Kanenobu, T.
Osaka J. Math.
29 (1992), 635–651

GENUS AND KAUFFMAN POLYNOMIAL OF A 2-BRIDGE KNOT

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(Received September 4, 1991)

In [6, Theorem 6], arbitrarily many 2-bridge knots sharing the same Jones polynomial are constructed. The construction is as follows (See Example 2.): Two 2-bridge knots 10_{22} and 10_{35} [15, Table] are obtained as symmetric skew unions [11] of 5_2. They have the same Jones polynomial [4] but have distinct genera, and so have distinct Alexander polynomials [3,14]. From these knots, we get four 2-bridge knots as symmetric skew unions. Continuing this construction, we have 2^N distinct 2-bridge knots with the same Jones polynomial for any positive integer N. The question is whether the Alexander polynomials, or genera, of these 2-bridge knots are mutually distinct or not.

In [7, Theorem 5], we constructed a pair of 2-bridge knots which have the same Kauffman polynomial and so have the same Jones polynomial, but have distinct Alexander polynomials. In fact, in the set of all the 2-bridge knots through 22 crossings, there are 239 pairs sharing the same Kauffman polynomial, among which 58 pairs also share the same homfly polynomial and the rest have distinct Alexander polynomials [9]. Also in [8], we constructed arbitrarily many skein equivalent 2-bridge knots with the same Kauffman polynomial, and so they have the same homfly, Jones and Alexander polynomials. We refer [13] for the definition of the skein equivalence and the homfly polynomial, and [10] for the Kauffman and L polynomials and the writhe. In this paper, we prove:

Theorem, *For any positive integer N, there exist N 2-bridge knots with the same Kauffman polynomial but distinct genera.*

Also for 2-bridge links, we can construct a similar example. In Section 1, we give a geometric algorithm ((8) and Proposition 2) deforming a 2-bridge knot or link in the form of a 4-plat or Conway's normal form $C(a_1, a_2, \cdots, a_n)$ [2] into the special form $D(b_1, b_2, \cdots, b_{2g+\mu-1})$, where g is the genus and μ the number of the components. Throughout the deformation, the 2-bridge knot or link

* This work was supported in part by Grant-in-Aid for Encouragement of Young Scientist (No. 02740046), Ministry of Education, Science and Culture.

is kept in a 4-plat. In Section 2, we prove the theorem using this algorithm and the method to construct many 2-bridge knots sharing the same Kauffman polynomial given in [8].

1. Genus of a 2-bridge link

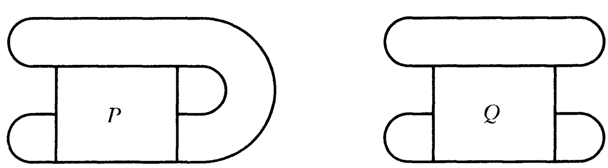
Let S_1 and S_2 be the elementary braids generating the 3-braid group as shown in Fig. 1. We denote by $C(a_1, a_2, \cdots, a_n)$ the unoriented 2-bridge knot or link (or diagram, according to the context) as shown in Fig. 2 [2]. There P and Q are the 3-braids $S_2^{a_1} S_1^{-a_2} S_2^{a_3} \cdots S_1^{-a_n}$ and $S_2^{a_1} S_1^{-a_2} S_2^{a_3} \cdots S_2^{a_n}$, respectively, depending as whether n is even or odd. However, allowing the augmentation

$$C(x_1, x_2, \cdots, x_n) \approx C(x_1, x_2, \cdots, x_n - \varepsilon, \varepsilon), \quad \varepsilon = \pm 1,$$

we may fix the parity of n.

S_1 S_2

Fig. 1



n even n odd

Fig. 2

We shall use the following without mention: If $a_i = 0$, then

$$C(a_1, a_2, \cdots, a_n) = \begin{cases} C(a_3, \cdots, a_n), & i = 1; \\ C(a_1, \cdots, a_{i-1} + a_{i+1}, \cdots, a_n), & 2 \leq i \leq n-1; \\ C(a_1, \cdots, a_{n-2}), & i = n, \end{cases}$$

We get the coprime integers $p(>0)$ and q from the continued fraction

$$\text{(1)} \qquad \frac{p}{q} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{}{\ddots + \cfrac{1}{a_n}}}}$$

which we shall denote by $[a_1, a_2, \cdots, a_n]$. Then the following is well-known (cf. 1, Chap. 12]): The 2-fold covering space of S^3 branched over $C(a_1, a_2, \cdots, a_n)$ is the lens space $L(p, q)$. $C(a_1, a_2, \cdots, a_n)$ is a knot iff p is odd. $C(a_1, a_2, \cdots, a_n)$ is ambient isotopic to the 2-bridge knot or link in the Schubert's normal form $S(p, q')$ [16] as an unoriented knot or link, where $q' \equiv q \bmod p$ with q' odd and $|q'| < p$.

These integers p, q are also calculated as follows: For integers $a_1, a_2, \cdots, a_n$, let $E[a_1, a_2, \cdots, a_n]$ be the function defined by $E[\emptyset] = E[\ ] = 1$, $E[a_1] = a_1$ and

$$\text{(2)} \qquad E[a_1, a_2, \cdots, a_n] = E[a_1, a_2, \cdots, a_{n-2}] + a_n E[a_1, a_2, \cdots, a_{n-1}]\,,$$

which is called the Euler bracket function in [12] and is denoted by $p_{12\cdots k}$ in [17]. Then the following hold:

$$\text{(3)} \qquad [a_1, a_2, \cdots, a_n] = \frac{E[a_1, a_2, \cdots, a_n]}{E[a_2, \cdots, a_n]}\,;$$

$$\text{(4)} \qquad E[a_1, a_2, \cdots, a_n] = E[a_n, \cdots, a_2, a_1]\,;$$

$$\text{(5)} \qquad E[-a_1, -a_2, \cdots, -a_n] = (-1)^n E[a_1, a_2, \cdots, a_n]\,;$$

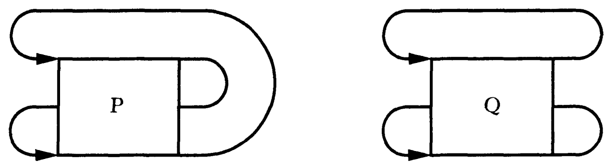
$$\text{(6)} \qquad \begin{pmatrix} 1 & a_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & a_3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_4 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & 0 \\ a_n & 1 \end{pmatrix}$$
$$= \begin{pmatrix} E[a_1, \cdots, a_n] & E[a_1, \cdots, a_{n-1}] \\ E[a_2, \cdots, a_n] & E[a_2, \cdots, a_{n-1}] \end{pmatrix}, \quad n \text{ is even;}$$

$$\text{(7)} \qquad \begin{pmatrix} 1 & a_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & a_3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a_4 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & a_n \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} E[a_1, \cdots, a_{n-1}] & E[a_1, \cdots, a_n] \\ E[a_2, \cdots, a_{n-1}] & E[a_2, \cdots, a_n] \end{pmatrix}, \quad n \text{ is odd.}$$

