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Osaka J. Math.
29 (1992), 161–174

SINGULAR PERTURBATION OF SYMBOLIC FLOWS AND POLES OF THE ZETA FUNCTIONS. ADDENDUM

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(Received June 14, 1991)

1. Introduction

In the previous paper [3] we considered singular perturbations of symbolic flows and showed the existence of poles of the zeta functions associated with perturbed symbolic flows. The purpose of the present paper is to remove some conditions required in the previous paper.

Our aim in studying poles of the zeta functions is to show the validity of the modified Lax-Phillips conjecture for obstacles consisting of several small balls. The modified Lax-Phillips conjecture is concerned with the distribution of poles of scattering matrices. About this conjecture, see Lax-Phillips [8, Epilogue] and Ikawa [5]. When we want to apply Theorem 1 of the previous paper to this conjecture, we have to require some additional conditions on the configuration of the centers of balls, that is, the conditions (A.2) and (A.3) of [3, Section 4]. As a consequence of the improvement of the theorem, we can show the validity of the modified Lax-Phillips conjecture for all obstacles consisting of small balls whose centers satisfy only (A.1) of [3].

Now we shall introduce notations for the statement of our main theorem. Let L be an integer≥ 2, and let $A=[A(i,j)]_{i,j=1,2,\cdots,L}$ and $B=[B(i,j)]_{i,j=1,2,\cdots,L}$ be zero-one $L\times L$ matrices.

For $i,j\in\{1,2,\cdots,L\}$, we denote $i\underset{B}{\rightarrow}j$ when there is a sequence $i_1,i_2,\cdots,i_p$ such that $B(i_1,i)=1$, $B(i_{q+1},i_q)=1$ for $q=1,2,\cdots,p-1$ and $B(j,i_p)=1$.

We assume on B the following:

There is $1<K\leq L$ such that

$$B(i,j)=0 \qquad \text{for all } j \quad \text{if } i\geq K+1\,, \tag{1.1}$$

$$i\underset{B}{\rightarrow}i \qquad \text{for all} \quad 1\leq i\leq K\,, \tag{1.2}$$

$$i\underset{B}{\rightarrow}j \text{ implies } j\underset{B}{\rightarrow}i \qquad \text{if} \quad i,j\leq K\,. \tag{1.3}$$

We assume also the following relation between A and B:

$$B(i,j)=1 \text{ implies } A(i,j)=1\,. \tag{1.2}$$

Let f_ε, h_ε be functions with parameter $\varepsilon \geq 0$ satisfying

$$f_\varepsilon,\, h_\varepsilon \in \mathcal{F}_\theta(\Sigma_A^+) \qquad \text{for all} \quad 0 \leq \varepsilon \leq \varepsilon_1,$$

where ε_1 is a positive constant, and let $k \in \mathcal{F}_\theta(\Sigma_A^+)$ satisfy

$$\begin{cases} k(\xi) = 0 & \text{if} \quad B(\xi_1, \xi_2) = 1 \\ k(\xi) > 0 & \text{if} \quad B(\xi_1, \xi_2) = 0\,. \end{cases} \tag{1.3}$$

We assume that

$$|||f_\varepsilon - f_0|||_\theta,\ |||h_\varepsilon - h_0|||_\theta \to 0 \quad \text{as} \quad \varepsilon \to 0\,. \tag{1.4}$$

For $0 < \varepsilon \leq \varepsilon_1$, we define zeta function $Z_\varepsilon(s)$ by

$$Z_\varepsilon(s) = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \sum_{\sigma_A^n \xi = \xi} \exp(S_n r_\varepsilon(\xi, s))\right)$$

where

$$r_\varepsilon(\xi, s) = -s f_\varepsilon(\xi) + h_\varepsilon(\xi) + k(\xi) \log \varepsilon$$

and

$$S_n r_\varepsilon(\xi, s) = r_\varepsilon(\xi, s) + r_\varepsilon(\sigma_A \xi, s) + \cdots + r_\varepsilon(\sigma_A^{n-1} \xi, s)\,.$$

Our main theorem is the following

Theorem 1. *Suppose that* (1.1)$\sim$(1.4) *are satisfied, and that*

$$f_0(\xi) > 0 \qquad \textit{for all} \quad \xi \in \Sigma_A^+\,, \tag{1.5}$$

$$h_0(\xi) \textit{ is real for all } \xi \in \Sigma_A^+ \textit{ such that } B(\xi_1, \xi_2) = 1. \tag{1.6}$$

Then there exist $s_0 \in \boldsymbol{R}$, D a neighborhood of s_0 in $\boldsymbol{C}$ and $\varepsilon_0 > 0$ such that, for every $0 < \varepsilon \leq \varepsilon_0$, $Z_\varepsilon(s)$ is meromorphic in D and has a pole s_ε in D with

$$s_\varepsilon \to s_0 \qquad \textit{as} \quad \varepsilon \to 0\,.$$

Compared with Theorem 1 of the previous paper, the present one requires neither the condition (1.2) nor (1.4) nor (1.10) in [3]. The removement of the additional conditions gives us the following result on the modified Lax-Phillips conjecture:

Theorem 2. *Let P_j, $j = 1, 2, \cdots, L$, be points in $\boldsymbol{R}^3$, and set for $\varepsilon > 0$*

$$\mathcal{O}_\varepsilon = \cup_{j=1}^{L} \mathcal{O}_{j,\varepsilon}, \quad \mathcal{O}_{j,\varepsilon} = \{x;\, |x - P_j| < \varepsilon\}\,.$$

Suppose that

$$\textit{any triple of } P_j\textit{'s does not lie on a straight line.} \tag{A.1}$$

Then, the modified Lax-Phillips conjecture is valid for $\mathcal{O}_\varepsilon$ if $0 < \varepsilon \leq \varepsilon_0$.

