Shiomi, T.
Osaka J. Math.
28 (1991) 649–661

ON IMBEDDING 3-MANIFOLDS INTO 4-MANIFOLDS

TATSUYUKI SHIOMI

(Received June 26, 1990)
(Revised November 15, 1990)

Introduction

We discuss an imbedding problem of a closed, connected, oriented 3-manifold into a given compact connected 4-manifold, which arises from certain signature invariants of 3-manifold associated with its cyclic coverings. Our main result is the following:

Theorem. *For any compact, connected (orientable or non-orientable) 4-manifold W (with or without boundary), there exist infinitely many closed, connected, orientable 3-manifolds M which cannot be imbedded in W.*

For a closed orientable 4-manifold W, this is a direct consequence of [**8**, Theorem 3.2] and, for an orientable 4-manifold W with boundary, we can prove it by using the doubling technique for W. Thus the main concern in this paper is for a non-orientable 4-manifold W.

The proof of Theorem is given in §3. In §2, a classification of the types of imbeddings of M into a closed 4-manifold W is given. Section 1 is devoted to the calculation of the signatures of the finite cyclic covers of a homology handle M. We can express these signatures in terms of the local signatures of M under a certain condition on the Alexander polynomial of M, where the Alexander polynomial of a homology handle is defined in the same way as in the case of knots (*cf.* [**3**, Definition 1.3]). Let $\sigma_a(M)$ be the local signature of M at $a\in[-1,1]$, which is an analogue of the Milnor signature of a knot (*cf.* [**9**]). Let $\sigma^{(n)}(M)$ be the signature of n-fold cyclic cover of M (whose definition is given in Section 1 where $\sigma^{(n)}(M)$ is denoted by $\sigma^{\dot{\gamma}^{(n)}}(M_{\dot{\gamma}(n)})$). Then the following will be shown.

Proposition 1.3. *If the Alexander polynomial of M has no 2n-th root of unity, then*

$$\sigma^{(n)}(M)=\sum_{j=0}^{n-1}(-1)^j\sum_{a_{j+1}<a<a_j}\sigma_a(M),$$

where $a_j = \cos(j\pi/n)$, $j = 0, 1, \cdots, n$.

This result reveals a connection between the signatures of finite cyclic covers of a homology handle and the local signatures of its infinite cyclic cover. When $n=2$ the assumption of the above proposition is always satisfied. So we have the following formula, which will be used in §3 to prove Theorem for a non-orientable 4-manifold W.

Corollary 1.4. $\sigma^{(2)}(M) = \sum_{-1<a<1} \operatorname{sign}(a)\, \sigma_a(M)$.

Throughout this paper, all manifolds and all maps between manifolds will be assumed to be smooth.

I would like to thank my advisor Professor Akio Kawauchi for suggesting the problem to me and for his advice and encouragement.

1. Signatures of Finite Cyclic Covers of a Homology Handle

In this section, we consider the signature of the n-fold cyclic cover of a homology handle.

Throughout this paper, we use Kawauchi's notations for signatures and local signatures of a 3-manifold; for a closed oriented 3-manifold M equipped with an element $\dot{\gamma} \in H^1(M; \boldsymbol{Z})$, $\sigma^{\dot{\gamma}}(M)$ denotes the *signature* of $(M, \dot{\gamma})$ and $\sigma_a^{\dot{\gamma}}(M)$, $a \in [-1, 1]$, denotes the *local signature* of $(M, \dot{\gamma})$ at a. For the definitions of these invariants, see [**6**] and also [**4**], [**5**], [**7**]. (Local singatures were first considered in [**9**, Section 5] for the exterior of a knot in S^3.) In this section, $\boldsymbol{Z}\langle t\rangle$ (resp. $\boldsymbol{R}\langle t\rangle$) denotes the group ring over the infinite cyclic group $\langle t\rangle$ generated by t with coefficient ring the ring $\boldsymbol{Z}$ of integers (resp. the field $\boldsymbol{R}$ of real numbers).

Now let M be an oriented homology handle, that is, a compact oriented 3-manifold having the homology isomorphic to that of $S^2 \times S^1$ (*cf.* [**3**]), and $\dot{\gamma}$ be a fixed generator of $H^1(M; \boldsymbol{Z}) = [M, S^1]$. Using the transversality of a map $M \to S^1$ representing $\dot{\gamma}$, we can find a closed, connected, oriented surface V in M representing the Poincaré dual of $\dot{\gamma}$. V is called a *leaf* of $\dot{\gamma}$ (*cf.* [**6**]).

We choose an orientation of $M \times [-1, 1]$ so that $M \times 1$ with the induced orientation is identified with M. Let $N(V)$ be a bicollar neighborhood of V in M. Let $W_c = M \times [-1, 1] - \operatorname{int}(N(V) \times [-1/2, 1/2])$ (*cf.* [**7**]). There is a natural diffeomorphism $N(V) \times [-1/2, 1/2] \cong V \times D^2$. Let $\bar{V}$ be a handlebody such that $\partial\bar{V}$ is diffeomorphic to V. By identifying $\partial(\bar{V} \times S^1)$ with $V \times S^1 = \partial(N(V) \times [-1/2, 1/2]) \subset W_c$, we get a compact 4-manifold $\bar{W}_c = W_c \cup \bar{V} \times S^1$ with boundary diffeomorphic to $M \cup -M$. By the Pontrjagin/Thom construction, we have an element $\bar{\gamma}_c \in H^1(\bar{W}_c; \boldsymbol{Z})$ such that $\bar{\gamma}_c | M \times 1 = \dot{\gamma}$, $\bar{\gamma}_c | M \times (-1) = 0$ and $\bar{\gamma}_c | \bar{V} \times S^1$ is represented by the natural projection $\bar{V} \times S^1 \to S^1$. Taking a compact, oriented 4-manifold W_0 bounded by M, we can cap the component

$M\times(-1)$ of $\partial\bar{W}_c$ and finally get a 4-manifold $W=\bar{W}_c\cup W_0$ with boundary M. Define an element $\gamma\in H^1(W;\boldsymbol{Z})$ by $\gamma|\bar{W}_c=\bar{\gamma}_c$ and $\gamma|W_0=0$. Note that $\partial(W,\gamma)=(M,\dot{\gamma})$ and γ has a leaf $U_\gamma=(V\times[1/2,1])\cup(\bar{V}\times x_0)$, where $x_0\in S^1$ is the point such that $\partial(\bar{V}\times x_0)\equiv V\times(1/2)\subset\partial W_c$.