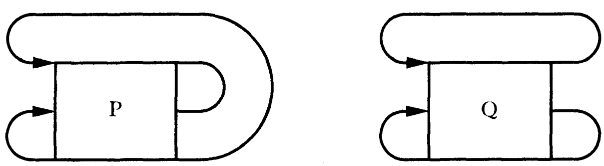
Since a 2-bridge knot or link is invertible (cf. [1, Proposition 12.5]), if $C(a_1, a_2, \cdots, a_n)$ is endowed with any orientation, it is ambient isotopic to either $\vec{C}(a_1, a_2, \cdots, a_n)$ or $\overleftarrow{C}(a_1, a_2, \cdots, a_n)$ as shown in Fig. 3.

Proposition 1. *For an oriented 2-bridge knot* $\vec{C}(a_1, a_2, \cdots, a_n)$ *with* (1), q *is even, and for* $\overleftarrow{C}(a_1, a_2, \cdots, a_n)$ *with* (1), q *is odd.*

Proof. We prove for the case n is even. Let $\mathfrak{S}_3$ be the symmetric group



$\vec{C}(a_1, a_2, \ldots, a_n)$



$\overleftarrow{C}(a_1, a_2, \ldots, a_n)$

Fig. 3

on 3 letters. There is a homomorphism $\Phi\colon B_3 \to \mathfrak{S}_3$ defined by $\Phi(S_i) = (i\ i+1)$, $i=1, 2$. Let $SL(2, 2)$ be the group of all 2×2 matrices of integer (mod 2) of determinant 1. Then the map $\Psi\colon \mathfrak{S}_3 \to SL(2, 2)$ given by

$$\Psi(1\ 2) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \Psi(2\ 3) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

$$\Psi(1\ 3\ 2) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \Psi(1\ 2\ 3) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad \Psi(1\ 3) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is an isomorphism. Consider the equation (6) in $SL(2, 2)$, we have

$$\Psi\Phi(P) = \begin{pmatrix} p_0 & r_0 \\ q_0 & s_0 \end{pmatrix},$$

where $p_0 \equiv p$, $q_0 \equiv q \bmod 2$. If p is odd and q is even (resp. odd), then $\Phi(P)=1$ or $(2\ 3)$ (resp. $(1\ 2)$ or $(1\ 3\ 2)$), and so $C(a_1, a_2, \cdots, a_n)$ is oriented as $\vec{C}(a_1, a_2, \cdots, a_n)$ (resp. $\overleftarrow{C}(a_1, a_2, \cdots, a_n)$). See Fig. 4. This completes the proof.

Let us consider the ambient isotopy:

$$\overleftarrow{C}(x_1, x_2, \cdots, x_k) \approx \vec{C}(-\varepsilon, \varepsilon - x_1, -x_2, \cdots, -x_k)\,, \tag{8}$$

where $\varepsilon = \pm 1$. Fig. 5 illustrates the case where $x_1=3$, $x_2=4$, $\varepsilon=1$, and R and

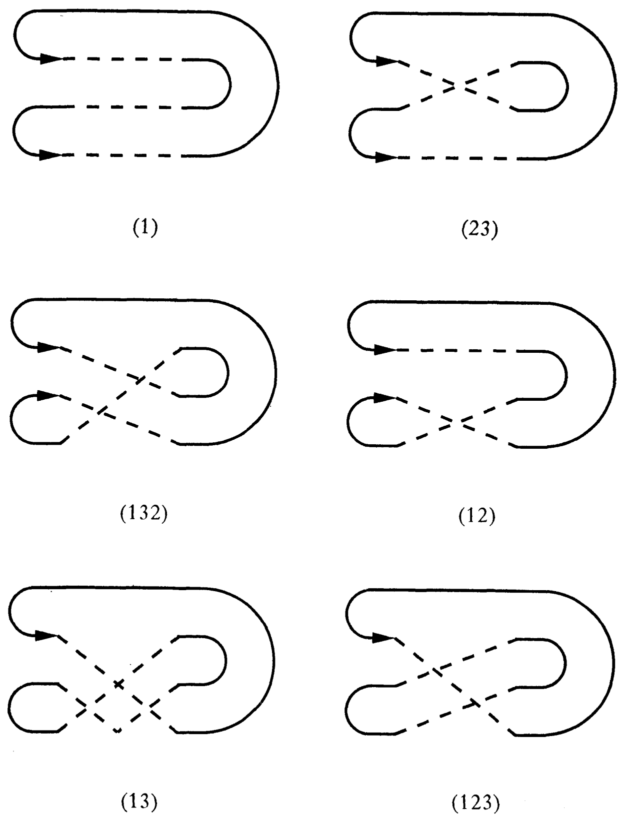


Fig. 4

$\tilde{R}$ are the 3-braids $S_2^{x_3}S_1^{-x_4}\cdots S_2^{x_{k-1}}S_1^{-x_k}$ and $S_1^{x_3}S_2^{-x_4}\cdots S_2^{x_{k-1}}S_1^{-x_k}$, respectively, with k even. Note that if $\varepsilon=x_1/|x_1|$, then this isotopy does not change the crossing number of the diagram.