The amelioration of the theorem is done with the aid of the results of Adachi-Sunada [1] and of Pollicott [10], Haydn [2].

The main reason why we had to assume the additional conditions was to guarantee the Property P of Parry [9] for $\mathcal{L}_{\varepsilon,s}$. Indeed, this property was essentially used in [9] for the proof of meromorphic extension of the zeta function. But Pollicott [10] and Hadyn [2] proved Theorems on the meromorphic extension without using the Property P. If we use the argument of [10,2], it suffices to consider the spectrum of the Perron-Frobenius operator $\mathcal{L}_{\varepsilon,s}$. To get informations of the perturbed operator $\mathcal{L}_{\varepsilon,s}$, first we have to consider the unperturbed operator $\tilde{\mathcal{L}}_s$. But we cannot apply the standard Perron-Frobenius theorem to the unperturbed operator because we do not assume the unperturbed system to be topological mixing. To overcome this difficulty, we decompose the unperturbed dynamical system into a direct sum of irreducible subsystems, and apply the generalized Perron-Frobenius theorem in [1] to each subsystem. In order to extract informations of the spectrum of perturbed operator from those of the unperturbed operator, we shall follow the argument done in [3].

2. Decomposition of $\mathcal{L}'_s$

Hereafter we shall use freely notations used in [3]. As in [3], we introduce an operator $\mathcal{L}'_s$ in $C(\Sigma_A^+)$ defined by

$$\mathcal{L}'_s u(\xi) = \begin{cases} \sum_{B(\eta_1,\xi_1)=1} \exp(r_0(\eta;s))u(\eta) & \text{for } \xi\in\Sigma(1), \\ 0 & \text{for } \xi\in\Sigma(2), \end{cases} \tag{2.1}$$

where

$$r_0(\xi;s) = -sf_0(\xi)+h_0(\xi), \tag{2.2}$$

and $\sum_{B(\eta_1,\xi_1)=1}$ indicates the summation taken over all $\eta\in\Sigma_A^+$ such that $\sigma_A\eta=\xi$ and $B(\eta_1,\xi_1)=1$.

In this section we shall consider the spectrum of $\mathcal{L}'_s$ in the space $\mathcal{F}_\theta(\Sigma_A^+)$. To this end, as mentioned in the introduction, we consider the spectrum of the operator $\tilde{\mathcal{L}}_s$ in the unperturbed dynamical system Σ_C^+, and compare the spectrum of $\mathcal{L}'_s$ with that of $\tilde{\mathcal{L}}_s$.

2.1. On the decomposition of $\tilde{\mathcal{L}}_s$

Let us say that i and j are equivalent when $i \underset{B}{\rightarrow} j$. Then the conditions (1.2) and (1.3) on B imply that this gives an equivalent relation in $\{1,2,\cdots,K\}$. Therefore, by changing the numbering of the elements of $\{1,2,\cdots,K\}$, we may assume that the set $\{1,2,\cdots,K\}$ is decomposed into equivalents classes

$$M_j = \{i_j, i_j+1, \cdots, i_{j+1}-1\} \quad (j=1,2,\cdots,l).$$

We shall denote by C_j the $(i_{j+1}-i_j)\times(i_{j+1}-i_j)$ matrix $[B(i,j)]_{i,j\in M_j}$. Note that each C_j is irreducible. We set

$$\Sigma_{C_j}^+ = \{\xi = (\xi_1, \xi_2, \cdots);\ \xi_i \in M_j \text{ and } B(\xi_i, \xi_{i+1}) = 1 \quad \text{for all } i\}$$

and

$$\Sigma_C^+ = \{\xi = (\xi_1, \xi_2, \cdots);\ 1\leq\xi_i\leq K \text{ and } B(\xi_i, \xi_{i+1}) = 1 \quad \text{for all } i\}.$$

Regarding $\Sigma_{C_j}^+$ and Σ_C^+ as subsets of Σ_A^+, we have a decomposition

$$C(\Sigma_C^+) = C(\Sigma_{C_1}^+)\oplus C(\Sigma_{C_2}^+)\oplus\cdots\oplus C(\Sigma_{C_l}^+)\,. \tag{2.3}$$

For $u\in C(\Sigma_A^+)$ we denote by $[u]$ and $[u]_j$ the restrictions of u to Σ_C^+ and $\Sigma_{C_j}^+$ respectively. Conversely, for functions in Σ_C^+ or in $\Sigma_{C_j}^+$ we shall often treat them as functions defined in Σ_A^+ by extending them by zero in the outside of Σ_C^+ or of $\Sigma_{C_j}^+$.

