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SINGULAR FOLIATIONS ON CROSS-SECTIONS OF EXPANSIVE FLOWS ON 3-MANIFOLDS

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1. Introduction

The notion of cross-sections is one of useful methods to investigate the behaviors of flows. H.B. Keynes and M. Sears [6] constructed a family of cross-sections and a first return map for a non-singular flow. In this paper we shall construct singular foliations on cross-sections invariant under the first return maps of flows furnishing expansiveness on three dimensional closed manifolds.

Recently K. Hiraide [5] showed the existence of invariant singular foliations for expansive homeomorphisms of closed surfaces. We shall construct singular foliations on cross-sections by using the method mentioned in [5]. However the first return maps are not continuous and we shall prepare supplementary tools to get our conclusion.

Let X be a closed topological manifold with metric d. By $(X, \boldsymbol{R})$ we denote a real continuous flow (abbrev. flow) without fixed points and the action of $t\in\boldsymbol{R}$ on $x\in X$ is written xt. $(X, \boldsymbol{R})$ is called an *expansive* flow if for any $\varepsilon>0$ there exists $\delta>0$ with the property that if $d(xt, yh(t))<\delta$ $(t\in\boldsymbol{R})$ for a pair of points $x, y\in X$ and for an increasing homeomorphism $h\colon \boldsymbol{R}\to\boldsymbol{R}$ such that $h(0)=0$ and $h(\boldsymbol{R})=\boldsymbol{R}$, then $y=xt$ for some $|t|<\varepsilon$. Every non-trivial expansive flow has no fixed points (see [1]). Hereafter the natural numbers, the integers and the real number will be denoted by $\boldsymbol{N}$, $\boldsymbol{Z}$ and $\boldsymbol{R}$ respectively.

Let $SI=\{xt; x\in S \text{ and } t\in I\}$ for an interval I and $S\subset X$. A subset $S\subset X$ is called a *local cross-section* of time $\zeta>0$ for a flow $(X, \boldsymbol{R})$ if S is closed and $S\cap x[-\zeta, \zeta]=\{x\}$ for all $x\in S$. If S is a local cross-section of time ζ, the action maps $S\times[-\zeta, \zeta]$ homeomorphically onto $S[-\zeta, \zeta]$. By the interior S^* of S we mean $S\cap \operatorname{int}(S[-\zeta, \zeta])$. Note that $S^*(-\varepsilon, \varepsilon)$ is open in X for any $\varepsilon>0$. Put $\varepsilon_0=\inf\{t>0; xt=x \text{ for some } x\in X\}$. Under the above assumptions and notations we have the following

Fact 1.1 ([6], Lemma 2.4). There is $0<\zeta<\varepsilon_0/2$ satisfying that for each $0<\alpha<\zeta/3$ we can find a finite family $\mathcal{S}=\{S_1, S_2, \cdots, S_k\}$ of pairwise disjoint local cross-sections of time ζ and diameter at most α and a family of local corsssections $\mathcal{T}=\{T_1, T_2, \cdots, T_k\}$ with $T_i \subset S_i^*$ $(i=1, 2, \cdots k)$ such that

$$X = T^+[0, \alpha] = T^+[-\alpha, 0] = S^+[0, \alpha] = S^+[-\alpha, 0]$$

where $\quad T^+ = \bigcup_{i=1}^{k} T_i \quad$ and $\quad S^+ = \bigcup_{i=1}^{k} S_i$.

Take $\zeta>0$ as in Fact 1.1 and fix $0<\alpha<\zeta/3$. $\mathcal{S}$ and $\mathcal{T}$ are families of local cross-sections of time ζ as in Fact 1.1. Put $\beta=\sup\{\delta>0;\ x(0, \delta)\cap S^+=\phi$ for $x\in S^+\}$. Obviously $0<\beta\leq\alpha$. Take and fix ρ with $0<\rho<\alpha$.

For $x\in T^+$ let $t\in \boldsymbol{R}$ be the smallest positive time such that $xt\in T^+$. Then obviously $\beta\leqq t\leqq\alpha$ and a map $\varphi(x)=xt$ is defined. It is easily checked that φ: $T^+\rightarrow T^+$ is bijective.