Let us consider another ambient isotopy:

$$\text{(9)} \qquad \vec{C}(x_1, \cdots, x_k, a+\varepsilon, -(b+\varepsilon), -y_1, \cdots, -y_l) \approx \vec{C}(x_1, \cdots, x_k, a, \varepsilon, b, y_1, \cdots, y_l)\,,$$

where $\varepsilon=\pm 1$. Fig. 6 illustrates the case where $a=3$, $b=2$, $\varepsilon=1$, and P, Q, and $\tilde{Q}$ are the 3-braids $S_2^{x_1}S_1^{-x_2}\cdots S_1^{-x_k}$, $S_1^{y_1}S_2^{-y_2}\cdots S_1^{y_l}$, and $S_2^{y_1}S_1^{-y_2}\cdots S_2^{y_l}$, respectively. Using (9) twice, we obtain

$$\vec{C}(x_1, \cdots, x_k, -y_1, \cdots, -y_l, z_1, \cdots, z_m)$$
$$\approx \begin{cases} \vec{C}(x_1, \cdots, x_{k-1}, x_k-\varepsilon, \varepsilon, y_1-2\varepsilon, \varepsilon, z_1-\varepsilon, z_2, \cdots, z_m) & \text{if } l=1; \\ \vec{C}(x_1, \cdots, x_{k-1}, x_k-\varepsilon, \varepsilon, y_1-\varepsilon, y_2, \cdots, y_{l-1}, y_l-\varepsilon, \varepsilon, z_1-\varepsilon, z_2, \cdots, z_m) & \\ & \text{if } l\geq 2, \end{cases}$$

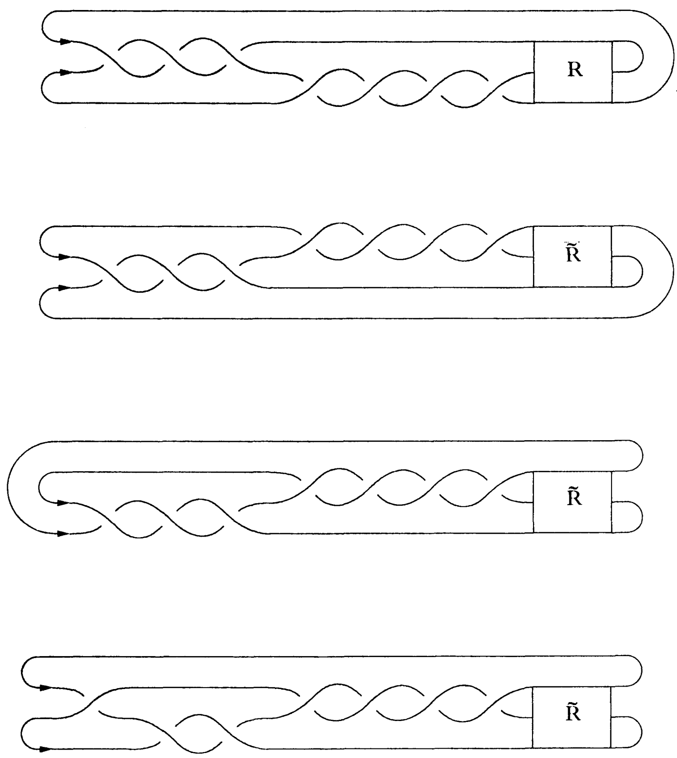


Fig. 5

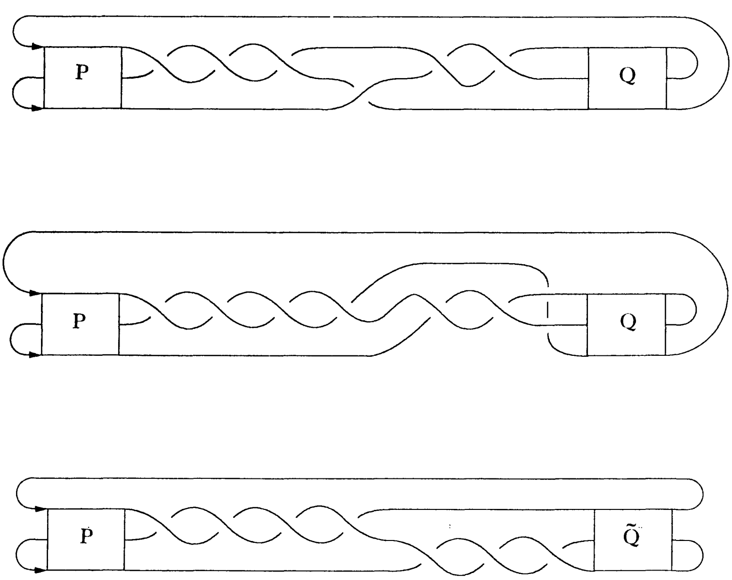


Fig. 6

where $\varepsilon=\pm 1$. In particular, if $l=1$ and $y_1=2\varepsilon$, then

$$(10)\qquad \vec{C}(x_1, \cdots, x_k, -2\varepsilon, z_1, \cdots, z_m) \approx \vec{C}(x_1, \cdots, x_{k-1}, x_k-\varepsilon, 2\varepsilon, z_1-\varepsilon, z_2, \cdots, z_m)\,.$$

Lemma 1. *The following isotopies are realized by a finite sequence of the deformation* (9).

$$(11)\qquad \vec{C}(x_1, \cdots, x_k, x, a, y, y_1, \cdots, y_l) \approx \vec{C}(x_1, \cdots, x_k, x+\alpha, ||a||, (-1)^a(y+\alpha), (-1)^a y_1, \cdots, (-1)^a y_l),$$

$$(12)\qquad \vec{C}(x_1, \cdots, x_k, x, a) \approx \vec{C}(x_1, \cdots, x_k, x+\alpha, ||a||),$$

where $\alpha|a|=a\neq 0$ *and* $||a||=(\underbrace{-2\alpha, 2\alpha, \cdots, (-1)^{a-1}2\alpha}_{|a|-1})$.

Proof. We only prove (11) for $a>0$. Using (10), we have

$$\begin{aligned}
&\vec{C}(x_1, \cdots, x_k, x, a, y, y_1, \cdots, y_l)\\
&\approx \vec{C}(x_1, \cdots, x_k, x, 2, 0, a-2, y, y_1, \cdots, y_l)\\
&\approx \vec{C}(x_1, \cdots, x_k, x+1, -2, 1, a-2, y, y_1, \cdots, y_l)\\
&\approx \vec{C}(x_1, \cdots, x_k, x+1, -2, 1, 2, 0, a-4, y, y_1, \cdots, y_l)\\
&\approx \vec{C}(x_1, \cdots, x_k, x+1, -2, 2, -2, 1, a-4, y, y_1, \cdots, y_l)
\end{aligned}$$

If a is even, then we have

$$\begin{aligned}
&\vec{C}(x_1, \cdots, x_k, x, a, y, y_1, \cdots, y_l)\\
&\approx \vec{C}(x_1, \cdots, x_k, x+1, \underbrace{-2, 2, \cdots, -2}_{a-1}, 1, 0, y, y_1, \cdots, y_l)\\
&\approx \vec{C}(x_1, \cdots, x_k, x+1, ||a||, y+1, y_1, \cdots, y_l).
\end{aligned}$$

If a is odd, then by (9), we have

$$\begin{aligned}
&\vec{C}(x_1, \cdots, x_k, x, a, y, y_1, \cdots, y_l)\\
&\approx \vec{C}(x_1, \cdots, x_k, x+1, \underbrace{-2, 2, \cdots, -2}_{a-2}, 1, 1, y, y_1, \cdots, y_l)\\
&\approx \vec{C}(x_1, \cdots, x_k, x+1, ||a||, -y-1, -y_1, \cdots, -y_l),
\end{aligned}$$

and this completes the proof.