Let $\tilde{\mathcal{L}}_s$ be the operator in $C(\Sigma_C^+)$ defined by

$$\tilde{\mathcal{L}}_s v(\xi) = \sum_{\sigma_C\eta=\xi} \exp(r_0(\eta; s))v(\eta) \qquad \text{for} \quad v\in C(\Sigma_C^+)\,,$$

and let $\tilde{\mathcal{L}}_{j,s}$, be the operators in $C(\Sigma_{C_j}^+)$ defined by

$$\tilde{\mathcal{L}}_{j,s} v(\xi) = \sum_{\sigma_{C_j}\eta=\xi} \exp(r_0(\eta; s))v(\eta) \qquad \text{for} \quad v\in C(\Sigma_{C_j}^+)\,,$$

where σ_C and σ_{C_j} denote the restrictions of σ_A to Σ_C^+ and $\Sigma_{C_j}^+$ respectively. Then $\tilde{\mathcal{L}}_s$ has a decomposition

$$\tilde{\mathcal{L}}_s = \tilde{\mathcal{L}}_{1,s}\oplus\tilde{\mathcal{L}}_{2,s}\oplus\cdots\oplus\tilde{\mathcal{L}}_{l,s}\,. \tag{2.4}$$

By using the notation introduced in the above, we have for all $u\in\Sigma_A^+$

$$\tilde{\mathcal{L}}_s[u] = \tilde{\mathcal{L}}_{1,s}[u]_1\oplus\tilde{\mathcal{L}}_{2,s}[u]_2\oplus\cdots\oplus\tilde{\mathcal{L}}_{l,s}[u]_l\,.$$

Note that the conditions (1.5) and (1.6) imply that r_0 is real valued in $\Sigma_{C_j}^+$ for $s\in\boldsymbol{R}$. Thus, taking account of the indecomposability of C_j we can apply Theorem 3.8 and Lemma 3.11 of [1] to $\tilde{\mathcal{L}}_{j,s}$ and get the followong

Lemma 2.1. *For $s\in\boldsymbol{R}$, $\tilde{\mathcal{L}}_{j,s}$ has a decomposition*

$$\tilde{\mathcal{L}}_{j,s} = \sum_{k=1}^{k_j} \tilde{\lambda}_{j,k,s}\tilde{E}_{j,k,s} + \tilde{S}_{j,s}\,,$$

with the following properties:

(i) $$\tilde{\mathcal{L}}_{j,s}\tilde{E}_{j,k,s} = \tilde{\lambda}_{j,k,s}\tilde{E}_{j,k,s}\,.$$

(ii) $$\tilde{\lambda}_{j,1,s}>0 \quad \textit{and} \quad -\frac{d\tilde{\lambda}_{j,1,s}}{ds}>0\,.$$

(iii) $$|\tilde{\lambda}_{j,k,s}| = \tilde{\lambda}_{j,1,s} \quad \textit{and} \quad \tilde{\lambda}_{j,k,s}\neq\tilde{\lambda}_{j,k',s} \quad \textit{if } k\neq k'.$$

(iv) $$\tilde{E}_{j,k,s}u(\xi) = \nu_{j,k,s}(u)p_{j,k,s}(\xi),$$
where $\nu_{j,k,s} \in \bigcap_{\theta'>0}\mathcal{F}_{\theta'}(\Sigma_{C_j}^+)^*$ *satisfying* $\nu_{j,k,s}(p_{j,k,s})=1$,

(v) $$\tilde{E}_{j,k,s}\tilde{E}_{j,k',s} = \delta_{k,k'}\tilde{E}_{j,k,s}, \quad \tilde{E}_{j,k,s}\tilde{S}_{j,s} = \tilde{S}_{j,s}\tilde{E}_{j,k,s} = 0,$$

(vi) *the spectral radius of* $\tilde{S}_{j,s} < \tilde{\lambda}_{j,1,s}$.

Hereafter, we shall denote often $\tilde{\lambda}_{j,1,s}$ as $\tilde{\lambda}_{j,s}$. Note that we have for each j

$$\tilde{\lambda}_{j,s} \to \infty \quad \text{as} \quad s \to -\infty,$$
$$\tilde{\lambda}_{j,s} \to 0 \quad \text{as} \quad s \to \infty.$$

Thus, by changing the numbering of $\tilde{\lambda}_{j,s}$ if necessary, we may suppose that for some $s_0 \in \boldsymbol{R}$

(2.5) $$1 = \tilde{\lambda}_{1,s_0} = \tilde{\lambda}_{2,s_0} = \cdots = \tilde{\lambda}_{h,s_0} > \tilde{\lambda}_{h+1,s_0} \geq \cdots \geq \tilde{\lambda}_{l,s_0}.$$

Then, by using the perturbation theory we have immediately the following

Lemma 2.2. *There are a neighborhood of* s_0 *in* $\boldsymbol{C}$ *and a constant* $\delta>0$ *such that for all* $s \in D$ *we have a decomposition*

$$\tilde{\mathcal{L}}_s = \sum_{j=1}^{h}\sum_{k=1}^{k_j} \tilde{\lambda}_{j,k,s}\tilde{E}_{j,k,s} + \tilde{S}_s$$

with the following properties:

(i) $$\tilde{E}_{j,k,s}(\xi) = \nu_{j,k,s}([u]_j)p_{j,k,s}(\xi),$$

(ii) $$\tilde{E}_{j,k,s}\tilde{E}_{j',k',s} = \delta_{j,j'}\delta_{k,k'}\tilde{E}_{j,k,s}$$

(iii) $$\tilde{E}_{j,l,s}\tilde{S}_s = \tilde{S}_s\tilde{E}_{j,k,s} = 0,$$

(iv) $$|\tilde{\lambda}_{j,s} - 1| < \delta,$$

(v) $$|\tilde{\lambda}_{j,k,s} - 1| > 2\delta, \quad 1-\delta < |\tilde{\lambda}_{j,k,s}| < 1+\delta \qquad \text{for} \quad k \geq 2,$$

(vi) *the spectral radius of* $\tilde{S}_s < 1-2\delta$.