For each positive integer n, let $p_n\colon M_{\dot{\gamma}(n)}\to M$ (resp. $P_n\colon W_{\gamma(n)}\to W$) be the n-fold cyclic covering of M (resp. W) associated with the mod n reduction $\dot{\gamma}(n)$ (resp. $\gamma(n)$) of $\dot{\gamma}$ (resp. γ). If $f_{\dot{\gamma}}\colon M\to S^1$ (resp. $f_\gamma\colon W\to S^1$) is a map representing $\dot{\gamma}$ (resp. γ), then the covering $p_n\colon M_{\dot{\gamma}(n)}\to M$ (resp. $P_n\colon W_{\gamma(n)}\to W$) is defined to be the fibered product of $f_{\dot{\gamma}}$ (resp. f_γ) with the natural n-fold covering $q_n\colon S^1\to S^1$, $z\mapsto z^n$, where $z\in S^1$ is considered as a complex number with unit norm. The lift $f_{\dot{\gamma}}^{(n)}\colon M_{\dot{\gamma}(n)}\to S^1$ (resp. $f_\gamma^{(n)}\colon W_{\gamma(n)}\to S^1$) of $f_{\dot{\gamma}}$ (resp. f_γ) by q_n is determined by $\dot{\gamma}$ (resp. γ) up to homotopy. The homotopy calss of $f_{\dot{\gamma}}^{(n)}$ (resp. $f_\gamma^{(n)}$) is denoted by $\dot{\gamma}^{(n)}\in[M_{\dot{\gamma}(n)},S^1]=H^1(M_{\dot{\gamma}(n)};\boldsymbol{Z})$ (resp. $\gamma^{(n)}\in[W_{\gamma(n)},S^1]=H^1(W_{\gamma(n)};\boldsymbol{Z})$). Note that $\partial(W_{\gamma(n)},\gamma^{(n)})=(M_{\dot{\gamma}(n)},\dot{\gamma}^{(n)})$ and that $\dot{\gamma}^{(n)}$ (resp. $\gamma^{(n)}$) has as its leaf a component of the pre-image of V (resp. U_γ) by the projection $p_n\colon M_{\dot{\gamma}(n)}\to M$ (resp. $P_n\colon W_{\gamma(n)}\to W$).

Since $W_{\gamma(2n)}$ is the 2-fold cyclic cover of $W_{\gamma(n)}$ associated with the mod 2 reduction of $\gamma^{(n)}$, we have, by [7, Lemma 4.3],

$$\sigma^{\dot{\gamma}^{(n)}}(M_{\dot{\gamma}(n)})=\operatorname{sign}W_{\gamma(2n)}-2\operatorname{sign}W_{\gamma(n)}\,.$$

To calculate sign $W_{\gamma(m)}$, note that $W_{\gamma(m)}=W_c^{(m)}\cup\bar{V}\times S^1\cup(\overset{m}{\cup}W_0)$, where $W_c^{(m)}$ denotes the m-fold cyclic cover of W_c associated with the mod m reduction of $\bar{\gamma}_c|W_c$. Since sign $\bar{V}\times S^1=0$, the Novikov additivity implies sign $W_{\gamma(m)}=$ sign $W_c^{(m)}+m$ sign W_0. Therefore

$$\sigma^{\dot{\gamma}^{(n)}}(M_{\dot{\gamma}(n)})=\operatorname{sign}W_c^{(2n)}-2\operatorname{sign}W_c^{(n)}\,.$$

Thus the calculation is reduced to that of sign $W_c^{(m)}$. For the calculation, we use, instead of $W_c^{(m)}$, the m-fold cyclic branched cover $\hat{W}_c^{(m)}=W_c^{(m)}\cup V\times D^2$ of $M\times[-1,1]=W_c\cup V\times D^2$ branched along $V\times 0$. Note that, by the Novikov additivity and sign $V\times D^2=0$, sign $\hat{W}_c^{(m)}=$sign $W_c^{(m)}$.

Let $L\colon H_1(V;\boldsymbol{R})\times H_1(V;\boldsymbol{R})\to\boldsymbol{R}$ be the *linking form* defined by $L(x,y)=\operatorname{Link}_M(c_x,c_y^+)$ for $x=[c_x]$, $y=[c_y]\in H_1(V;\boldsymbol{R})$, where c_y^+ denotes the translation of the cycle c_y in the positive normal direction and $\operatorname{Link}_M(c_x,c_y^+)$ is the linking number of c_x with c_y^+ (*cf.* [**6**, p. 53 and p.77]). A matrix representing L for some basis of $H_1(V;\boldsymbol{R})$ is called a *linking matrix* on $H_1(V;\boldsymbol{R})$. Let $T\colon\hat{W}_c^{(m)}\to\hat{W}_c^{(m)}$ be the natural extension of the generator $T\colon W_c^{(m)}\to W_c^{(m)}$ of the group of covering transformations of the covering $P_m|W_c^{(m)}\colon W_c^{(m)}\to W_c$ which is specified by $\bar{\gamma}_c|W_c$. Let $\operatorname{Int}_{\hat{W}_c^{(m)}}\colon H_2(\hat{W}_c^{(m)};\boldsymbol{R})\times H_2(\hat{W}_c^{(m)};\boldsymbol{R})\to\boldsymbol{R}$ be the intersection form on $\hat{W}_c^{(m)}$. Take a basis $\{e_1,e_2,\cdots,e_r\}$ for $H_1(V;\boldsymbol{R})$. By a standard argument due to [**11**] or [2] and used in [7, Lemma 3.3], we have the following.

Lemma 1.1. *There exist elements* $\bar{e}_1, \cdots, \bar{e}_r, \bar{e}_{r+1}, \cdots, \bar{e}_s$ *in* $H_2(\hat{W}_c^{(m)}; \boldsymbol{R})$ *such that* $\bar{e}_1, \cdots, \bar{e}_r, T_*\bar{e}_1, \cdots, T_*\bar{e}_r, \cdots, T_*^{m-2}\bar{e}_1, \cdots, T_*^{m-2}\bar{e}_r, \bar{e}_{r+1}, \cdots, \bar{e}_s$ *form a basis for* $H_2(\hat{W}_c^{(m)}; \boldsymbol{R})$ *and such that, for* $i, j \leq r$ *and* $p, q=0, 1, \cdots, m-2$,

$$\mathrm{Int}_{\hat{W}_c^{(m)}}(T_*^p \bar{e}_i, T_*^q \bar{e}_j) = \begin{cases} 0 & \text{if } |p-q|>1\,, \\ -L(e_i, e_j) & \text{if } p=q+1\,, \\ -L(e_j, e_i) & \text{if } q=p+1\,, \\ L(e_i, e_j)+L(e_j, e_i) & \text{if } p=q\,, \end{cases}$$

and, for $i=1, 2, \cdots, s$, $j>r$ *and* $k=0, 1, \cdots, m-2$, $\mathrm{Int}_{\hat{W}_c^{(m)}}(T_*^k \bar{e}_i, \bar{e}_j)=0$.

