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COMPLETE MINIMAL HYPERSURFACES IN $S^4(1)$ WITH CONSTANT SCALAR CURVATURE

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1. Introduction

Let M be an n-dimensional closed minimally immersed hypersurface in the unit sphere $S^{n+1}(1)$. If the square S of the length of the second fundamental form h on M satisfies $0 \leqslant S \leqslant n$, then $S \equiv 0$ or $S \equiv n$. In [3], S.S. Chern, M. do Carmo and S. Kobayashi proved that the Clifford tori are the only minimal hypersurfaces with $S=n$. C. K. Peng and C. L. Terng [6] studied the case $S=$ constant and shown, among other things, that if $n=3$ and $S>3$, then $S \geqslant 6$. The condition $S=6$ is also assumed in the examples of Cartan [1] and Hsiang [4]. On the other hand, in Otsuki's examples of minimal hypersurface in $S^{n+1}(1)$ (see [5]), H. D. Hu proved that there exist complete and non-compact minimal hypersurfaces in $S^{n+1}(1)$. Hence, it is interesting to study complete minimal hypersurfaces in $S^{n+1}(1)$. In [2], the author considered a compete minimally immersed hypersurface M in $S^{n+1}(1)$ with $S=$constant, and proved that if $0 \leqslant S \leqslant n$, then $S=0$ or $S=n$.

In this paper, we generalize the above theorem due to C. K. Peng and C. L. Terng [6] to complete minimal hypersurfaces. That is, we obtain the following.

Theorem. *Let M^3 be a complete minimally immersed hypersuface in $S^4(1)$ with $S=$constant. If $S>3$, then $S \geqslant 6$.*

Corollary. *Let M^3 be a complete minimally immersed hypersurface in $S^4(1)$ with $S=$constant. If $0 \leqslant S \leqslant 6$, then $S=0$, $S=3$ or $S=6$.*

Proof. According to Theorem and the result of the author [2], Corollary is true obviously.

2. Preliminaries

Let M be an n-dimensional immersed hypersurface in the $n+1$-dimensional unit sphere $S^{n+1}(1)$. We choose a local field of orthonormal frames $e_1, \cdots, e_{n+1}$ in $S^{n+1}(1)$ such that, restricted to M, the vectors $e_1, \cdots, e_n$ are tangent to M. We use the following convention on the range of indices unless otherwise stated:

$A, B, C, \cdots = 1, 2, \cdots, n+1$, $i, j, k, \cdots = 1, 2, \cdots, n$. And we agree the repeated indices under a summation sign without indication are summed over the respective range. With respect to the frame field of $S^{n+1}(1)$ chosen above, let $\omega_1, \cdots, \omega_{n+1}$ be the dual frame. Then the structure equations of $S^{n+1}(1)$ are given by

$$d\omega_A = -\sum \omega_{AB} \wedge \omega_B, \; \omega_{AB} + \omega_{BA} = 0\,, \tag{2.1}$$

$$d\omega_{AB} = -\sum \omega_{AC} \wedge \omega_{CB} + \Omega_{AB}\,, \tag{3.2}$$

$$\Omega_{AB} = \frac{1}{2} \sum K_{ABCD}\, \omega_C \wedge \omega_D\,. \tag{2.3}$$

Restricting these forms to M, we have the structure equations of the immersion.

$$\omega_{n+1} = 0\,. \tag{2.4}$$

$$\omega_{n+1,i} = \sum h_{ij}\, \omega_j, \; h_{ij} = h_{ji}\,, \tag{2.5}$$

$$d\omega_{ij} = -\sum \omega_{ij} \wedge \omega_j, \; \omega_{ij} + \omega_{ji} = 0\,, \tag{2.6}$$

$$d\omega_{ij} = -\sum \omega_{ik} \wedge \omega_{jk} + \frac{1}{2} \sum R_{ijkl}\, \omega_k \wedge \omega_l\,. \tag{2.7}$$

The symmetric 2-form

$$h = \sum h_{ij}\, \omega_i\, \omega_j$$

and the scalar

$$H = \frac{1}{3} \sum h_{ii}$$

are called the second fundamental form and the mean curvature of M respectively. If $H=0$, then M is said to be minimal.