2.2. On eigenvalues of $\mathcal{L}'_s$

With the aid of the results of the previous subsection, we shall consider the decomposition of $\mathcal{L}'_s$. First remark that for any positive integer m and for $\xi \in \Sigma(1)$ we have an expression

(2.6) $$\mathcal{L}'^m_s u(\xi) = \sum_{\eta_1,\cdots,\eta_m} \exp(S_m r_0(\eta_m, \eta_{m-1}, \cdots, \eta_1, \xi; s)) \cdot u(\eta_m, \eta_{m-1}, \cdots, \eta_1, \xi),$$

where the summation is takenover all $\eta_1, \eta_2, \cdots, \eta_m$ satisfying $B(\eta_1, \xi_1)=1$, $B(\eta_2, \eta_1) =1, \cdots, B(\eta_m, \eta_{m-1})=1$. If $\xi \in \Sigma_C^+$, all $(\eta_m, \cdots, \eta_1, \xi)$ in the right hand side of (2.6) belong to Σ_C^+. Thus we have

(2.7) $$\mathcal{L}'^m_s u(\xi) = \tilde{\mathcal{L}}^m_s u(\xi) \qquad \text{for all} \quad \xi \in \Sigma_C^+.$$

Lemma 2.3. *For each pair j, k in Lemma 2.2, there is a function $w_{j,k,s}(\xi)\in\mathcal{F}_\theta(\Sigma_A^+)$ satisfying*

$$(2.8)\qquad |(\tilde{\lambda}_{j,k,s})^{-m}\sum_{\eta_m,\cdots,\eta_2,l}\exp(S_q r_0(\eta_m,\cdots,\eta_2,l,\xi))\,p_{j,l,s}(\eta_q,\cdots l,\eta^{(l)}) - w_{j,k,s}(\xi)|\le C\gamma_1^m \qquad \text{for}\quad m=1,2,\ldots,$$

and

$$\mathcal{L}'_s w_{j,k,t}=\tilde{\lambda}_{j,k,s}\,w_{j,k,s}\,.$$

Here γ_1 is a constant such that $0\le\gamma_1<1$.

Proof. Let $\xi\in\Sigma(1)$ be an element such that $\xi_1\in C_h$. Then all the η_j in the summation of the right hand side of (2.6) belong to C_h. Thus the argument in Section 2 of [3] can be applied and we see that (2.8) holds for all ξ such that $\xi_1\le K$. It follows from (2.8) that

$$\mathcal{L}'_s w_{i,k,s}(\xi)=\tilde{\lambda}_{j,k,s}w_{j,k,s}(\xi)\quad\text{for all }\xi\in\Sigma_A^+\text{ such that }\xi_1\le K.$$

Define $w_{j,k,s}(\xi)$ for $\xi\in\Sigma(1)$ such that $\xi_1\ge K+1$ by

$$(2.9)\qquad w_{j,k,s}(\xi)=(\tilde{\lambda}_{j,k,s})^{-1}\sum_{\substack{\eta_1\in M_j\\ B(\eta_1,\xi_1)=1}}\exp(r_0(\eta_1,\xi))w_{j,k,s}(\eta_1,\xi)\,.$$

We have immediately

$$\mathcal{L}'_s w_{j,k,s}(\xi)=\tilde{\lambda}_{j,k,s}w_{j,k,s}(\xi)\qquad\text{for all}\quad \xi\in\Sigma_A^+\,.$$

Concerning the converging estimate (2.8) for $\xi_1\ge K+1$, we use the following relation:

$$\begin{aligned}&(\tilde{\lambda}_{j,k,s})^{-m}\sum_{\eta_m,\cdots,\eta_2,l}\exp(S_m r_0(\eta_m,\cdots,\eta_2,l,\xi))\,p_{j,k,s}(\eta_m,\cdots,l,\eta^{(l)})\\ &=(\tilde{\lambda}_{j,k,s})^{-1}\sum_{l\in M_j\ B(l,\xi_1)=1}\exp(r_0(l,\xi))\{(\tilde{\lambda}_{j,k,s})^{-m+1}\\ &\quad\cdot\sum_{\eta_m,\cdots,\eta_2}\exp(S_{m-1}r_0(\eta_m,\cdots,\eta_2,l,\xi))\,p_{j,k,s}(\eta_m,\cdots,l,\eta^{(l)})\}.\end{aligned}$$

By using the fact that (2.8) holds for ξ satisfying $\xi_1\le K$, we see immediately from (2.9) that (2.8) holds for all $\xi\in\Sigma(1)$.

Remark that we have from (2.7)

$$w_{j,k,s}(\xi)=p_{j,k,s}(\xi)\qquad\text{for all}\quad \xi\in\Sigma_C^+\,,$$

from which it follows that

$$\nu_{j,k,s}([w_{j',k',s}]_j)=\delta_{j,j'}\delta_{k,k'}\,.$$

Define $E'_{j,k,s}$ by

$$(2.10)\qquad E'_{j,k,s}u(\xi)=\nu_{j,k,s}([u]_j)w_{j,k,s}(\xi)\,.$$

Then, we have

$$E'_{j,l,s}E'_{j',k',s}=\delta_{j,j'}\delta_{k,k'}E'_{j,k,s}\,, \tag{2.11}$$

and

$$\mathcal{L}'_sE'_{j,k,s}=\tilde{\lambda}_{j,k,s}E'_{j,k,s}\,. \tag{2.12}$$

Now use the following expression

$$\begin{aligned}\mathcal{L}'^m_s u(\xi)&=\sum_{\eta_1,\cdots,\eta_b}\exp(S_q r_0(\eta_q,\cdots,\eta_1,\xi))\mathcal{L}'^p_s u(\eta_q,\cdots,\eta_1,\xi)\\&=\sum_{\eta_1,\cdots,\eta_q}\exp(S_q r_0(\eta_q,\cdots,\eta_1,\xi))\,\{\mathcal{L}'^p_s u(\eta_q,\cdots,\eta_1,\eta^{(l)})\\&\qquad+(\mathcal{L}'^p_s u(\eta_q,\cdots,\eta_1,\xi)-\mathcal{L}'^p_s u(\eta_q,\cdots,\eta_1,\eta^{(l)}))\}\\&=\mathrm{I}+\mathrm{II}\,.\end{aligned}$$

We get immediately

$$|\mathrm{II}|\leq C\theta^q\exp(q\tilde{P}(\operatorname{Re}r_0))\exp(p\tilde{P}(\operatorname{Re}r_0))|||u|||_\theta\,,$$

where we set $\tilde{P}(r_0)=\max\tilde{P}(r_j)$ and C is a positive constant independent of p and q.