Let $\mathcal{E}$ be the subspace of $H_2(\hat{W}_c^{(m)}; \boldsymbol{R})$ generated by $T_*^j \bar{e}_i$, $i=1, \cdots, r$, $j=0, 1, \cdots, m-2$. It is easily seen that the form $(\mathrm{Int}_{\hat{W}_c^{(m)}}|\mathcal{E}, T_*|\mathcal{E})$ is isomorphic to the symmetric $\boldsymbol{Z}_m$-form of L defined in [**11**] (although the coefficient in [**11**] is rational). Recall that the *symmetric* $\boldsymbol{Z}_m$*-form* of L is the pair $(L^{(m)}, \tau_m)$ of symmetric bilinear form $L^{(m)}: H^{m-1}\times H^{m-1}\to\boldsymbol{R}$ and isometry $\tau_m: H^{m-1}\to H^{m-1}$ of $L^{(m)}$ of order m, defined by

$$\begin{aligned} L^{(m)}(x, y) = \sum_{i=1}^{m-1} (L(\pi_i(x), \pi_i(y))+L(\pi_i(y), \pi_i(x))) \\ - \sum_{i=1}^{w-2} (L(\pi_{i+1}(x), \pi_i(y))+L(\pi_{i+1}(y), \pi_i(x))) \end{aligned}$$

and

$$\tau_m(x) = \sum_{i=1}^{m-2} \iota_{i+1}\, \pi_i(x) - \sum_{i=1}^{m-1} \iota_i\, \pi_{m-1}(x)$$

for $x, y\in H^{m-1}$, where H^{m-1} denotes the $(m-1)$th Cartesian product of the real vector space $H=H_1(V; \boldsymbol{R})$, and $\pi_i: H^{m-1}\to H$ and $\iota_i: H\to H^{m-1} (i=1, 2, \cdots, m-1)$ are the i-th coordinate projection and imbedding respectively.

Thus we have proved the following.

Proposition 1.2. $\sigma^{\dot{\gamma}^{(n)}}(M_{\dot{\gamma}(n)})=\mathrm{sign}\, L^{(2n)}-2\mathrm{sign}\, L^{(n)}$.

By using Proposition 1.2, we can express $\sigma^{\dot{\gamma}^{(n)}}(M_{\dot{\gamma}(n)})$ in terms of local signatures $\sigma_a^{\dot{\gamma}}(M)$ of $(M, \dot{\gamma})$.

Proposition 1.3. *If the Alexander polynomial* $A_{\dot{\gamma}}(t)\in\boldsymbol{Z}\langle t\rangle$ *of the homology handle* $(M, \dot{\gamma})$ *has no* $2n$*-th root of unity, then*

$$\sigma^{\dot{\gamma}^{(n)}}(M_{\dot{\gamma}(n)}) = \sum_{j=0}^{n-1} (-1)^j \sum_{a_{j+1}<a<a_j} \sigma_a^{\dot{\gamma}}(M)\,,$$

where $a_j=\cos(j\pi/n)$, $j=0, 1, \cdots, n$.

Since $|A_{\dot{\gamma}}(1)|=1$ for any homology handle $(M, \dot{\gamma})$ (*cf.* [**3**, Theorem 1.4]),

$A_{\dot{\gamma}}(t)$ always has no 4-th root of unity. Thus the following simple formula is given.

Corollary 1.4. *For any homology handle* $(M, \dot{\gamma})$,

$$\sigma^{\dot{\gamma}(2)}(M_{\dot{\gamma}(2)}) = \sum_{-1<a<1} \operatorname{sign}(a)\, \sigma_a^{\dot{\gamma}}(M)\,.$$

To prove Proposition 1.3, we need some lemmas. Let $H_C^{m-1}=H^{m-1}\otimes \boldsymbol{C}(m\geq 2)$ and $L_C^{(m)}\colon H_C^{m-1}\times H_C^{m-1}\to \boldsymbol{C}$ be the Hermitian form of $L^{(m)}$ in the usual sense (*cf.* [**11**, 3.6. Note]). The isometry $\tau_m\colon H^{m-1}\to H^{m-1}$ of $L^{(m)}$ extends to the isometry (also denoted by τ_m) $H^{m-1}\otimes\boldsymbol{C}\to H^{m-1}\otimes\boldsymbol{C}$ of $L_C^{(m)}$ naturally. Let $E_m(\zeta)$ be the eigenspace of H_C^{m-1} corresponding to the eigenvalue $\zeta\in\boldsymbol{C}$ of $\tau_m\colon H_C^{m-1}\to H_C^{m-1}$.

Lemma 1.5. *If $m=pq$, $p, q>0$, and ζ_p is a primitive p-th root of unity, then*

$$\mu\colon E_p(\zeta_p)\to E_m(\zeta_p)\,,\quad \mu(z) = \frac{1}{\sqrt{q}}\sum_{l=0}^{q-1}\sum_{j=1}^{p-1} \iota_{j+pl}^{(m)}\pi_j^{(p)}(z)$$

is an isometry between $L_C^{(p)}|_{E_p(\zeta_p)}$ and $L_C^{(m)}|_{E_m(\zeta_p)}$, where $\pi_j^{(k)}\colon H_C^{k-1}\to H_C$ and $\iota_j^{(k)}\colon H_C\to H_C^{k-1}$ are the j-th coordinate projection and imbedding respectively on H_C^{k-1}.

Proof. First we show that

$$\bar{\mu}\colon E_p(\zeta_p)\to H_C^{m-1}\,,\quad \bar{\mu}(z) = \sum_{l=0}^{q-1}\sum_{j=1}^{p-1} \iota_{j+pl}^{(m)}\pi_j^{(p)}(z)$$

is an injection and the image of $\bar{\mu}$ is $E_m(\zeta_p)$. In fact, by solving the equation $\tau_k\, z=\zeta_p\, z\,(k=p, m)$ directly, we can check that

$$E_p(\zeta_p) = \left\{(x, \sum_{j=0}^{1}\bar{\zeta}_p^j\, x, \cdots, \sum_{j=0}^{p-2}\bar{\zeta}_p^j\, x)\in H_C^{p-1};\ x\in H_C = H\otimes\boldsymbol{C}\right\}$$

and $E_m(\zeta_p)=\bar{\mu}(E_p(\zeta_p))$, from which the injectivity of $\bar{\mu}$ is obvious.

Since spaces $E_p(\zeta_p)$ and $E_m(\zeta_p)$ are such ones as described above, we can easily calculate $L_C^{(m)}(\bar{\mu}(x), \bar{\mu}(y))$ for $x, y\in E_p(\zeta_p)$ and have

$$L_C^{(m)}(\bar{\mu}(x), \bar{\mu}(y)) = qL_C^{(p)}(x, y)\,,$$

which means that $\mu=(1/\sqrt{q})\cdot\bar{\mu}$ is an isometry between $L_C^{(p)}|_{E_p(\zeta_p)}$ and $L_C^{(m)}|_{E_m(\zeta_p)}$. This completes the proof.