We denote by $D(b_1, b_2, \cdots, b_m)$ the oriented 2-bridge knot or link $\vec{C}(2b_1, 2b_2, \cdots, 2b_m)$. This is of genus $(m-\mu+1)/2$, μ is the number of the components, if $b_i\neq 0$ for any i [17].

Proposition 2. *An oriented 2-bridge knot or link $\vec{C}(a_1, a_2, \cdots, a_n)$ can be deformed into $D(b_1, b_2, \cdots, b_m)$ by a finite sequence of the deformation* (9).

Proof. If each a_i is even, then we have nothing to do. Suppose that $a_1, a_2, \cdots, a_{p-1}$ are even and a_p is odd. If we deform $\vec{C}(a_1, a_2, \cdots, a_n)$ into $\vec{C}(e_1, e_2, \cdots, e_l)$ such that $e_1, e_2, \cdots, e_{q-1}$ are even and e_q is odd, and $n-p>l-q$, then the proof is complete by induction.

Considering the orientation, we see $p \neq n$. If $p=n-1$, then by (12), $\vec{C}(a_1, a_2, \cdots, a_n)$ is deformed into $\vec{C}(a_1, a_2, \cdots, a_{n-2}, a_{n-1}+\alpha_n, ||a_n||)$, where $\alpha_n=a_n/|a_n|$. Thus $1=n-p>l-q=0$. If $1 \leq p \leq n-2$, then by (11), $\vec{C}(a_1, a_2, \cdots, a_n)$ is deformed into $\vec{C}(a_1, a_2, \cdots, a_{p-1}, a_p+\alpha_{p+1}, ||a_{p+1}||, (-1)^{a_{p+1}}(a_{p+2}+\alpha_{p+1}), (-1)^{a_{p+1}}a_{p+3}, \cdots, (-1)^{a_{p+1}}a_n)$, where $\alpha_{p+1}=a_{p+1}/|a_{p+1}|$. Thus $n-p-1>l-q$, and the proof is complete.

REMARK. In a similar way, we can prove: A 2-bridge knot or link diagram $C(a_1, a_2, \cdots, a_n)$ can be deformed into an alternating diagram $C(e_1, e_2, \cdots, e_m)$, $e_i>0$, by a finite sequence of the ambient isotopy:

$$\begin{aligned} &C(x_1, \cdots, x_k, a, \varepsilon, b, y_1, \cdots, y_l) \\ &\qquad \approx C(x_1, \cdots, x_k, a+\varepsilon, -(b+\varepsilon), -y_1, \cdots, -y_l), \end{aligned}$$

which is an unoriented version of (9).

Proposition 3. (i) *If $[x_1, x_2, \cdots, x_k]=p/q$, then*

$$[-\varepsilon, \varepsilon-x_1, -x_2, \cdots, -x_k] = p/(q-\varepsilon p)\,.$$

(ii) $[x_1, \cdots, x_k, a, \varepsilon, b, y_1, \cdots, y_l]=[x_1, \cdots, x_k, a+\varepsilon, -(b+\varepsilon), -y_1, \cdots, -y_l]$.

Proof. (i) By using (2), (4) and (5), we have

$$E[\varepsilon-x_1, -x_2, \cdots, -x_k] = (-1)^k(E[x_1, \cdots, x_k]-\varepsilon E[x_2, \cdots, x_k])\,,$$

and

$$E[-\varepsilon, \varepsilon-x_1, -x_2, \cdots, -x_k] = (-1)^{k+1}\varepsilon E[x_1, \cdots, x_k]\,.$$

Then by (3), the proof is complete.

(ii) We can prove the following formula [12, Lemma 9] by induction on l:

$$\begin{aligned} &E[x_1, \cdots, x_k, a, 1, b, y_1, \cdots, y_l] \\ &\qquad = (-1)^{l+1}E[x_1, \cdots, x_k, a+1, -(b+1), -y_1, \cdots, y_l]\,. \end{aligned}$$

Using this and (3), (5), we obtain the result.

Suppose that a 2-bridge knot or link L is given in the form $\overleftarrow{C}(a_1, a_2, \cdots, a_n)$ or $\vec{C}(a_1, a_2, \cdots, a_n)$ with (1), from Propositions 1, 2 and 3, we have:

Proposition 4. *Let*

$$q' \equiv \begin{cases} q-p & \text{if } L=\overleftarrow{C}(a_1, a_2, \cdots, a_n)\,; \\ q & \text{if } L=\vec{C}(a_1, a_2, \cdots, a_n)\,, \end{cases}$$

mod $2p$, *with* $|q'|<p$. *Then*

$$L \approx D(b_1, b_2, \cdots, b_m)\,,$$

where

$$p/q' = [2b_1, 2b_2, \cdots, 2b_m]\,.$$

EXAMPLE 1. We apply (8) and Propositions 2 and 4:

$$\text{(i)} \qquad \begin{aligned} \vec{C}(1, -1, 2, 3) &\approx \vec{C}(0, ||-1||, -(2-1), -3) = \vec{C}(0, -1, -3) \quad (11) \\ &\approx \vec{C}(0, -2, ||-3||) = \vec{C}(0, -2, 2, -2) \quad (12) \\ &\approx \vec{C}(2, -2) = D(1, -1)\,. \end{aligned}$$

On the other hand, $[1, -1, 2, 3]=3/(-4)=[0, -2, 2, -2]$.

$$\text{(ii)} \qquad \begin{aligned} \overleftarrow{C}(1, 2, 3) &\approx \vec{C}(-1, 0, -2, -3) = \vec{C}(-3, -3) \quad (8) \\ &\approx \vec{C}(-4, ||-3||) = \vec{C}(-4, 2, -2) = D(-2, 1, -1)\,. \end{aligned}$$

On the other hand, $[1, 2, 3]=10/7$, and $10/(-3)=[-4, 2, -2]$.