By using (2.7) and Lemma 2.2 we have

$$\begin{aligned}\mathcal{L}'^p_s u(\eta_q,\cdots,\eta_1,\eta^{(l)})=\sum_{j=1}^{h}\sum_{k=1}^{kj}(\tilde{\lambda}_{j,k,s})^p\,\nu_{j,k,s}([u]_j)\cdot p_{j,k,s}(\eta_q,\cdots,\eta_1,\eta^{(l)})\\+\bar{S}^p_s[u](\eta_q,\cdots,\eta_1,\eta^{(l)})\,.\end{aligned}$$

Applying the argument in Section 2 of [3] to the above expression we have

$$|\,\mathrm{I}-\sum_{j=1}^{h}\sum_{k=1}^{kj}(\tilde{\lambda}_{j,k,s})^p\,E'_{j,k,s}u(\xi)\,|\leq C|||u|||_\theta\{(1-2\delta)^p\exp(q\tilde{P}(\operatorname{Re}r_0))+\gamma_1^m\}$$

Then, by exchanging D_1 by a smaller neighborhood of s_0 if necessary, we have

Lemma 2.4. *There exist a neighborhood D_1 of s_0 in* **C** *and a positive constant δ_2 such that we have for all $s\in D_1$*

$$1-\delta_2/2\leq|\tilde{\lambda}_{j,k,s}|\leq 1+\delta_2/2\,, \tag{2.13}$$

$$|||\mathcal{L}'^m_s u-\sum_{j=1}^{h}\sum_{k=1}^{kj}(\tilde{\lambda}_{j,k,s})^m E'_{j,k,s}u(\xi)|||_\theta\leq C|||u|||_\theta(1-2\delta_2)^m. \tag{2.14}$$

2.3. On the decomposition of $\mathcal{L}'_s$.

By using the same argument as in [3], we have the following two estimates concerinng $\mathcal{L}'_{s_0}$ for any $u\in\mathcal{F}_\theta(\Sigma_A^+)$

$$\|\mathcal{L}'^m_{s_0}u\|_\infty \leq C_1\|u\|_\infty ,$$
$$|||\mathcal{L}'^m_{s_0}u|||_\theta \leq C_2\theta^m|||u|||_\theta + C_3\|u\|_\infty .$$

Thus, by applying the theorem of Ionescu Turcia, Marinescu [6] to the pair of the spaces $C(\Sigma_A^+)$ and $\mathcal{F}_\theta(\Sigma_A^+)$, we have from the above inequalities the following decomposition of $\mathcal{L}'_{s_0}$ in $\mathcal{F}_\theta(\Sigma_A^+)$

$$\mathcal{L}'_{s_0} = \sum_{j=1}^{J} c_j E'_j + S' = E' + S', \tag{2.15}$$

where

$$\mathcal{L}'_{s_0} E'_j = c_j E'_j \quad \text{and} \quad |c_j| = 1 \quad \text{for all } j,$$
$$E'_j E'_l = \delta_{jl} E'_j \quad \text{for all } j, l,$$
$$E'_j S' = S' E'_j = 0 \quad \text{for all } j,$$
$$\text{the speatral radius of } S' < 1.$$

We shall show that there is no eigenvalue of E' besides $\tilde{\lambda}_{j,k,s_0}$. Suppose that c such that $|c|=1$ is an eigenvalue and $w \in \mathcal{F}_\theta(\Sigma_A^+)$ is its associated eigenfunction. Note that $w \not\equiv 0$ implies that $\tilde{w} = [w] \not\equiv 0$. Indeed, suppose that $\tilde{w} \equiv 0$. Note that (2.10) gives us $E'_{j,k,s_0} w = 0$ for all j, k. Then, the application of (2.14) to w implies that for all $\xi \in \Sigma_A^+$

$$|w(\xi)| = |c^{-m}\mathcal{L}'^m_s w(\xi)| \leq C|||w|||_\theta (1-2\delta_2)^m.$$

By letting m tend to the infinity, we have $w(\xi)=0$. This implies that $w \equiv 0$. This contradicts $w \not\equiv 0$. Thus our assertion is proved.