For $\omega\in\boldsymbol{C}$, $|\omega|=1$, $\omega\neq 1$, define a Hermitian form $L_{(\omega)}\colon (H\otimes\boldsymbol{C})\times(H\otimes\boldsymbol{C})\to \boldsymbol{C}$ by

$$L_{(\omega)}(x\otimes\alpha, y\otimes\beta) = \alpha\bar{\beta}((1-\bar{\omega})\, L(x, y)+(1-\omega)\, L(y, x))$$

for $x, y \in H$ and $\alpha, \beta \in \boldsymbol{C}$. The following lemma is well-known (*cf.* [**11**, 4.7]).

Lemma 1.6. *Let $p(\geq 2)$ be an integer. If ζ_p is a primitive p-th root of unity, then the form $L_{(\zeta_p)}$ is isomorphic to the restriction to $E_p(\zeta_p)$ of the form $L_C^{(p)}$.*

Let $\omega_x = x + \sqrt{1-x^2}\, i \in \boldsymbol{C}$, $x \in [-1, 1]$. For any real square matrix A, define a t-Hermitian $\boldsymbol{R}\langle t\rangle$-matrix

$$A^-(t) = (2-(t+t^{-1}))\,((1-t)\,A+(1-t^{-1})\,A^{\mathrm{T}})\,.$$

Kawauchi [**6**, §5] considered the "local signatures" $\sigma_a^-(A)$, $a \in [-1, 1]$, of A which are defined by $\sigma_a^-(A) = \lim_{x\to a-0} \operatorname{sign} A^-(\omega_x) - \lim_{x\to a+0} \operatorname{sign} A^-(\omega_x)$ for $a \in (-1, 1)$ and $\sigma_1^-(A) = \lim_{x\to 1-0} \operatorname{sign} A^-(\omega_x)$, $\sigma_{-1}^-(A) = \operatorname{sign}(A+A^{\mathrm{T}}) - \lim_{x\to -1+0} \operatorname{sign} A^-(\omega_x)$.

Lemma 1.7. *For $\omega_a(\neq 1)$ satisfying* $\operatorname{rank}_{\boldsymbol{C}}(A-\omega_a A^{\mathrm{T}}) = \operatorname{rank}_{\boldsymbol{R}\langle t\rangle}(A-tA^{\mathrm{T}})$,

$$\operatorname{sign}((1-\bar{\omega}_a)\,A+(1-\omega_a)\,A^{\mathrm{T}}) = \sum_{a<x\leq 1} \sigma_x^-(A)\,.$$

Proof. Note that $A^-(t) = (1-t)^2(1-t^{-1})\,(A-t^{-1}A^{\mathrm{T}})$. Let $x_1 < x_2 < \cdots < x_r$ be the all points in the interval $(a, 1)$ satisfying $\operatorname{rank}_{\boldsymbol{C}}(A-\bar{\omega}_{x_i} A^{\mathrm{T}}) < \operatorname{rank}_{\boldsymbol{R}\langle t\rangle}(A-t^{-1}A^{\mathrm{T}})$. By assumption, $\operatorname{rank}_{\boldsymbol{C}}(A-\bar{\omega}_x A^{\mathrm{T}}) = \operatorname{rank}_{\boldsymbol{R}\langle t\rangle}(A-t^{-1}A^{\mathrm{T}})$ on $x \in [a, 1) - \{x_1, x_2, \cdots, x_r\}$. Then by [**6**, Corollary 5.2],

$$\operatorname{sign} A^-(\omega_a) = \lim_{x\to x_1-0} \operatorname{sign} A^-(\omega_x)\,,$$
$$\lim_{x\to x_i+0} \operatorname{sign} A^-(\omega_x) = \lim_{x\to x_{i+1}-0} \operatorname{sign} A^-(\omega_x), \quad i = 1, \cdots, r-1$$

and

$$\lim_{x\to x_r+0} \operatorname{sign} A^-(\omega_x) = \lim_{x\to 1-0} \operatorname{sign} A^-(\omega_x) = \sigma_1^-(A)\,.$$

Thus

$$\operatorname{sign}((1-\bar{\omega}_a)\,A+(1-\omega_a)\,A^{\mathrm{T}}) = \operatorname{sign} A^-(\omega_a) = \operatorname{sign} \overline{A^-(\omega_a)} = \sum_{a<x\leq 1} \sigma_x^-(A)\,.$$

This completes the proof.

1.8. Proof of Proposition 1.3. For simplicity, we use the following notations:

$$\langle k\rangle_m = E_m(e^{2\pi ik/m})\,, \quad k = 0, 1, \cdots, m-1\,,$$
$$\sigma\langle k\rangle_m = \operatorname{sign}(L_C^{(m)} \,|\, \langle k\rangle_m)\,, \quad k = 0, 1, \cdots, m-1\,,$$
$$s_j = \sum_{a_{j+1}<a<a_j} \sigma_a^{\dot{\gamma}}(M)\,, \quad j = 0, 1, \cdots, n-1\,.$$

Note that $\langle 0\rangle_m = \{0\}$ for all m. We have to show $\sigma^{\dot{\gamma}(n)}(M_{\dot{\gamma}(n)}) = \sum_{j=0}^{n-1}(-1)^j s_j$.

First we consider the case when n is odd. In this case, H_C^{2n-1} and H_C^{n-1}

split into the orthogonal sums

$$H_C^{2n-1} = (\mathop{\perp}_{k=1}^{n-1} (\langle k\rangle_{2n} \perp \langle -k\rangle_{2n})) \perp \langle n\rangle_{2n}$$

and

$$H_C^{n-1} = \mathop{\perp}_{k=1}^{(n-1)/2} (\langle k\rangle_n \perp \langle -k\rangle_n)$$

with respect to $L_C^{(2n)}$ and $L_C^{(n)}$ respectively. By Proposition 1.2 and the fact

(†) $\sigma\langle 2k\rangle_{2n} = \sigma\langle k\rangle_n = \sigma\langle q\rangle_p$, where $0<q<p, (p, q) = 1$ and $q/p = k/n$,

which is derived from Lemma 1.5, we have

$$\begin{aligned}\sigma^{\dot{\gamma}(n)}(M_{\dot{\gamma}(n)}) &= \operatorname{sign} L_C^{(2n)} - 2\operatorname{sign} L_C^{(n)} \\ &= (\sum_{k=1}^{n-1} 2\sigma\langle k\rangle_{2n} + \sigma\langle n\rangle_{2n}) - 2\sum_{k=1}^{(n-1)/2} 2\sigma\langle k\rangle_n \\ &= 2\sum_{k=1}^{(n-1)/2} (\sigma\langle 2k-1\rangle_{2n} - \sigma\langle 2k\rangle_{2n}) + \sigma\langle n\rangle_{2n} .\end{aligned}$$