For

$$A = (2a_1, 2b_1, 2a_2, 2b_2, \cdots, 2a_n, 2b_n)$$

and $\varepsilon=\pm 1$, let

$$A^{-1} = (-2b_n, -2a_n, \cdots, -2b_2, -2a_2, -2b_1, -2a_1)$$

and

$$\begin{aligned} &A(\varepsilon) = \\ &(||2a_1||, 2b_1+\alpha_1+\alpha_2, ||2a_2||, \cdots, 2b_{n-1}+\alpha_{n-1}+\alpha_n, ||2a_n||, 2b_n+\alpha_n+\varepsilon, \\ &2b_n+\alpha_n-\varepsilon, ||2a_n||, 2b_{n-1}+\alpha_{n-1}+\alpha_n, \cdots, ||2a_2||, 2b_1+\alpha_1+\alpha_2, ||2a_1||), \end{aligned}$$

where $\alpha_i=a_i/|a_i|$. Then we have:

Lemma 2. $\overleftarrow{C}(A, \varepsilon, A^{-1})\approx\vec{C}(A(\varepsilon))$.

Proof. By (8),

$$\overleftarrow{C}(A, \varepsilon, A^{-1})$$
$$\approx \vec{C}(-\alpha_1, \alpha_1-2a_1, -2b_1, -2a_2, -2b_2, \cdots, -2a_n, -2b_n, -\varepsilon, 2b_n, 2a_n, \cdots, 2b_2, 2a_2, 2b_1, 2a_1).$$

Using (9), (11), and (12), we deform this as follows:

$$\vec{C}(-2\alpha_1, ||\alpha_1-2a_1||, 2b_1+\alpha_1, 2a_2, 2b_2, \cdots, 2a_n, 2b_n, \varepsilon, -2b_n, -2a_n, \cdots, -2b_2, -2a_2, -2b_1, -2a_1)$$
$$\approx \vec{C}(||2a_1||, 2b_1+\alpha_1+\alpha_2, ||2a_2||, 2b_2+\alpha_2, 2a_3, 2b_3, \cdots, 2a_n, 2b_n, \varepsilon, -2b_n, -2a_n, \cdots, -2b_2, -2a_2, -2b_1, -2a_1)$$
$$\cdots$$
$$\approx \vec{C}(||2a_1||, 2b_1+\alpha_1+\alpha_2, ||2a_2||, 2b_2+\alpha_2+\alpha_3, ||2a_3||, \cdots, 2b_{n-1}+\alpha_{n-1}+\alpha_n, ||2a_n||, 2b_n+\alpha_n, \varepsilon, -2b_n, -2a_n, \cdots, -2b_2, -2a_2, -2b_1, -2a_1)$$
$$\approx \vec{C}(||2a_1||, 2b_1+\alpha_1+\alpha_2, ||2a_2||, 2b_2+\alpha_2+\alpha_3, ||2a_3||, \cdots, 2b_{n-1}+\alpha_{n-1}+\alpha_n, ||2a_n||, 2b_n+\alpha_n+\varepsilon, 2b_n-\varepsilon, 2a_n, 2b_{n-1}, 2a_{n-1}, \cdots, 2b_2, 2a_2, 2b_1, 2a_1)$$
$$\approx \vec{C}(||2a_1||, 2b_1+\alpha_1+\alpha_2, ||2a_2||, 2b_2+\alpha_2+\alpha_3, ||2a_3||, \cdots, 2b_{n-1}+\alpha_{n-1}+\alpha_n, ||2a_n||, 2b_n+\alpha_n+\varepsilon, 2b_n+\alpha_n-\varepsilon, ||2a_n||, 2b_{n-1}+\alpha_n, 2a_{n-1}, \cdots, 2b_2, 2a_2, 2b_1, 2a_1)$$
$$\approx \vec{C}(||2a_1||, 2b_1+\alpha_1+\alpha_2, ||2a_2||, 2b_2+\alpha_2+\alpha_3, ||2a_3||, \cdots, 2b_{n-1}+\alpha_{n-1}+\alpha_n, ||2a_n||, 2b_n+\alpha_n+\varepsilon, 2b_n+\alpha_n-\varepsilon, ||2a_n||, 2b_{n-1}+\alpha_{n-1}+\alpha_n, ||2a_{n-1}||, \cdots, 2b_2+\alpha_2+\alpha_3, ||2a_2||, 2b_1+\alpha_2, 2a_1)$$
$$\approx \vec{C}(A(\varepsilon)).$$

This completes the proof.

EXAMPLE 2. Let K be a 2-bridge knot $D(a_1, b_1, a_2, b_2, \cdots, a_n, b_n)=\overleftarrow{C}(A)$ of genus n. Then $\overleftarrow{C}(A, \varepsilon, A^{-1})$and $\overleftarrow{C}(A^{-1}, \varepsilon, A)$, $\varepsilon=\pm 1$, have the same Jones polynomial [6, Lemma 6.2], which are symmetric skew union [11] of K. Let g_1 and g_2 be the genera of these knots. Then by Lemma 2, we have

$$2\sum_{i=1}^{n} |a_i|-2n+1\leq g_1\leq 2\sum_{i=1}^{n} |a_i|,$$

and

$$2\sum_{i=1}^{n} |b_i|-2n+1\leq g_2\leq 2\sum_{i=1}^{n} |b_i|.$$

Let

$$X=(x_1, x_2, \cdots, x_k),$$
$$Y=(y_1, y_2, \cdots, y_l),$$
$$-Y=(-y_1, -y_2, \cdots, -y_l).$$

Lemma 3. *If* $\varepsilon_i=\pm 1$, *then*

$$\vec{C}(X, \varepsilon_1, A, \varepsilon_2, A^{-1}, \varepsilon_3, Y) \approx \vec{C}(X, \alpha_1+\varepsilon_1, A(\varepsilon_2), \alpha_1-\varepsilon_3, -Y).$$