It is evident that $\tilde{w} \not\equiv 0$ satisfies

$$\tilde{\mathcal{L}}_s \tilde{w} = c\tilde{w} .$$

Lemma 2.2 shows that c must be one of $\tilde{\lambda}_{j.k,s_0}$'s. Therefore, the eigenvalues of E' are $\tilde{\lambda}_{j,k,s_0}$'s. Then we have

$$E' = \sum_{j=1}^{h} \sum_{k=1}^{kj} \tilde{\lambda}_{j,k,s_0} E'_{j,k,s_0} ,$$

$$\dim \text{Range } E' = \sum_{j=1}^{h} k_j = \tilde{k} .$$

The decomposition (2.15) shows that for all $u \in \mathcal{F}_\theta(\Sigma_A^+)$ and m

$$\mathcal{L}u'^m_{s_0} = \sum_{j=1}^{h} \sum_{k=1}^{kj} (\tilde{\lambda}_{j,k,s_0})^m E'_{j,k,s_0} u + S'^m u ,$$

and (2.14) implies that

$$|||S'^m u|||_\theta \leq C|||u|||_\theta (1-2\delta_2)^m.$$

This shows that

$$\text{the spectral radius of } S' \leq 1-2\delta_2 .$$

By means of perturbation theory, we see that there are a neighborhood $D_2 \subset D_1$ of s_0 and a constant $0<\delta_3 \leq \delta_2$ such that for all $s \in D_2$

$$\mathcal{L}'_s = E'_s + S'_s ,$$
$$\dim \text{Range } E'_s = \tilde{k} ,$$
$$\text{all the eigenvalues of } E'_s \in \{\lambda ; 1-\frac{1}{3}\delta_3 \leq |\lambda| \leq 1+\frac{1}{3}\delta_3\}$$
$$\text{the spectral radius of } S'_s < 1-3\delta_3 .$$

On thd the other hand, it is proved that $\tilde{\lambda}_{j,k,s}$ and $w_{j,k,s}$ are eigenpairs of $\mathcal{L}'_s$, which satisfy $1-\frac{1}{3}\delta_3 \leq |\tilde{\lambda}_{j,k,s}| \leq 1+\frac{1}{3}\delta_3$ and that $w_{j,k,s}$ are linearly independent. This fact shows that

$$\dim \text{Range} \sum_{j=1}^{h} \sum_{k=1}^{kj} \tilde{\lambda}_{j,l,s} E'_{j,k,s} = \tilde{k} .$$

Therefore it follows that

$$E'_s = \sum_{j=1}^{h} \sum_{k=1}^{kj} \tilde{\lambda}_{j,k,s} E'_{j,k,s} .$$

Denote by μ_k^0, $l=1, 2, \cdots, l_0$ the distinct values of $\tilde{\lambda}_{j,k,s_0}$, $k=1, 2, \cdots, k_j$, $j=1, 2, \cdots, h$, and rename all the $\tilde{\lambda}_{j,k,s}$ such that $\tilde{\lambda}_{j,k,s_0}=\mu_l^0$ as $\mu_{(l,i),s}$, $i=1, 2, \cdots, i_l$. Hereafter we denote by $F'_{(l,i),s}$ and $w_{(l,i),s}$ the corresponding $E'_{j,k,s}$ and $w_{j,k,s}$.

We set

$$F'_{0,s} = \sum_{j=1}^{h} \tilde{\lambda}_{j,s} E'_{j,1,s}$$

and

$$F'_{l,s} = \sum_{i=1}^{i_l} \mu_{(l,i),s} F'_{(l,i),s} .$$

Then, summing up the argument in this subsection we have the following

Proposition 2.5. *There are $s_0 \in \boldsymbol{R}$, a neighborhood D_2 of s_0 in $\boldsymbol{C}$ and a positive constant δ_3 such that, for all $s \in D_2$, $\mathcal{L}'_s$ has a decomposition*

$$\mathcal{L}'_s = \sum_{l=1}^{l_0} F'_{l,s} + S'_s$$

satisfying the following:

(1) *$F'_{l,s} S' = S'_s F'_{s,s} = 0$, for all $l=0, 1, \cdots, l_0$.*

(2) *$F'_{l,s} F'_{k,s} = F'_{k,s} F'_{l,s} = 0$ for all $l, k=0, 1, \cdots, l_0$ such that $l \neq k$.*

(3) *For $0 \leq l \leq l_0$, the dimension of the range of $F'_{l,s} = i_l$ for all $s \in D_2$ and the*

eigenvalues of $F'_{l,s}$ *are* $\mu_{(l,i),s}$ $i=1, 2, \cdots, i_l$, *which satisfy*

$$|\mu_{(l,i),s}-\mu_l^0|<\frac{1}{3}\delta_3 \quad |\mu_l^0-\mu_{l'}^0|>\delta_3\ (l\neq l')\,.$$

Especially, $\mu_0^0=1$, $i_0=h$ *and* $\mu_{(0,j),s}=\tilde{\lambda}_{j,s}$, $(j=1, 2, \cdots, h)$.

(4) *the spectral radius of* $S_s^l<3-\delta_3$.

3. Spectrum of $\mathcal{L}_{\varepsilon,s}$

Let $\mathcal{L}_{\varepsilon,s}$ be the operator in Σ_A^+ defined by

$$\mathcal{L}_{\varepsilon,s}u(\xi) = \sum_{\sigma_A\eta=\xi}\exp(r_\varepsilon(\eta, s))u(\eta)\,.$$

We shall show the existence of s such that $\mathcal{L}_{\varepsilon,s}$ has 1 as an eigenvalue.

Even though the following is a well known fact on perturbations of linear operators, we shall mention it in the form of lemma to make clear the argument below.