Note that $\sigma\langle k\rangle_m = \sigma\langle -k\rangle_m$ by (†) and Lemma 1.7. If the Alexander polynomial $A_{\dot{\gamma}}(t) \doteq \det(A - tA^{\mathrm{T}}) \in \boldsymbol{R}\langle t\rangle$ has no $2n$-th root of unity, then, by (†) and Lemmas 1.6 and 1.7, we have

$$\begin{aligned}\sigma\langle k\rangle_{2n} &= \operatorname{sign} L(e^{\pi ik/n}) \\ &= \operatorname{sign} ((1-e^{-\pi ik/n}) A + (1-e^{\pi ik/n}) A^{\mathrm{T}}) \\ &= \sum_{j=0}^{k-1} s_j ,\end{aligned}$$

for all $k=1, 2, \cdots, n$, where A is a linking matrix on $H=H_1(V; \boldsymbol{R})$. So we have $\sigma\langle 2k-1\rangle_{2n} - \sigma\langle 2k\rangle_{2n} = -s_{2k-1}$, $k=1, 2, \cdots, (n-1)/2$. Furthermore, by [**6**, Main Theorem], $\sigma\langle n\rangle_{2n} = \sigma\langle 1\rangle_2 = \sigma^{\dot{\gamma}}(M) = \sum_{j=0}^{n-1} s_j$. Therefore

$$\sigma^{\dot{\gamma}(n)}(M_{\dot{\gamma}(n)}) = 2\sum_{k=1}^{(n-1)/2} (-s_{2k-1}) + \sum_{j=0}^{n-1} s_j = \sum_{j=0}^{n-1} (-1)^j s_j .$$

Next we consider the case when n is even. In this case, H_C^{2n-1} and H_C^{n-1} split into the orthogonal sums

$$H_C^{2n-1} = (\mathop{\perp}_{k=1}^{n-1} (\langle k\rangle_{2n} \perp \langle -k\rangle_{2n})) \perp \langle n\rangle_{2n}$$

and

$$H_C^{n-1} = (\mathop{\perp}_{k=1}^{(n-2)/2} (\langle k\rangle_n \perp \langle -k\rangle_n)) \perp \langle n/2\rangle_n$$

respectively. By the same argument as in the odd case, we have

$$\begin{aligned}\sigma^{\dot\gamma(n)}(M_{\dot\gamma(n)}) &= \operatorname{sign} L_C^{(2n)} - 2\operatorname{sign} L_C^{(n)} \\ &= \Big(\sum_{k=1}^{n-1} 2\sigma\langle k\rangle_{2n} + \sigma\langle n\rangle_{2n}\Big) - 2\Big(\sum_{k=1}^{(n-2)/2} 2\sigma\langle k\rangle_n + \sigma\langle n/2\rangle_n\Big) \\ &= 2\Big(\sum_{k=1}^{(n-2)/2} (\sigma\langle 2k-1\rangle_{2n} - \sigma\langle 2k\rangle_{2n}) + \sigma\langle n-1\rangle_{2n}\Big) - \sigma\langle 1\rangle_2 \\ &= 2\Big(-\sum_{k=1}^{(n-2)/2} s_{2k-1} + \sum_{j=0}^{n-2} s_j\Big) - \sum_{j=0}^{n-1} s_j \\ &= 2\sum_{k=0}^{(n-2)/2} s_{2k} - \sum_{j=0}^{n-1} s_j \\ &= \sum_{j=0}^{n-1} (-1)^j s_j\,.\end{aligned}$$

This completes the proof.

EXAMPLE 1.9. Let k be a knot in S^3 and $M=M(k)$ denote S^3 surgered along k with framing zero. Then M is a homology handle. Let $\tilde{M}$ be the infinite cyclic cover of M associated with any generator $\dot\gamma$ of $H^1(M; Z)$. The quadratic form of $\tilde{M}$ on $H^1(\tilde{M}; \boldsymbol{R})$ (see [4, p. 186] for the definition) in the present case is non-singular (*cf.* [5, p. 99]).

If k is a trefoil knot, then $H^1(\tilde{M}; \boldsymbol{R}) \cong \boldsymbol{R}\langle t\rangle/(t^2-t+1)$. Thus $\sigma^{\dot\gamma}_{1/2}(M) = \pm 2$ and $\sigma^{\dot\gamma}_a(M)=0$ for $a \neq 1/2$ (*cf.* [9, Assertion 11] or [5, Lemma 1.4]). By Corollary 1.4, we have $\sigma^{\dot\gamma(2)}(M_{\dot\gamma(2)}) = \sigma^{\dot\gamma}_{1/2}(M) = \pm 2$. This result can be obtained from a direct calculation of the quadratic form by using a mapping torus structure of $M(k)_{\dot\gamma(2)}$ (*cf.* [10, p. 333]). Furthermore, if k is the g-fold connected sum of trefoil knot, then the quadratic form of $\tilde{M}$ is the orthogonal sum of g copies of the form of trefoil knot. Thus $\sigma^{\dot\gamma}_{1/2}(M(k)) = \pm 2g$ and $\sigma^{\dot\gamma}_a(M(k))=0$ for $a \neq 1/2$. By Corollary 1.4, $\sigma^{\dot\gamma(2)}(M(k)_{\dot\gamma(2)}) = \sigma^{\dot\gamma}_{1/2}(M(k)) = \pm 2g$, which of course coincides with the result obtained from the calculation using the mapping torus structure of $M(k)_{\dot\gamma(2)}$.

2. Types of Imbeddings

Throughout this section, M is a closed, connected, oriented 3-manifold and W a closed, connected 4-manifold. We consider imbeddings $f\colon M \to W$.

First note that f has at least two types according to whether $W-fM$ is connected or not. We say that f is of *type* I (resp. *type* II) if $W-fM$ is connected (resp. disconnected). We can characterize the type I or II imbedding by examining the homomorphism $f_*\colon H_3(M; \boldsymbol{Z}_2) \to H_3(W; \boldsymbol{Z}_2)$. If $f_* \neq 0$ then f is of type I, and if $f_*=0$ then f is of type II and $W-fM$ has exactly two components. This is stated in [8] in the case when W is orientable, and Kawauchi's proof is valid for non-orientable 4-manifold W. Note that the coefficient of the (co-)homology in [**8**, p. 171] is $\boldsymbol{Z}_2$.

For the rest of this section we assume that W is *non-orientable*, and classify the types of $f: M \to W$ more in detail. Let $p: \tilde{W} \to W$ be the orientation double covering of W.

Type I *imbedding*. A type I imbedding f is called *two-sided* or *one-sided* according as the normal bundle of f is trivial or not.

If f is of type I and one-sided (called *type* I_1), we have two cases according as $W-fM$ is orientable or not. These two cases may be characterized by the types of the imbedding $\tilde{M}=p^{-1}(fM) \subset \tilde{W}$. That is, $W-fM$ is non-orientable (resp. orientable) if and only if $\tilde{M} \subset \tilde{W}$ is of type I (resp. type II). Thus we say that f is of *type* I_1–1 (resp. *type* I_1–2) if $W-fM$ is non-orientable (resp. orientable).