For $S_i\in\mathcal{S}$ set $D_\rho^i=S_i[-\rho, \rho]$ and define a projective map P_ρ^i: $D_\rho^i\rightarrow S_i$ by $P_\rho^i(x)=xt$, where $xt\in S_i$ and $|t|\leq\rho$. Then P_ρ^i is continuous and surjective. We write $D_\rho^i=D_\rho$ and $P_\rho^i=P_\rho$ if there is no confusion. From continuity of $(X, \boldsymbol{R})$ we have

Fact 1.2. There exists $\delta_0>0$ such that if $d(x, y)\leq\delta_0$ $(x, y\in S^+)$ and $xt\in T_j$ $(|t|\leq 3\alpha)$ for some $T_j\in\mathcal{T}$, then $yt\in D_\rho^j$.

We can set up a shadowing orbit of $y\in S^+$ relative to a φ-orbit of $x\in T^+$ as follows. If $d(x, y)\leq\delta_0$, then $y_x^1=P_\rho(yt)$ for the time t with $\varphi(x)=xt$ by Fact 1.2. Whenever $\varphi^i(x)$ and y_x^i are defined such that $d(\varphi^i(x), y_x^i)<\delta_0$, we write $y_x^{i+1}=P_\rho(y_x^i\, t)$ where $\varphi(\varphi^i(x))=\varphi^i(x)\, t$. Thus we obtain a time delayed y shadowing orbit along a piece of the orbit of x. Also the negative powers of φ is constructed as above and so we obtain $\{y_x^i;\ i\in\boldsymbol{Z}\}$. For simplicity write

$$y = \varphi_x^0(y),\ \varphi_x(y)=\varphi_x^1(y) \quad \text{and} \quad y_x^i = \varphi_x^i(y) \quad (i\in\boldsymbol{Z})$$

and to avoid complication $\varphi_*^l(\varphi_x^k(y))$ instead of $\varphi_{\varphi^k(x)}^l(\varphi_x^k(y))$.

For $x\in T^+$ the *η-stable* (*η-unstable*) *set*

$$W_\eta^s(x) = \{y\in S^+;\ d(\varphi^i(x), \varphi_x^i(y))\leq\eta \qquad \text{for all} \quad i\geq 0\}$$
$$(W_\eta^u(x) = \{y\in S^+;\ d(\varphi^i(x), \varphi_x^i(y))\leq\eta \qquad \text{for all} \quad i\leq 0\})$$

is defined. Remark that $W_\eta^\sigma(x)\subset S^+$ for $x\in T^+$ $(\sigma=s, u)$.

The complex numbers will be denoted by $\boldsymbol{C}$. For $p\in\boldsymbol{N}$, let π_p: $\boldsymbol{C}\rightarrow\boldsymbol{C}$ be the map which sends z to z^p. We define the domains $\mathcal{D}_p(p=1, 2, \cdots)$ of $\boldsymbol{C}$ by

$$\mathcal{D}_2 = \{z\in\boldsymbol{C}:\ |Re\, z|<1,\ |Im\, z|<1\}\ ,$$

$\mathcal{D}_1=\pi_2(\mathcal{D}_2)$ and $\mathcal{D}_p=\pi_p^{-1}(\mathcal{D}_1)$. It is easily checked that π_p: $\mathcal{D}_p\rightarrow\mathcal{D}_1$ is a p-fold branched cover for every $p\in\boldsymbol{N}$. Denote by $\mathcal{H}_2$ and $\mathcal{V}_2$ the horizontal and vertical foliations on $\mathcal{D}_2$ respectively. We define the decomposition $\mathcal{H}_1$ (resp. $\mathcal{V}_1$) of $\mathcal{D}_1$ as the projection of $\mathcal{H}_2$ (resp. $\mathcal{V}_1$) by π_2: $\mathcal{D}_2\rightarrow\mathcal{D}_1$, and define the decomposition $\mathcal{H}_p$(resp. $\mathcal{V}_p$) of $\mathcal{D}_p$ as the lifting of $\mathcal{H}_1$(resp. $\mathcal{V}_1$) by π_p: $\mathcal{D}_p \to \mathcal{D}_1$.