Lemma 3.1. *Let T be a bounded operator in Banach space B with norm* $\|\cdot\|$. *Suppose that* $\{\lambda;\ |\lambda-\mu|=\alpha\}$ $(\mu\in \boldsymbol{C},\ \alpha>0)$ *is contained in the resolvent set of T, and that the projection*

$$P=\frac{1}{2\pi\sqrt{-1}}\oint_{|\lambda-\mu|=\alpha}(\lambda-T)^{-1}\,d\lambda$$

is of finite rank h. Let $\{w_1, w_2, \cdots, w_h\}$ *is a basis of the range of P.*

Then there is $a>0$ such that

$$\|T'-T\|\leq a$$

implies the following:

(i) $\{\lambda;|\lambda-\mu|=\alpha\}$ *is also contained in the resolvent set of T'.*

(ii) *The projection*

$$P'=\frac{1}{2\pi\sqrt{-1}}\oint_{|\lambda-\mu|=\alpha}(\lambda-T')^{-1}d\lambda$$

is of rank h and

$$w_j'=P'w_j\ (j=1, 2, \cdots, h)$$

form a basis of the range of P'.

iii)
$$\|\sum_{j=1}^{h}a_kw_k-\sum_{j=1}^{h}a_k'w_k'\|\leq\|T'-T\|$$

implies that

$$\sum_{j=1}^{h}|a_k'-a_k|\leq C_1\|T'-T\|\,,$$

where C_1 is a constant depending on μ and α but independent of T'.

Suppose that Lemma 2.4 and Proposition 2.5 hold for the open disk $D_2=\{s;\,|s-s_0|<\alpha_0\}$ $(\alpha_0>0)$. Recall that $\tilde{\lambda}_{j,s}, j=1,2,\cdots,h$ are analytic in D_2, and satisfies

$$\tilde{\lambda}_{j,s_0}=1\,,\quad -\frac{d}{ds}\tilde{\lambda}_{j,s}\Big|_{s=s_0}>0\,.$$

Thus, by exchanging α_0 by a smaller one if necessary, we may assume the following:

$$|\tilde{\lambda}_{s,j}-1|\leq\delta_3/3 \qquad \text{for all} \quad s\in D_2\,,$$

(3.1) $$|\tilde{\lambda}_{s,j}-1|\geq c_1|s-s_0| \qquad \text{for all} \quad s\in\{s;\,|s-s_0|\leq\alpha_0\}\ (c_1>0)\,.$$

By the same argument as in [3, Section 3] we have

$$|||\mathcal{L}'_{0,s}-\mathcal{L}_{\varepsilon,s}|||_\theta\to 0 \quad \text{uniformly in } s\in D_2 \text{ as } \varepsilon\to 0\,.$$

Therefore by applying Lemma 3.1 to the pair of operators $T=\mathcal{L}'_s$, $T'=\mathcal{L}_{\varepsilon,s}$ we have

Lemma 3.2. *There are positive constants ε_0 and δ_4 such that for all $0<\varepsilon\leq\varepsilon_0$ and $s\in D_2$ we have the following decomposition of $\mathcal{L}_{\varepsilon,s}$:*

(3.2) $$\mathcal{L}_{\varepsilon,s}=\sum_{l=0}^{l_0}\mathcal{E}_{(l),\varepsilon,s}+\mathcal{S}_{\varepsilon,s}$$

where

(3.3) $$\mathcal{E}_{(l),\varepsilon,s}\mathcal{E}_{(k),\varepsilon,s}=\mathcal{E}_{(k),\varepsilon,s}\mathcal{E}_{(l),\varepsilon,s}=0 \qquad \textit{if} \quad l\neq k\,,$$

(3.4) $$\mathcal{E}_{(l),\varepsilon,s}\mathcal{S}_{\varepsilon,s}=\mathcal{S}_{\varepsilon,s}\mathcal{E}_{(l),\varepsilon,s}=0\,,$$

(3.5) $$\textit{the spectral radius of } \mathcal{S}_{\varepsilon,s}<1-2\delta_3\,,$$
$$\dim \operatorname{Range}\mathcal{E}_{(l),\varepsilon,s}=i_l \qquad \textit{for all} \quad 0<\varepsilon<\varepsilon_0\,,$$

(3.6) $$\sum_{\sigma_A^n\xi=\xi}\exp\left(\operatorname{Re} r_\varepsilon(\xi,s)\right)\leq C(1+\delta_3)^n \qquad \textit{for all} \quad n\,.$$

Moreover, denoting the eigenvalues of $\mathcal{E}_{(l),\varepsilon,s}$ by $\lambda_{l,i}(\varepsilon,s)$, $i=0,1,\cdots,i_l$, $l=1,2,\cdots,h$ we have for all $0<\varepsilon\leq\varepsilon_0$

(3.7) $$|\lambda_{l,j}(\varepsilon,s)-\mu_l^0|\leq\frac{2}{3}\delta_3 \qquad \textit{for all} \quad s\in D_2,\quad l=0,1,\cdots,l_0\,,$$

(3.8) $$|\lambda_{0,j}(\varepsilon,s)-1|>\delta_4 \qquad \textit{for all} \quad s\in\{s;\,|s-s_0|=\alpha_0\}\,.$$

By using the decomposition (3.2) with the properties (3.6) and (3.7), we have the expression

$$\mathcal{E}_{(k),\varepsilon,s}=\frac{1}{2\pi i}\oint_{|z-\mu_j^0|=\delta_1} z(z-\mathcal{L}_{\varepsilon,s})^{-1}\,dz$$

and

$$\mathcal{P}_{(l),\varepsilon,s}=\frac{1}{2\pi i}\oint_{|z-\mu_l^0|=\delta_1}(z-\mathcal{L}_{\varepsilon,s})^{-1}\,dz\,.$$