If f is of type I and two-sided (called *type* I_2), then f can be lifted to two imbeddings $\tilde{f}: M \to \tilde{W}$, each of which is of type I. [To see that $\tilde{f}$ is of type I, note that there is a loop α in W which intersects fM transversely in a single point. If α preserves orientation, then one of the lifts of α to $\tilde{W}$ intersects $\tilde{f}M$ transversely in a single point. Thus $\tilde{f}_* \neq 0: H_3(M; \boldsymbol{Z}_2) \to H_3(\tilde{W}; \boldsymbol{Z}_2)$, which means $\tilde{f}$ is of type I. If α reverses orientation, then, by using the loop $p^{-1}\alpha$, we can do the same argument as above and have the same conclusion.]

Type II *imbedding*. Assume f is of type II. Let W_1, W_2 be the components of $W-fM$. Since W is non-orientable and M is connected, we have the following two cases:

a) both W_1 and W_2 are non-orientable,

b) one of W_1 and W_2 is orientable and the other is non-orientable.

The type II imbedding f can be lifted to two imbeddings $\tilde{f}: M \to \tilde{W}$. Take any one of them. Then it is easily seen that a) (resp. b)) is equal to the condition that $\tilde{f}$ is of type I (resp. $\tilde{f}$ is of type II). From this, in case a) (resp. b)) we say that f is of *type* II–1 (resp. *type* II–2).

3. Proof of Theorem

Throughout this section, for a manifold X with boundary, DX denotes the double of X. For a closed oriented 3-manifold M equipped with an element $\dot{\gamma} \in H^1(M; \boldsymbol{Z})$, we define $\tau_a^{\dot{\gamma}}(M)=\sum_{x \in (a,1]} \sigma_x^{\dot{\gamma}}(M)$ for all $a \in [-1, 1]$ (*cf.* [**8**]). We denote by $\kappa_1^{\dot{\gamma}}(M)$ the rank of the kernel of the homomorphism $t-1: H_1(\tilde{M}; \boldsymbol{Z}) \to H_1(\tilde{M}; \boldsymbol{Z})$, where $\tilde{M}$ is the infinite cyclic cover of M associated with $\dot{\gamma}$ and t: $H_1(\tilde{M}; \boldsymbol{Z}) \to H_1(\tilde{M}; \boldsymbol{Z})$ is the automorphism induced from the generator specified by $\dot{\gamma}$ of the group of covering transformations on $\tilde{M}$ (*cf.* [**8**]).

For the rest of this section, M denotes a closed, connected, oriented 3-manifold and W denotes a compact, connected 4-manifold. Let M^0 denote the once punctured M. Recall that an element $\dot{\gamma} \in H^1(DM^0; \boldsymbol{Z})$ is called $\boldsymbol{Z}_2$-*asym-*

metric if the mod 2 reduction $\dot\gamma(2)\in H^1(DM^0; \boldsymbol{Z}_2)$ of $\dot\gamma$ satisfies $\rho_*(\dot\gamma(2))\neq\dot\gamma(2)$ for the standard reflection ρ of DM^0 ([**8**, p. 179]). Theorem 3.1 of [**8**] can be extended to the case of orientable 4-manifold W *with boundary*.

Lemma 3.1. *Assume that W is orientable and $\partial W\neq\emptyset$. If M^0 is imbedded in W, then $\beta_1(M; \boldsymbol{Z})\leq\beta_2(W; \boldsymbol{Z}_2)$ or there is a $\boldsymbol{Z}_2$-asymmetric indivisible element $\dot\gamma\in H^1(DM^0; \boldsymbol{Z})$ such that for all $a\in[-1, 1]$*

$$|\tau_a^{\dot\gamma}(DM^0)|-\kappa_1^{\dot\gamma}(DM^0)\leq 2\beta_2(W; \boldsymbol{Z}).$$

Proof. Applying [**8**, Theorem 3.1] to the imbedding $M^0\subset W\subset DW$, we have the above conclusion. Note that $\beta_2(DW; \boldsymbol{Z})=2\beta_2(W; \boldsymbol{Z})$, $\beta_2(DW; \boldsymbol{Z}_2)=2\beta_2(W; \boldsymbol{Z}_2)$ and $\operatorname{sign} DW=0$.

We then think of non-orientable case.

Lemma 3.2. *Assume W is non-orientable and closed. Let $f\colon M\to W$ be an imbedding.*

(1) *If f is of type I_2 or II–1, then $\beta_1(M; \boldsymbol{Z})\leq\beta_2(W; \boldsymbol{Z}_2)$ or there is a $\boldsymbol{Z}_2$-asymmetric indivisible element $\dot\gamma\in H^1(DM^0; \boldsymbol{Z})$ such that for all $a\in[-1, 1]$*

$$|\tau_a^{\dot\gamma}(DM^0)|-\kappa_1^{\dot\gamma}(DM^0)\leq\beta_2(W; \boldsymbol{Z})+\beta_2(W; \boldsymbol{Z}_2).$$

(2) *If f is of type II–2, then $2\beta_1(M; \boldsymbol{Z})\leq\beta_2(W; \boldsymbol{Z})+\beta_2(W; \boldsymbol{Z}_2)$ or there is an indivisible element $\dot\gamma\in H^1(M; \boldsymbol{Z})$ such that for all $a\in[-1, 1]$*

$$|\tau_a^{\dot\gamma}(M)|-\kappa_1^{\dot\gamma}(M)\leq\beta_2(W; \boldsymbol{Z})+\beta_2(W; \boldsymbol{Z}_2).$$

Proof. Let $\tilde W$ be the orientation double cover of W. As seen in section 2, each imbedding of above types has a lift $\tilde f\colon M\to\tilde W$. Applying [**8**, Theorems 2.1, 3.1] to $\tilde f$ and noting the following lemma and the fact that $\operatorname{sign}\tilde W=0$ [because $\tilde W$ admits an orientation-reversing involution], we have the result.

Lemma 3.3. *Let X be a compact manifold and $\tilde X$ be any double cover of X. Then $\beta_k(X; \boldsymbol{Z})\leq\beta_k(\tilde X; \boldsymbol{Z})\leq\beta_k(X; \boldsymbol{Z})+\beta_k(X; \boldsymbol{Z}_2)$ and $\beta_k(\tilde X; \boldsymbol{Z}_2)\leq 2\beta_k(X, \boldsymbol{Z}_2)$ for all k.*

Proof. By the transfer argument, we have $\beta_k(X; \boldsymbol{Z})\leq\beta_k(\tilde X; \boldsymbol{Z})$. The inequality $\beta_k(\tilde X; \boldsymbol{Z})\leq\beta_k(X; \boldsymbol{Z})+\beta_k(X; \boldsymbol{Z}_2)$ is the case $d=2$ of [**1**, Proposition 1.3]. The inequality $\beta_k(\tilde X; \boldsymbol{Z}_2)\leq 2\beta_k(X; \boldsymbol{Z}_2)$ is readily obtained from the exact sequence of Smith homology groups used in the proof of [**1**, Proposition 1.3].