Proof. Using (9) and (11), we calculate as follows:

$$\begin{aligned}
&\vec{C}(X, \varepsilon_1, A, \varepsilon_2, A^{-1}, \varepsilon_3, Y)\\
&\quad\approx \vec{C}(X, \varepsilon_1+\alpha_1, ||2a_1||, 2b_1+\alpha_1, 2a_2, 2b_2, \cdots, 2a_n, 2b_n, \varepsilon_2, A^{-1}, \varepsilon_3, Y)\\
&\quad\approx \vec{C}(X, \varepsilon_1+\alpha_1, ||2a_1||, 2b_1+\alpha_1+\alpha_2, ||2a_2||, 2b_2+\alpha_2, 2a_3, 2b_3, \cdots,\\
&\qquad\qquad 2a_n, 2b_n, \varepsilon_2, A^{-1}, \varepsilon_3, Y)\\
&\quad\cdots\\
&\quad\approx \vec{C}(X, \varepsilon_1+\alpha_1, ||2a_1||, 2b_1+\alpha_1+\alpha_2, ||2a_2||, 2b_2+\alpha_2+\alpha_3, \cdots,\\
&\qquad\qquad ||2a_n||, 2b_n+\alpha_n, \varepsilon_2, A^{-1}, \varepsilon_3, Y)\\
&\quad\approx \vec{C}(X,\varepsilon_1+\alpha_1, ||2a_1||, 2b_1+\alpha_1+\alpha_2, ||2a_2||, 2b_2+\alpha_2+, \alpha_3, \cdots,\\
&\qquad\qquad ||2a_n||, 2b_n+\alpha_n+\varepsilon_2, 2b_n-\varepsilon_2, 2a_n, 2b_{n-1}, 2a_{n-1}, \cdots,\\
&\qquad\qquad 2b_1, 2a_1, -\varepsilon_3, -Y)\\
&\quad\approx \vec{C}(X, \varepsilon_1+\alpha_1, ||2a_1||, 2b_1+\alpha_1+\alpha_2, ||2a_2||, 2b_2+\alpha_2+\alpha_3, \cdots,\\
&\qquad\qquad ||2a_n||, 2b_n+\alpha_n+\varepsilon_2, 2b_n+\alpha_n-\varepsilon_2, ||2a_n||, 2b_{n-1}+\alpha_n, 2a_{n-1}, \cdots,\\
&\qquad\qquad 2b_1, 2a_1, -\varepsilon_3, -Y)\\
&\quad\cdots\\
&\quad\approx \vec{C}(X, \alpha_1+\varepsilon_1, A(\varepsilon_2), \alpha_1-\varepsilon_3, -Y).
\end{aligned}$$

This completes the proof.

Using this lemma twice, we get:

Corollary. *If* $\varepsilon_i=\pm 1$, *then*

$$\begin{aligned}
&\vec{C}(X, \varepsilon_1, A^{-1}, \varepsilon_2, A, \varepsilon_3, A^{-1}, \varepsilon_4, A, \varepsilon_5, A^{-1}, \varepsilon_6, Y)\\
&\quad\approx \vec{C}(X, \varepsilon_1-\beta_m, (A^{-1})(\varepsilon_2), -\varepsilon_3-\beta_m, -A^{-1}, -\varepsilon_4-\alpha_1, (-A)(-\varepsilon_5), \varepsilon_6-\alpha_1, Y).
\end{aligned}$$

EXAMPLE 3. The oriented 2-bridge link $L^\wedge=\overleftarrow{C}(2a_1, 2b_1, \cdots, 2a_g, 2b_g, 2a_{g+1})$ is obtained from the 2-bridge link $L=\vec{C}(2a_1, 2b_1,\cdots, 2a_g, 2b_g, 2a_{g+1})$ of genus g by reversing the orientation of one of the two components. As in Example 2, let us compute $g^\wedge$, the genus of $L^\wedge$:

$$\begin{aligned} L^\wedge &\approx \vec{C}(-\alpha_1, -2a_1+\alpha_1, -2b_1, \cdots, -2a_g, -2b_g, -2a_{g+1}) \\ &\approx \vec{C}(||2a_1||, 2b_1+\alpha_1, 2a_2, 2b_2, \cdots, 2a_g, 2b_g, 2a_{g+1}) \\ &\approx \vec{C}(||2a_1||, 2b_1+\alpha_1+\alpha_2, ||2a_2||, 2b_2+\alpha_2+\alpha_3, \cdots, \\ &\qquad\qquad ||2a_g||, 2b_g+\alpha_g, 2a_{g+1}) \\ &\approx \vec{C}(||2a_1||, 2b_1+\alpha_1+\alpha_2, ||2a_2||, 2b_2+\alpha_2+\alpha_3, \cdots, \\ &\qquad\qquad ||2a_g||, 2b_g+\alpha_g+\alpha_{g+1}, ||2a_{g+1}||), \end{aligned}$$

where $\alpha_i = a_i/|a_i|$. Let $I = \#\{i \mid 2b_i+\alpha_i+\alpha_{i+1}=0\}$. Then $g^\wedge = \sum_{i=1}^{g+1}|a_i| - I - 1$. Since $0 \leq I \leq g$, we obtain

$$g, g^\wedge \leq \sum_{i=1}^{g+1} |a_i| - 1 \leq g + g^\wedge .$$

Note that $\sum_{i=1}^{g+1}|a_i| - 1$ is just the t_1-degree ($=t_2$-degree) of the Alexander polynomial of L $\Delta_L(t_1, t_2)$, that is, (maximum t_1-power of any term of $\Delta_L(t_1, t_2)$) minus (minimum t_1-power of any term of $\Delta_L(t_1, t_2)$). Cf. [5, Corollary 2].

2. Proof of Theorem

For $A=(2a_1, 2a_g, 2a_2, 2a_{g-1}, \cdots, 2a_g, 2a_1)$, $a_i \neq 0$, we define $(2g-1)(k+1)-1$-tuple of integers

$$A[\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_k] = (A, \varepsilon_1, -A, \varepsilon_2, \cdots, (-1)^{k-1}A, \varepsilon_k, (-1)^k A),$$

where $\varepsilon_i = \pm 1$ and

$$-A = A^{-1} = (-2a_1, -2a_g, -2a_2, -2a_{g-1}, \cdots, -2a_g, -2a_1).$$

Then $C(A[\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_k])$ is a 2-bridge knot iff $k \equiv 0, 1 \bmod 3$, which we may orient as follows:

$$\begin{aligned} &\vec{C}(A[\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_k]) \qquad \text{if } k \equiv 0 \bmod 3; \\ &\overleftarrow{C}(A[\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_k]) \qquad \text{if } k \equiv 1 \bmod 3. \end{aligned}$$

We define $A_n = A\langle p_0, p_1, \cdots, p_{n-1}\rangle$, $p_i = \pm 1$, as follows:

$$\begin{aligned} &A_0 = A; \\ &A_{i+1} = (A_i, p_i - A_i, p_i, A_i, p_i - A_i, p_i, A_i), \qquad i = 0, 1, \cdots, n. \end{aligned}$$

For $j \equiv 0 \bmod 5^i$ and $j \not\equiv 0$ omd 5^{i+1}, put

$$q_j = \begin{cases} p_i & \text{if } j/5^i \equiv 1, 2, 3, 4 \bmod 10; \\ -p_i & \text{if } j/5^i \equiv 6, 7, 8, 9 \bmod 10. \end{cases}$$

Then $A_n = A[q_1, q_2, \cdots, q_m]$, $m = 5^n - 1$. We shall consider the oriented 2-bridge knots $\vec{C}(A_n)$ (n is even) and $\overleftarrow{C}(A_n)$ (n is odd).

