \Gamma_{x_{ijk}}$.*
(4) *If x is a left extension of both x_{ijk} and $x_{i'j'k'}$ with $\{i,j,k\}=\{i',j',k'\}$ and $k \neq k'$, then $x \in \Gamma_{z_{mnp}}$, where $\{m,n,p\}=\{i,j,k\}$ and $z_{mnp} \in \Sigma$.*

6.6 When $x \in W_a$ is in the exceptional case (3), there are two elements $y, y' \in \Sigma$ which have the same generalized τ-invariant as x, where $y \in M(\mathbf{121})$ and $y' \in M(\mathbf{013420})$. There are two ways to determine the left cell Γ_x. One is to see whether graph $\mathscr{M}(x)$ is finite or not. We have

$$\Gamma_x = \begin{cases} \Gamma_{y'} & \text{if } \mathscr{M}(x) \text{ is finite;} \\ \Gamma_y & \text{otherwise.} \end{cases} \tag{6.6.1}$$

Another is to see whether x is a left extension of y'. We have

$$\Gamma_x = \begin{cases} \Gamma_{y'} & \text{if } x \text{ is a left extension of } y'; \\ \Gamma_y & \text{otherwise.} \end{cases} \tag{6.6.2}$$

The second way is based on the fact that y' is the unique shortest element in the left cell $\Gamma_{y'}$ and that any element of $\Gamma_{y'}$ is a left extension of y'.

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