In the case of type I_1 imbedding, we cannot use [**8**, Theorem 2.1, 3.1] as in the proof of Lemma 3.2. But for certain M an estimation like Lemma 3.2 can be obtained by using the consequence of Section 1. For each positive integer r, consider the class $\mathcal{M}(r)$ of 3-manifolds consisting of the connected sums of r homology handles:

$$\mathcal{M}(r)=\{M=\mathop{\#}_{i=1}^{r} M_i;\ M_i \text{ is a 3-manifold with } H_*(M_i;\boldsymbol{Z})\cong H_*(S^2\times S^1;\boldsymbol{Z}),\ \forall i\}.$$

Especially we have a subclass $\mathcal{M}'(r)$ of $\mathcal{M}(r)$ consisting of all $M=\#_{i=1}^{r} M_i$ such that each M_i is S^3 surgered along a knot with framing zero (*cf.* Example 1.9). Note that, for any $M\in\mathcal{M}(r)$ and any $\dot{\gamma}\in H^1(M;\boldsymbol{Z})$, $\tau_{-1}^{\dot{\gamma}}(M)=\sigma^{\dot{\gamma}}(M)$ and $\kappa_1^{\dot{\gamma}}(M)=0$ (*cf.* [**3**]). For an (oriented) homology handle M, we denote by $\sigma(M)$ (resp. $\sigma_a(M)$) the signature $\sigma^{\dot{\gamma}}(M)$ (resp. the local signature $\sigma_a^{\dot{\gamma}}(M)$) associated with any generator $\dot{\gamma}$ of $H^1(M;\boldsymbol{Z})$. [Note that $\sigma^{\dot{\gamma}}(M)=\sigma^{-\dot{\gamma}}(M)$ and $\sigma_a^{\dot{\gamma}}(M)=\sigma_{\bar{a}}^{-\dot{\gamma}}(M)$.]

Lemma 3.4. *Let W be as in Lemma 3.2. Let $M=\#_{i=1}^{r} M_i$ be the connected sum of homology handles M_i, $i=1,2,\cdots,r$. If M is type I_1 imbedded in W, then $r\le\beta_2(W;\boldsymbol{Z}_2)$ or there are numbers $(1\le)\, i_1, i_2, \cdots, i_p, i_{p+1}, \cdots, i_q(\le r)$ such that*

$$(*)\quad \Big|\sum_{j=1}^{p}\sum_{-1<a<1}\varepsilon_j\,\mathrm{sign}\,(a)\,\sigma_a(M_{i_j})+\sum_{j=p+1}^{q}\varepsilon_j\,\sigma(M_{i_j})\Big|\le\beta_2(W;\boldsymbol{Z})+\beta_2(W;\boldsymbol{Z}_2),$$

where $\varepsilon_j=1$, or -1, $j=1,2,\cdots,q$.

Proof. Assume that M is type I_1 imbedded in W. We think M is a submanifold of W. If $p\colon \tilde{W}\to W$ is the orientation double covering of W, then $M^{(2)}=p^{-1}M\subset\tilde{W}$ is a double cover of M.

Since the mod 2 reduction $H^1(M;\boldsymbol{Z})\cong\oplus_{i=1}^{r}\boldsymbol{Z}\to H^1(M;\boldsymbol{Z}_2)\cong\oplus_{i=1}^{r}\boldsymbol{Z}_2$ is onto, any double cover of M is associated with the mod 2 reduction $\psi(2)\in H^1(M;\boldsymbol{Z}_2)$ of some $\psi\in H^1(M;\boldsymbol{Z})$. For each $i=1,2,\cdots,r$, the restriction $\psi(2)\,|\,M_i$ is the δ_i multiple of the generator of $H^1(M_i;\boldsymbol{Z}_2)\cong\boldsymbol{Z}_2$, where $\delta_i=0$ or 1. Thus we denote $\psi(2)$ by $\psi[\delta_1,\cdots,\delta_r]$.

We may assume $M^{(2)}$ is the double cover corresponding to $\psi[\overbrace{1,\cdots,1}^{m},0,\cdots,0]$ by permuting the indices if necessary. Then $M^{(2)}$ is diffeomorphic to

$$(\mathop{\#}_{i=1}^{m} M_i^{(2)})\,\#\,(\mathop{\#}_{i=m+1}^{r} M_i)\,\#\,(\mathop{\#}_{i=m+1}^{r} M_i)\,\#\,(\mathop{\#}^{m-1} S^2\times S^1),$$

where $M_i^{(2)}$ denotes the unique (up to equivalence) double cover of M_i.

Put $\hat{M}=(\#_{i=1}^{m} M_i^{(2)})\#(\#_{i=m+1}^{r} M_i)$. Since $\hat{M}^0$ is imbedded in $M^{(2)}$ naturally, $\hat{M}^0$ can be imbedded into $\tilde{W}$. Applying Theorem 3.1 of [**8**] and using Lemma 3.3, we have $r\le\beta_2(W;\boldsymbol{Z}_2)$ or there is a $\boldsymbol{Z}_2$-asymmetric indivisible element $\dot{\eta}\in H^1(D\hat{M}^0;\boldsymbol{Z})$ such that $|\sigma^{\dot{\eta}}(D\hat{M}^0)|\le\beta_2(W;\boldsymbol{Z})+\beta_2(W;\boldsymbol{Z}_2)$. (Note that $\tau_{-1}^{\dot{\eta}}(D\hat{M}^0)=\sigma^{\dot{\eta}}(D\hat{M}^0)$.)