We divide the proof of Theorem into:

Assertion 1. *All the* 2*-bridge knots*

$$\overleftarrow{C}(A\langle p_0, p_1, \cdots, p_{n-1}\rangle), \quad p_i=\pm 1,$$

share the same Kauffman polynomial.

Assertion 2. *Suppose that n is odd and $a_1 \geq 2$. Let $g_n=g\langle p_0, p_1, \cdots, p_{n-1}\rangle$ be the genus of $\overleftarrow{C}(A\langle p_0, p_1, \cdots, p_{n-1}\rangle)$. Then*

$$g_n=\frac{5^n-2}{3}g+\frac{5^n+1}{3}h+(p_0+p_1+\cdots+p_{n-1}),$$

where h is the genus of $\overleftarrow{C}(A(1))$.

Let n be an odd integer with $n \geq N$ and K_p be one of the 2-bridge knots $\overleftarrow{C}(A\langle p_0, p_1, \cdots, p_{n-1}\rangle)$ with $p=p_0+p_1+\cdots+p_{n-1}$. Then by these assertions, $\{K_{-n}, K_{-n+2}, \cdots, K_{-n+2N-2}\}$ is the desired set of the 2-bridge knots for Theorem.

We use the following notation:

$$U^{(n)}=\{1, 2, \cdots, 5^n-1\},$$

whence $U^{(n+1)}$ is the disjoint union $\coprod_{k=0}^4\{x+k\cdot 5^n \mid x\in U^{(n)}\}\coprod\{5^n, 2\cdot 5^n, 3\cdot 5^n, 4\cdot 5^n\}$;

$$T_i^{(n)}=\{x\in U^{(n)} \mid x\equiv i \bmod 6\},$$
$$T_{i,r}^{(n)}=T_i^{(n)}\cap 5^r Z,$$

whence $T_i^{(n)}=T_{i,0}^{(n)}\supset T_{i,1}^{(n)}\supset\cdots\supset T_{i,n-1}^{(n)}\supset T_{i,n}^{(n)}=\emptyset$;

$$S_{i,r}^{(n)}=T_{i,r}^{(n)}-T_{i,r+1}^{(n)}.$$

Lemma 4. *The set $S_{i,r}^{(n+1)}$ is equal to the disjoint union*

$$\coprod_{k=0}^{4}\{x+k\cdot 5^n \mid x\in S_{i+\varepsilon k,r}^{(n)}\},$$

where $r=0, 1, \cdots, n-1$, and $\varepsilon=(-1)^{n+1}$.

Proof. Suppose that $y\in S_{i,r}^{(n+1)}$. Then $y\in U^{(n+1)}$, $y\equiv i \bmod 6$, $y\equiv 0 \bmod 5^r$, and $y\not\equiv 0 \bmod 5^{r+1}$. Thus for some k, $0\leq k\leq 4$, $y=x+k\cdot 5^n$, $x\in U^{(n)}$, and so $x=y-k\cdot 5^n\equiv i-k\cdot(-1)^n=i+\varepsilon k \bmod 6$, $x\equiv 0 \bmod 5^r$ and $x\not\equiv 0 \bmod 5^{r+1}$, that is, $x\in S_{i+\varepsilon k,r}^{(n)}$. The converse can be seen similarly. The proof is complete.

Lemma 5. *If n is odd, then*

$$\sum_{x\in S_{i,r}^{(n)}} q_x=\begin{cases} 0 & \text{if } i=0, 5;\\ p_r & \text{if } i=1, 2, 3, 4,\end{cases}$$

and if n is even, then

$$\sum_{x\in S_{i,r}^{(n)}} q_x = \begin{cases} 0 & \textit{if } i=0,1; \\ p_r & \textit{if } i=2,3,4,5, \end{cases}$$

where $0\leq r\leq n-1$.

Proof. We prove by induction on n. Since

$$S_{i,0}^{(1)} = T_i^{(1)} = \begin{cases} \emptyset & \text{if } i=0,5; \\ \{i\} & \text{if } i=1,2,3,4, \end{cases}$$

the lemma is true for $n=1$.

Suppose that the lemma is true for $n=l$. By Lemma 4,

$$\sum_{x\in S_{i,r}^{(l+1)}} q_x = \sum_{k=0}^{4}\Big(\sum_{x\in S_{i+\varepsilon k,r}^{(l)}} q_{x+k\cdot 5^l}\Big),$$

where $\varepsilon=(-1)^{l+1}$. If $x\in S_{i,r}^{(l)}$, then

$$q_x = \begin{cases} p_r & \text{if } x/5^r\equiv 1,2,3,4 \bmod 10; \\ -p_r & \text{if } x/5^r\equiv 6,7,8,9 \bmod 10. \end{cases}$$

Put $x=j\cdot 5^r$, where $j\equiv 1,2,3,4,6,7,8,9 \bmod 10$. Then $x+k\cdot 5^l=5^r(j+k\cdot 5^{l-r})$, whence $j+k\cdot 5^{l-r}\equiv j+5k \bmod 10$. Thus

$$q_{x+k\cdot 5^l} = \begin{cases} -q_x & \text{if } k=1.3; \\ q_x & \text{if } k=2,4. \end{cases}$$

Therefore we obtain

$$\sum_{x\in S_{i,r}^{(l+1)}} q_x = \sum_{x\in S_{i,r}^{(l)}} q_x + \sum_{x\in S_{i+\varepsilon,r}^{(l)}} (-q_x) + \sum_{x\in S_{i+2\varepsilon,r}^{(l)}} q_x + \sum_{x\in S_{i+3\varepsilon,r}^{(l)}} (-q_x) + \sum_{x\in S_{i+4\varepsilon,r}^{(l)}} q_x\,.$$

We consider for $i=0$. If l is even, then $\varepsilon=-1$ and

$$\begin{aligned}\sum_{x\in S_{0,r}^{(l+1)}} q_x &= \sum_{x\in S_{0,r}^{(l)}} q_x - \sum_{x\in S_{5,r}^{(l)}} q_x + \sum_{x\in S_{4,r}^{(l)}} q_x - \sum_{x\in S_{3,r}^{(l)}} q_x + \sum_{x\in S_{2,r}^{(l)}} q_x \\ &= 0-p_r+p_r-p_r+p_r \\ &= 0,\end{aligned}$$

where we use inductive hypothesis. If l is odd, then $\varepsilon=1$ and

$$\begin{aligned}\sum_{x\in S_{0,r}^{(l+1)}} q_x &= \sum_{x\in S_{0,r}^{(l)}} q_x - \sum_{x\in S_{1,r}^{(l)}} q_x + \sum_{x\in S_{2,r}^{(l)}} q_x - \sum_{x\in S_{3,r}^{(l)}} q_x + \sum_{x\in S_{4,r}^{(l)}} q_x \\ &= 0-p_r+p_r-p_r+p_r \\ &= 0,\end{aligned}$$

For $i=1, 2, 3, 4, 5$, we can prove similarly. This completes the proof.