Recall that $w_{(l,i),s}$, $i=1,2,\cdots,i_l$ form a basis of Range $F'_{l,s}$. From the continuity of $\mathcal{L}_{\varepsilon,s}$ on ε,

$$w_{(l,i),\varepsilon,s}=\mathcal{P}_{(l),\varepsilon,s}w_{(l,i),s}\,,\qquad i=1,2,\cdots,i_l$$

are linearly independent for all $0<\varepsilon\leq\varepsilon_1$ and $s\in D_2$. This fact implies that $\{w_{(l,i),\varepsilon,s};\,i=1,2,\cdots,i_l\}$ is a basis of the range $\mathcal{P}_{(l),\varepsilon,s}$. It holds that

$$w_{(l,i),\varepsilon,s}\text{ is analytic in } s\in D_2\,,$$
$$w_{(l,i),\varepsilon,s}\to w_{(l,i),s}\text{ uniformly in } s\in D_2\text{ as }\varepsilon\to 0\,.$$

Therefore $\mathcal{E}_{(l),\varepsilon,s}w_{(l,i),\varepsilon,s}$ is a linear combination of $w_{(l,i),\varepsilon,s}$, $i=1,2,\cdots,i_l$, that is,

$$\mathcal{E}_{(l),\varepsilon,s}w_{(l,j),\varepsilon,s}=\sum_{i=1}^{i_l}a_{(l),jk}(\varepsilon,s)\,w_{(l,k),\varepsilon,s}\,.$$

Applying Lemma 3.1 we have

$$a_{(l),jk}\text{ depends on } s\in D_2\text{ analytically}\,,$$
$$a_{(l),jk}(\varepsilon,s)\to a_{(l),jk}(0,s)\text{ uniformly in } s\in D_2\text{ as }\varepsilon\to 0\,.$$

Let $\mathcal{A}_l(\varepsilon,s)$ be the $i_l\times i_l$ matrix defined by

$$\mathcal{A}_l(\varepsilon,s)=[a_{(l),jk}]_{j,k=1,2,\cdots,i_l}\,.$$

It is evident from Lemma 3.1 that the eigenvalues of $\mathcal{E}_{(k),\varepsilon,s}$ in the space $\mathcal{F}_\theta(\Sigma_A^+)$ and those of $\mathcal{A}_l(\varepsilon,s)$ in C^{i_l} coincide including the multiplicities. Set

$$\begin{aligned}(3.9)\qquad f_l(\lambda,s;\varepsilon)&=\det(\lambda I-\mathcal{A}_l(\varepsilon,s))\\&=\lambda^{i_l}+\lambda^{i_l-1}b_{l,1}(\varepsilon,s)+\lambda^{i_l-2}b_{l,2}(\varepsilon,s)+\cdots+b_{l,i_l}(\varepsilon,s)\,.\end{aligned}$$

Then it follows from the properties of $a_{(l),jk}$ that

(3.10) $\qquad b_{l,j}(\varepsilon,s)$ is analytic in $s\in D_2$,

(3.11) $\qquad b_{l,j}(\varepsilon,s)$ is continuous in $\varepsilon\in[0,\varepsilon_0]$ uniformly for $s\in D_2$.

Lemma 3.3. *The eigenvalues of $\mathcal{E}_{(l),\varepsilon,s}$ are the roots of $f_l(\lambda,s;\varepsilon)=0$, which is the polynomial given by* (3.9) *whose coefficients satisfy* (3.10) *and* (3.11). *Moreover, for each $0<\varepsilon\leq\varepsilon_0$, $f_0(1,s;\varepsilon)=0$ has exactly h zeros in $\{s;\,|s-s_0|\leq\alpha_0\}$, which converge to s_0 when ε tends to zero.*

Proof. Note that $i_0=h$ and

$$f_0(\lambda, s; 0) = \prod_{j=1}^{h} (\lambda - \tilde{\lambda}_{j,s}) .$$

Then (3.1) shows that

$$f_0(1, s; 0) \neq 0 \qquad \text{for all } 0 < |s - s_0| \leq \alpha_0 ,$$
$$s = s_0 \text{ is a zero point of } h\text{-th order of } f_0(1, s; 0) .$$

Thus, $f_0(1, s; 0)$ has exactly h zeros in $\{s; |s-s_0| \leq \alpha_0\}$. On the other hand, (3.11) and (3.8) imply that the number of zero points of $f_0(1, s; \mathcal{E})$ in $\{s; |s-s_0| \leq \alpha_0\}$ is invariant for all $0 \leq \mathcal{E} \leq \mathcal{E}_0$. Since the dependency on $\mathcal{E}$ of the zero points of $f_0(1, s; \mathcal{E})$ is continuous, they converge to those of $f_0(1, s; 0) = 0$, which are equal to s_0. Thus the assertion of the lemma is proved.

4. Proof of Theorems

In order to show Theorem 1, we apply Theorem 2 of [9, Section 4] or Theorem 4 of [2, Section 4] to $\mathcal{L}_{\varepsilon,s}$. By exchanging $\mathcal{E}_0$ by a smaller one if necessary we may assume that

$$\theta(1+\delta_3) < 1 .$$

Then, the application of the theorems of [9,2] to $\mathcal{L}_{\varepsilon,s}$ assures that

$$Z_\varepsilon(s) \text{ is meromorphic in } \operatorname{Re} s > s_0 + \alpha_0$$

and is of the form

$$Z_\varepsilon(s) = \exp(\phi(s, \mathcal{E})) \prod_{l=0}^{l_0} f_l(1, s; \mathcal{E})^{-1},$$

where $\phi(\cdot, \mathcal{E})$ is holomorphic in $\operatorname{Re} s > s_0 + \alpha_0$. From Lemma 3.3, we have Theorem 1.

As to Theorem 2, follow the argument in [3, Section 4] by using Theorem 1 of the present paper instead of Theorem 1 of [3], and we have Theorem 2.

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