Since $D\hat{M}^0=[\#_{i=1}^{m}(M_i^{(2)}\#-M_i^{(2)})]\#[\#_{i=m+1}^{r}(M_i\#-M_i)]$, we have

$$\sigma^{\dot{\eta}}(D\hat{M}^0)=\sum_{i=1}^{m}\sigma^{\dot{\eta}_i}(M_i^{(2)}\#-M_i^{(2)})+\sum_{i=m+1}^{r}\sigma^{\dot{\eta}_i}(M_i\#-M_i),$$

where $\dot{\eta}_i$ is the restriction of $\dot{\eta}$ to the i-th summand, $i=1, 2, \cdots, r$. Let $\{i_j;\ 1\leq j\leq p\}$ (resp. $\{i_j;\ p+1\leq j\leq q\}$) be the set of all integers i between 1 and m (resp. $m+1$ and r) such that the restriction $\dot{\eta}_i$ of $\dot{\eta}$ is still $\boldsymbol{Z}_2$-asymmetric. Then by [**8**, Lemma 1.3] we have

$$\sigma^{\dot{\eta}}(D\hat{M}^0)=\sum_{j=1}^{p}\varepsilon_j\,\sigma^{\dot{\gamma}_{i_j}^{(2)}}(M_{i_j}^{(2)})+\sum_{j=p+1}^{q}\varepsilon_j\,\sigma(M_{i_j})\,,$$

for some $\varepsilon_j\in\{1, -1\}$, $j=1, 2, \cdots, q$, where $\dot{\gamma}_{i_j}^{(2)}\in H^1(M_{i_j}^{(2)};\boldsymbol{Z})\cong\boldsymbol{Z}$ is the element defined, as in section 1, by a generator $\dot{\gamma}_{i_j}$ of $H^1(M_{i_j};\boldsymbol{Z})$, $j=1, 2, \cdots, p$. Compare the proof of [**8**, Theorem 3.2]. Since $\sigma^{\dot{\gamma}_i^{(2)}}(M_i^{(2)})=\sum_{-1<a<1}\operatorname{sign}(a)\,\sigma_a(M_i)$ by Corollary 1.4, this implies the inequality (*). This completes the proof.

We now prove Theorem.

3.5. Proof of Theorem for orientable 4-manifold W. Assume that W is compact, connected and orientable. If W is closed, then Theorem is an immediate consequence of [**8**, Theorem 3.2] showing that, for sufficiently large r_0 and for all $r>r_0$, certain elements of $\mathcal{M}'(r)$ cannot be imbedded in W. If W is bounded, then Lemma 3.1 implies Theorem by the same argument as the proof of [**8**, Theorem 3.2].

3.6. Proof of Theorem for non-orientable 4-manifold W. Assume first that W is closed, connected and non-orientable. Let $M[g]$ be S^3 surgered along the g-fold connected sum of trefoil knot with framing zero. Recall that, for any generator $\dot{\gamma}\in H^1(M[g];\boldsymbol{Z})\cong\boldsymbol{Z}$, $\sigma^{\dot{\gamma}}(M[g])=\sigma^{\dot{\gamma}^{(2)}}(M[g]_{\dot{\gamma}^{(2)}})=\pm 2g$ (*cf.* Example 1.9).

From now on, assume $M=\#_{i=1}^{r} M[g_i]$. We show that if M is imbedded in W, then one of the following conditions holds:

(1) $r\leq\beta_2(W;\boldsymbol{Z}_2)$.

(2) For some numbers $(1\leq)\, i_1, i_2, \cdots, i_s\,(\leq r)$ and for some choice of $\varepsilon_j\in\{1, -1\}$, $j=1, 2, \cdots, s$, the inequality

$$2\,|\varepsilon_1 g_{i_1}+\varepsilon_2 g_{i_2}+\cdots+\varepsilon_s g_{i_s}|\leq\beta_2(W;\boldsymbol{Z})+\beta_2(W;\boldsymbol{Z}_2)$$

holds.

In fact, if M is type I_1 imbedded in W, then, by Lemma 3.4, we obtain the desired result. If M is type I_2 or II–1 imbedded in W, then, by Lemma 3.2-(1), we have the above result. Compare the proof of Lemma 3.4. If M is type II–2 imbedded in W, then by Lemma 3.2–(2) we have $r\leq[\beta_2(W;\boldsymbol{Z})+\beta_2(W;\boldsymbol{Z}_2)]/2$ or the above condition (2) holds. Note that for an indivisible element $\dot{\gamma}\in H^1(M;\boldsymbol{Z})$, if $M[g_{i_j}]$, $j=1, 2, \cdots, s$, $1\leq i_1<i_2<\cdots<i_s\leq r$ are the all summands of M such that $\dot{\gamma}\,|\,M[g_{i_j}]$ is an odd multiple of a generator of $H^1(M[g_{i_j}];\boldsymbol{Z})\cong\boldsymbol{Z}$, we have

$$\tau_{-1}^{\dot{\gamma}}(M) = \sigma^{\dot{\gamma}}(M) = 2(\varepsilon_1 g_{i_1} + \varepsilon_2 g_{i_2} + \cdots + \varepsilon_s g_{i_s})$$

for some $\varepsilon_i \in \{1, -1\}$ (*cf.* [**8**, Lemma 1.3]).

Thus, if we take $r_0 = \beta_2(W; \boldsymbol{Z}_2)$, then for all $r > r_0$ and for $\{g_i\}_{i=1}^r$ such that

$$g_1 \geq \beta_2(W; \boldsymbol{Z}_2) \quad \text{and} \quad g_i \geq \beta_2(W; \boldsymbol{Z}_2) + \sum_{j=1}^{i-1} g_j, \ i = 2, 3, \cdots, r,$$

$M = \#_{i=1}^r M[g_i]$ cannot be imbedded in W. This implies Theorem for closed non-orientable 4-manifold W.

To have Theorem for non-orientable 4-manifold W *with boundary*, we have only to use the doubling technique as in the orientable case. The proof of Theorem is completed.

References

[1] P.M. Gilmer: *Configurations of surfaces in 4-manifolds*, Trans. Amer, Math. Soc. **264** (1981), 353–380.

[2] L.H. Kauffman: *Branched coverings, open books and knot periodicity*, Topology **13** (1974), 143–160.

[3] A. Kawauchi: *Three dimensional homology handles and circles*, Osaka J. Math. **12** (1975), 565–581.

[4] ———: *On quadratic forms of 3-manifolds*, Invent. Math. **43** (1977), 177–198.

[5] ———: *On a 4-manifold homology equivalent to a bouquet of surfaces*, Trans. Amer. Math. Soc. **262** (1980), 95–112.

[6] ———: *The signature invariants of infinite cyclic coverings of closed odd dimensional manifolds*, "Algebraic and Topological Theories-to the memory of Dr. MIYATA," Kinokuniya Co. Ltd., 1985, 52–85.

[7] ———: *On the signature invariants of infinite cyclic coverings of even dimensional manifolds*, "Homotopy Theory and Related Topics," Adavanced Studies in Pure Math., 1986, 177–188.

[8] ———: *The imbedding problem of 3-manifolds into 4-manifolds*, Osaka J. Math. **25** (1988), 171–183.

[9] J.W. Milnor: *Infinite cyclic coverings*, "Conference on the Topology of Manifolds," Prindle, Weber & Schmidt, Boston, Mass., 1968, 115–133.

[10] D. Rolfsen: "Knots and Links," Math. Lecture Series ♯7, Publish or Perish Inc., Berkeley, 1976.

[11] O.Ja. Viro: *Branched coverings of manifolds with boundary and link invariants. I*, Math. USSR Izv. **7** (1973), 1239–1256.

Department of Mathematics
Osaka City University
Sugimoto, Sumiyoshi-ku
Osaka 558, Japan