Define h_{ijk} by

$$\sum h_{ijk}\, \omega_k = dh_{ij} - \sum h_{im}\, \omega_{mj} - \sum h_{mj}\, \omega_{mi}\,, \tag{2.8}$$

Exterior differentiating (2.5) and using structure equations, we obtain

$$\sum_{k,j} h_{ijk}\, \omega_k \wedge \omega_j = 0\,.$$

Thus we have

$$h_{ijk} = h_{ikj}\,. \tag{2.9}$$

Similarly define h_{ijkl} by

$$\sum h_{ijkl}\, \omega_l = dh_{ijk} - \sum h_{ijm}\, \omega_{mk} - \sum h_{imk}\, \omega_{mj} - \sum h_{mjk}\, \omega_{mi}\,, \tag{2.10}$$

then,

$$h_{ijkl}-h_{ijlk} = \sum h_{im} R_{mjkl}+\sum h_{mj} R_{mikl}\,. \tag{2.11}$$

If the square S of length of h, *i.e.*, $S=\sum h_{ij}^2$, is constant and M is minimal, then the following formulas are well known (see [6]).

For any point $p\in M$, we can choose a frame field $e_1, \cdots, e_n$ so that $h_{ij}=\lambda_i \delta_{ij}$.

$$\sum h_{ijk}^2 = S(S-n)\,, \tag{2.12}$$

$$\sum h_{ijkl}^2 = S(S-n)(S-2n-3)+3(A-2B)\,. \tag{2.13}$$

where $A=\sum h_{ijk}^2 \lambda_i^2$, $B = \sum h_{ijk}^2 \lambda_i \lambda_j$.

$$t_{ij} = h_{ijij}-h_{jiji} = (\lambda_i-\lambda_j)(1+\lambda_i \lambda_j)\,. \tag{2.14}$$

Let $f_m=\sum \lambda_j^m$. Then we have

$$\sum t_{ij}^2 = 2[nS-2S^2+Sf_4-f_3^2]\,, \tag{2.15}$$

$$\Delta f_3 = 3[(n-S)f_3+2\sum h_{ijk}^2 \lambda_i]\,. \tag{2.16}$$

When $n=3$, we have

Lemma 1 (*see* [6]). (1) *f_3=constant if and only if M has constant principal curvature*; (2) $-\sqrt{S^3}/6\leqslant f_3\leqslant\sqrt{S^3}/6$ *and equality is reached if and only if two of the principal curvature are equal.*

Lemma 2 (*see* [7]). *Let M be a complete Riemannian manifold whose Ricci curvature is bounded from below. Let f be a C^2-function which is bounded from above on M. Then there exists a sequence $\{p_m\}$ such that*

$$\lim f(p_m) = \sup f,\ \lim \|\nabla f(p_m)\| = 0, \quad \lim\sup \Delta f(p_m)\leqslant 0 \tag{2.17}$$

3. Proof of Theorem

At first, we show the following two propositions.

Proposition 1. *Let M be a complete minimal hypersurface in $S^4(1)$ with S=constant. If $\inf f_3\cdot \sup f_3=0$, and $S>3$, then $S\geqslant 6$.*

Proof. Because of $\inf f_3\cdot\sup f_3=0$, we have $\inf f_3=0$ or $\sup f_3=0$. If $\inf f_3=\sup f_3=0$, namely, f_3 vanishes identically, then it follows from Lemma 1 (1) that M has constant principal curvature. Thus Proposition 1 is true. (cf. [6: Corollary 1])

Next we will only consider the case $f_3\neq$ constant. Without loss of generality, we can suppose $\sup f_3=0$. According to the Gauss' equation and the assumption that S is constnat, we see that the Ricci curvature of M is bounded from below. Hence we can apply Lemma 2 to f_3 and we have a sequence $\{p_m\}$ in M such that

$$\lim_{m\to\infty} f_3(p_m) = \sup f_3 = 0\,, \quad \lim_{m\to\infty} ||\nabla f_3(p_m)|| = 0\,. \tag{3.1}$$

$$\lim_{m\to\infty} \sup \Delta f_3(p_m) \leqslant 0\,. \tag{3.2}$$

Since λ_i, h_{ijk} and h_{ijkl} are bounded because of (2.12) and (2.13), we may assume that

$$\lim_{m\to\infty} \lambda_i(p_m) = \lambda_i^\circ \tag{3.3}$$

$$\lim_{m\to\infty} h_{ijk}(p_m) = h_{ijk}^\circ\,, \tag{3.4}$$

$$\lim_{m\to\infty} h_{ijkl}(p_m) = h_{ijkl}^\circ\,, \tag{3.5}$$

by taking a subsequence of $\{p_m\}$ if necessary. Hence

$$\begin{aligned} &\lambda_1^\circ+\lambda_2^\circ+\lambda_3^\circ = 0\,, \\ &\lambda_1^{\circ 2}+\lambda_2^{\circ 2}+\lambda_3^{\circ 2} = S\,, \\ &\lambda_1^{\circ 3}+\lambda_2^{\circ 3}+\lambda_3^{\circ 3} = 0\,, \end{aligned} \tag{3,6}$$

that is,

$$\lambda_1^\circ = -\sqrt{S}/2,\ \lambda_2^\circ = 0 \quad \text{and} \quad \lambda_3^\circ = \sqrt{S}/2\,. \tag{3.7}$$

Here we assume $\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3$.