Proof of Assertion 1. This is divided into two lemmas:

Lemma 6. *Let $\Lambda_{A\langle p_0,p_1,\cdots,p_{n-1}\rangle}$ be the L polynomial of the unoriented 2-bridge knot diagram $C(A\langle p_0, p_1, \cdots, p_{n-1}\rangle)=C(A_n)$. Then*

$$\Lambda_{A\langle p_0,p_1,\cdots,p_i,\cdots,p_{n-1}\rangle} = \Lambda_{A\langle p_0,p_1,\cdots,-p_i,\cdots,p_{n-1}\rangle} .$$

Lemma 7. *The writhes of $\overleftarrow{C}(A_n)$ (n odd) and $\vec{C}(A_n)$ (n even) are zero.*

Proof of Lemma 6. First we note

$$A\langle p_0, \cdots, p_{i-1}, p_i, \cdots, p_{n-1}\rangle = (A\langle p_0, \cdots, p_{i-1}\rangle)\langle p_i, \cdots, p_{n-1}\rangle .$$

By [8, Proposition 3], the 2-bridge knot diagrams $C(A[q_1, q_2, \cdots, q_m])$ and $C(A[-q_m, \cdots, -q_2, -q_1])$ have the same L polynomials. Thus we have

$$\Lambda_{A\langle p_0,p_1,\cdots,p_{n-1}\rangle} = \Lambda_{A\langle -p_0,-p_1,\cdots,-p_{n-1}\rangle} .$$

In the same way, we can prove

$$\Lambda_{(A\langle p_0,p_1,\cdots,p_{i-1}\rangle)\langle p_i,\cdots,p_{n-1}\rangle} = \Lambda_{(A\langle p_0,p_1,\cdots,p_{i-1}\rangle)\langle -p_i,\cdots,-p_{n-1}\rangle} .$$

So we have

$$\begin{aligned}\Lambda_{A\langle p_0,\cdots,p_{n-1}\rangle} &= \Lambda_{(A\langle p_0,\cdots,p_{i-1}\rangle)\langle -p_i,-p_{i+1},\cdots,-p_{n-1}\rangle}\\ &= \Lambda_{(A\langle p_0,\cdots,p_{i-1},-p_i\rangle)\langle -p_{i+1},\cdots,-p_{n-1}\rangle}\\ &= \Lambda_{(A\langle p_0,\cdots,p_{i-1},-p_i\rangle)\langle p_{i+1},\cdots,p_{n-1}\rangle}\\ &= \Lambda_{A\langle p_0,\cdots,p_{i-1},-p_i,p_{i+1},\cdots,p_{n-1}\rangle} ,\end{aligned}$$

and the proof is complete.

REMARK. This proof of Lemma 6 is essentially the same as that of [8, Theorem 1].

Proof of Lemma 7. We only prove when n is odd . It is easy to see that the writhe of $\overleftarrow{C}(A[\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{3k+1}])$ is

$$\varepsilon_1+\sum_{i=1}^{k}(-1)^i(-\varepsilon_{3i-1}+\varepsilon_{3i}+\varepsilon_{3i+1})$$

Thus the writhe of $\overleftarrow{C}(A_n)$ is

$$\sum_{x\in T_0^{(n)}\cup T_1^{(n)}\cup T_2^{(n)}} q_x - \sum_{x\in T_3^{(n)}\cup T_4^{(n)}\cup T_5^{(n)}} q_x$$

Since $T_i^{(n)}=\coprod_{r=0}^{n-1}S_{i,r}^{(n)}$, we obtain the result from Lemma 5.

Proof of Assertion 2. Suppose that n is odd. Then $m=5^n-1\equiv(-1)^n-1\equiv 4 \bmod 6$. Since $a_1\geq 2$, using (8) and Corollary, we have

$$
\begin{aligned}
\overleftarrow{C}(A_n) &= \overleftarrow{C}(A, q_1, -A, q_2, A, \cdots, q_m, A) \\
&\approx \vec{C}(-1, 1-2a_1, -2a_g, \cdots, -2a_g, -2a_1, -q_1, A, -q_2, \\
&\qquad\qquad -A, -q_3, A, \cdots, -q_m, -A) \\
&\approx \vec{C}(A(q_1), -q_2+1, \\
&\quad -A, -q_3, A, -q_4, -A, -q_5, A, -q_6, -A, -q_7, A, -q_8, \\
&\quad \cdots \\
&\quad -A, -q_{6k-3}, A, -q_{6k-2}, -A, -q_{6k-1}, A, -q_{6k}, -A, -q_{6k+1}, A, -q_{6k+2}, \\
&\quad \cdots \\
&\quad -A, -q_{m-1}, A, -q_m, -A) \\
&\approx \vec{C}(A(q_1), -q_2+1, \\
&\quad -A, -q_3+1, A(-q_4), q_5+1, -A, q_6+1, A(q_7), -q_8+1, \\
&\quad \cdots \\
&\quad -A, -q_{6k-3}+1, A(-q_{6k-2}), q_{6k-1}+1, -A, q_{6k}+1, A(q_{6k+1}), -q_{6k+2}+1, \\
&\quad \cdots \\
&\quad -A, -q_{m-1}+1, A(-q_m)).
\end{aligned}
$$

Since $a_1\geq 2$, the genus of $\overleftarrow{C}(A(-1))$ equals h, the genus of $\overleftarrow{C}(A(1))$.

Thus we have

$$g_n = \frac{m-1}{3}g+\frac{m+2}{3}h+\frac{1}{2}\Big(\sum_{x\in T_2^{(n)}\cup T_3^{(n)}} q_x - \sum_{x\in T_0^{(n)}\cup T_5^{(n)}} q_x\Big).$$

We obtain the desired formula by Lemma 5.

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