By differentiating $\sum h_{ii}=0$ and $\sum h_{ij}^2=S=\text{constant}$, we obtain

$$\sum h_{iik} = 0\,, \tag{3.8}$$

$$\sum h_{iik}\lambda_i = 0\,. \tag{3.9}$$

(3.3) and (3.4) imply

$$\sum h_{iik}^\circ = 0\,, \tag{3.10}$$

$$\sum h_{iik}^\circ \lambda_i^\circ = 0\,. \tag{3.11}$$

According to (3.1), we have $\lim_{m\to\infty} ||\nabla f_3||(p_m)=0$. Since

$$||\nabla f_3|| = [\sum_k (\sum_i h_{iik}\lambda_i^2)^2]^{1/2}\,,$$

we obtain

$$\lim_{m\to\infty} ||\nabla f_3||(p_m) = \lim_{m\to\infty} [\sum_k (\sum_i h_{iik}\lambda_i^2)^2]^{1/1}(p_m) = 0\,.$$

Thus, by (3.3), (3.4) and the above fact, we get

(3.12) $$\sum h^{\circ}_{iik}\lambda_i^{\circ 2}=0\,,\quad \text{for any } k.$$

Because λ_i° are distinct, (3.10), (3.13) and (3.12) yield

(3.13) $$h^{\circ}_{iik}=0\,,\quad \text{for any } i \text{ and } k.$$

On the other hand,

$$\begin{aligned}3(A-2B) &= \sum h^2_{ijk}[\lambda_i^2+\lambda_j^2+\lambda_k^2-2\lambda_i\lambda_j-2\lambda_i\lambda_k-2\lambda_j\lambda_k]\\ &= \sum_{i\neq j\neq k \atop i\neq k} h^2_{ijk}[2(\lambda_i^2+\lambda_j^2+\lambda_k^2)-(\lambda_i+\lambda_j+\lambda_k)^2]\\ &\quad +3\sum_{i\neq k} h^2_{iik}(\lambda_k^2-4\lambda_i\lambda_k)-3\sum h^2_{iii}\lambda_i^2\,.\end{aligned}$$

Hence

(3.14) $$\begin{aligned}&\lim_{m\to\infty} 3(A-2B)(p_m)\\ &= \sum_{i\neq j\neq k \atop i\neq k} h^{\circ 2}_{ijk}[2(\lambda_i^{\circ 2}+\lambda_j^{\circ 2}+\lambda_k^{\circ 2})-(\lambda_i^{\circ}+\lambda_j^{\circ}+\lambda_k^{\circ})^2]\\ &\quad +3\sum_{i\neq k} h^{\circ 2}_{iik}(\lambda_k^{\circ 2}-4\lambda_i^{\circ}\lambda_k^{\circ})-3\sum h^{\circ 2}_{iii}\lambda_i^{\circ 2}\\ &= 2S\sum h^{\circ 2}_{ijk}\quad (\text{by (3.13) and (3.6)})\\ &= 2S^2(S-3)\quad (\text{by (2.12)})\,;\end{aligned}$$

(3.15) $$\begin{aligned}\sum h^2_{ijkl} &\geqslant 3\sum_{i\neq j} h^2_{ijij}+\sum_i h^2_{iiii}\\ &\geqslant 3\sum_{i\neq j}(h_{ijij}-t_{ij}/2)^2+\frac{3}{4}\sum t^2_{ij}\,,\end{aligned}$$

where $t_{ij}=h_{ijij}-h_{jiji}=(\lambda_i-\lambda_j)(1+\lambda_i\lambda_j)$. From (3.2) and (3.5), we get

(3.16) $$\sum h^{\circ 2}_{ijkl}\geqslant 3\sum_{i\neq j}(h^{\circ}_{ijij}-t^{\circ}_{ij}/2)^2+\frac{3}{4}\sum t^{\circ 2}_{ij}\,,$$

where $t^{\circ}_{ij}=h^{\circ}_{ijij}-h^{\circ}_{jiji}=(\lambda_i^{\circ}-\lambda_j^{\circ})(1+\lambda_i^{\circ}\lambda_j^{\circ})$. (3.7) implies

(3.17) $$\sum t^{\circ 2}_{ij}=S^2-4S^3+6S\,.$$

Accordingly,

(3.18) $$\begin{aligned}&S(S-3)(S-9)+2S^2(S-3)\\ &\quad\geqslant 3\sum_{i\neq j}(h^{\circ}_{ijij}-t^{\circ}_{ij}/2)^2+\frac{3}{4}(S^3-4S^2+6S)\,.\end{aligned}$$

(3.19) $$\begin{aligned}&\sum_{i\neq j}(h^{\circ}_{ijij}-t^{\circ}_{ij}/2)^2\\ &\geqslant 2[(h^{\circ}_{1212}-t^{\circ}_{12}/2)^2+(h^{\circ}_{2323}-t^{\circ}_{23}/2)^2]\\ &= [h^{\circ}_{1212}+h^{\circ}_{323}-\frac{1}{2}(t^{\circ}_{2}-t^{\circ}_{3})]^2+[h^{\circ}_{1212}-h^{\circ}_{2323}-\frac{1}{2}(t^{\circ}_{12}-t^{\circ}_{23})]^2\\ &\geqslant [h^{\circ}_{1212}+h^{\circ}_{2323}]^2\,,\end{aligned}$$

here we make use of $\overset{\circ}{t}_{12}=\overset{\circ}{t}_{23}=-\sqrt{S}/2$. Differentiating $\sum h_{ij}^2=S$, we obtain

$$\sum_{i,j} h_{ijll}\, h_{ij}+\sum_{i,j} h_{ijl}^2=0\,, \qquad \text{for} \quad l=1,2,3,$$

which implies

$$\sum_{i} h_{iill}\, \lambda_i+\sum_{i,j} h_{ijl}^2=0\,, \qquad \text{for} \quad l=1,2,3.$$

Hence

$$\sum \overset{\circ}{h}_{iill}\, \overset{\circ}{\lambda}_i+\sum \overset{\circ}{h}{}_{ijl}^2=0\,, \qquad \text{for} \quad l=1,2,3. \tag{3.20}$$

We substitute (3.7) into (3.20). Then

$$\sqrt{S}/2[\overset{\circ}{h}_{11ll}-\overset{\circ}{h}_{33ll}]=\sum_{i,j} \overset{\circ}{h}{}_{ijl}^2\,, \qquad \text{for} \quad l=1,2,3.$$

In particular, we have

$$\sqrt{S}/2[\overset{\circ}{h}_{1122}-\overset{\circ}{h}_{3322}]=\sum \overset{\circ}{h}{}_{ij2}^2\,.$$

Hence

$$\begin{aligned}\overset{\circ}{h}_{1212}-\overset{\circ}{h}_{2323}&=\overset{\circ}{h}_{1122}-\overset{\circ}{h}_{2233}\\&=\overset{\circ}{h}_{1122}-\overset{\circ}{h}_{3322}-\overset{\circ}{t}_{23}=\sqrt{2}/S[\sum \overset{\circ}{h}{}_{ij2}^2+S/2]\\&=\sqrt{2}/S[\frac{1}{3}S(S-3)+S/2]\,,\end{aligned} \tag{3.21}$$

since we use $\sum \overset{\circ}{h}{}_{ij2}^2=2\overset{\circ}{h}{}_{123}^2=\frac{1}{3}S(S-3)$ by (3.13).

By means of (3.18), (3.19) and (3.21), we get

$$3S(S-3)^2\geqslant\frac{6}{S}[\frac{1}{3}S(S-3)+\frac{S}{2}]^2+\frac{3}{4}S(S^2-4S+6)\,.$$

Namely,

$$S(S-6)(19S-42)\geqslant 0\,.$$

It is clear that if $S>3$, then $S\geqslant 6$. We complete the proof of Proposition 1.

Proposition 2. *Let M be a complete minimal hypersurface in $S^4(1)$ with S=constant. If* $\inf f_3\cdot\sup f_3\neq 0$ *and $S>3$, then $S\geqslant 6$.*

Proof. If f_3=constant, then it follows from Lemma 1 (1) that M has constant principal curvature. Thus Proposition 2 is valid obviously (cf. [6: Corollary 1]). If $f_3\neq$constant, then we can prove that there exists a point $p\in M$ such that $f_3(p)=0$ because of $\inf f_3\cdot\sup f_3\neq 0$.

(1) If $\inf f_3\cdot\sup f_3<0$, then, from the continuation of f_3, we have that there exists a point $p\in M$ such that $f_3(p)=0$.

(2) If $\inf f_3 \cdot \sup f_3 > 0$, then $\inf f_3$ and $\sup f_3$ have the same sign, we shall prove that this does not occur. In fact, without loss of generality, we can assume $\sup f_3 < 0$. Lemma 1 (2) yields

$$-\sqrt{S^2/6} < \sup f_3 < 0\,; \tag{3.22}$$

$$\lim_{m\to\infty} f_3(p_m) = \sup f_3\,, \quad \lim_{m\to\infty} \|\nabla f_3(p_m)\| = 0, \tag{3.23}$$

$$\lim_{m\to\infty} \sup \Delta f_3(p_m) \leqslant 0\,, \tag{3.24}$$

$$\lim_{m\to\infty} \lambda_i(p_m) = \mathring{\lambda}_i\,, \tag{3.25}$$

$$\lim_{m\to\infty} h_{ijk}(p_m) = \mathring{h}_{ijk}\,, \tag{3.26}$$

$$\mathring{\lambda}_1 + \mathring{\lambda}_2 + \mathring{\lambda}_3 = 0\,, \tag{3.27}$$

$$\mathring{\lambda}_1^2 + \mathring{\lambda}_2^2 + \mathring{\lambda}_3^2 = S\,, \tag{3.28}$$

$$\mathring{\lambda}_1^3 + \mathring{\lambda}_2^3 + \mathring{\lambda}_3^3 = \sup f_3\,. \tag{3.29}$$

(3.22), (3.27), (3.28) and (3.29) imply that $\mathring{\lambda}_1$, $\mathring{\lambda}_2$ and $\mathring{\lambda}_3$ are distinct. By the same proof as in Proposition 1, we can obtain

$$\mathring{h}_{iik} = 0\,, \quad \text{for any } i \text{ and } k. \tag{3.30}$$

On the other hand, from

$$\Delta f_3 = 3\left[(3-S) f_3 + 2 \sum h_{ijk}^2 \lambda_i\right],$$

(3.23), (3.24), (3.25) and (3.26) yield

$$3\left[(3-S) \sup f_3 + 2 \sum \mathring{h}_{ijk}^2 \mathring{\lambda}_i\right] \leqslant 0\,. \tag{3.31}$$

$$\begin{aligned}
&3 \sum \mathring{h}_{ijk}^2 \mathring{\lambda}_i \\
&= \sum \mathring{h}_{ijk}^2 (\mathring{\lambda}_i + \mathring{\lambda}_j + \mathring{\lambda}_k) \\
&= \sum_{\substack{i \neq j \neq k \\ i \neq k}} \mathring{h}_{ijk}^2 (\mathring{\lambda}_i + \mathring{\lambda}_j + \mathring{\lambda}_k) \quad \text{(by (3.13))} \\
&= 0 \quad (\text{by } \mathring{\lambda}_1 + \mathring{\lambda}_3 + \mathring{\lambda}_2 = 0)\,.
\end{aligned}$$

Hence

$$(3-S) \sup f_3 \leqslant 0\,.$$

Because of $S > 3$ and $\sup f_3 < 0$, we know that this is impossible. Hence there exists a point $p \in M$ such that $f_3(p) = 0$.

Next by the same proof as in [6], we know that Proposition 2 is valid.

Proof of Theorem. From Propositions 1 and 2, Theorem is obvious.

References

[1] E. Cartan: *Sur des familles remarquables d'hypersurfaces isoparametriques dans les espaces spheriques,* Math. Z., **45** (1939), 335–367.

[2] Q.M. Cheng: *A characterization of complete Riemannian manifold minimally immersed in a unit sphere,* to appear in Nagoya Math. J..

[3] S.S. Chern, M. do Carmo and S. Kobayashi: *Minimal submanifolds of a sphere with second fundamental form of constant length,* In functional analysis and related fields, Springer Verlag, Berlin, 1970, 59–75.

[4] W.Y. Hsiang: *Remarks on closed minimal submanifolds in the standard Riemannian m-sphere,* J. Diff. Geom., **1** (1967), 257–267.

[5] W.Y. Hsiang and H.B. Lawson: *Minimal submanifolds of low cohomogeneity,* J. Diff. Geom., **5** (1971), 1–38.

[6] C.K. Peng and C.L. Terng: *Minimal hypersurfaces of spheres with constant scalar curvature,* Seminar on minimal submanifolds, 1983, Princeton Univ. Press, 177–198.

[7] S.T. Yau: *Harmonic functions on complete Riemannian manifolds,* Comm. Pure and Appl. Math., **28** (1975), 201-228.

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