Let $U_x(x \in T^+)$ be a neighborhood of x in S^+. A decomposition $\mathcal{F}_{U_x}$ of U_x is called a C^0 *local singular foliation* if every $L \in \mathcal{F}_{U_x}$ is arcwise connected and if there are $p(x) \in \boldsymbol{N}$ and a C^0 chart h_x: $U_x \to \boldsymbol{C}$ around x such that

(1) $h_x(x) = 0$,

(2) $h_x(U_x) = \mathcal{D}_{p(x)}$,

(3) h_x sends each $L \in \mathcal{F}_{U_x}$ onto some element of $\mathcal{H}_p(x)$.

The number $p(x)$ is called the ***number of separatrices*** at x. We asy that x is a *regulra point* if $p(x)=2$, and x is a *singular point* with $p(x)$*-singularities* (or $p(x)$*-prong singularity*) if $p(x) \neq 2$. A neighborhood U_x of x equipped with a C^0 local singular foliation is called a C^0 *singular foliated neighborhood.*

Let $\mathcal{F}_{U_x}$ and $\mathcal{F}'_{U_x}$ be local singular foliations on U_x. We say that $\mathcal{F}'_{U_x}$ is *transverse* to $\mathcal{F}_{U_x}$ if $\mathcal{F}_{U_x}$ and $\mathcal{F}'_{U_x}$ have the same number $p(x)$ at x and if there is a C^0 chart h_x: $U_x \to \boldsymbol{C}$ such that

(1) $h_x(x) = 0$,

(2) $h_x(U_x) = \mathcal{D}_{p(x)}$,

(3) h_x sends each $L \in \mathcal{F}_{U_x}$ onto some element of $\mathcal{H}_{p(x)}$,

(4) h_x sends each $L' \in \mathcal{F}'_{U_x}$ onto element of $\mathcal{V}_{p(x)}$.

If there are C^0 transvversal singular foliations on U_x, then U_x is called a C^0 *transversal singular foliated neighborhood.* Our aim is to prove the following

Theorem. *Let $(X, \boldsymbol{R})$ be an expansive flow on a closed 3-manifold X. Then there is a sufficiently small η such that for every $x \in T^+$ there is a C^0 transversal singular foliated neighborhood U_x such that if $L \in \mathcal{F}_{U_x}(\mathcal{F}'_{U_x})$ contains $y \in T^+$, then $L = W^s_\eta(y) \cap U_x(W^u_\eta(y) \cap U_x)$.*

For the proof we need that $W^\sigma_\eta(x)$ $(\sigma = s, u)$ is arcwise connected. However it is difficult to directly verify the connectendness of $W^\sigma_\eta(x)$. In §4 we shall prove the following proposition, which plays an important role through the paper. We denote by $C^\sigma_\eta(x)$ the connected component of x in $W^\sigma_\eta(x)$ $(\sigma = s, u)$. Let $S^\#_\delta(x)$ be a circle in S^+ with the radius δ and the center x.

Proposition A. *For any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in T^+$*

$$C^\sigma_\varepsilon(x) \cap S^\#_\delta(x) \neq \phi \quad (\sigma = s, u).$$

Hereafter int $W^\sigma_\varepsilon(x)$ denotes the interior of $W^\sigma_\varepsilon(x)$ in S^+. Proposition A is obtained by the following

Proposition B. *There exists $c_1 > 0$ such that if $0 < \varepsilon \leq c_1/4$, then*

$$\text{int } W^\sigma_\varepsilon(x) = \phi \quad (x \in T^+, \sigma = s, u).$$

In §2 we shall prepare some notations and establish several properties for the first return map φ. In §3 and §4 Proposition B and A will be proved. To find constants $c_1>0$ and $\delta>0$ in Propositions A and B we need to treat the first return map φ like an expansive homeomorphism. However φ is not continuous as mentioned above. So we shall introduce a new first return map ψ defined on an extended domain V^+ containing T^+. It will be shown that $C_\varepsilon^\sigma(x)$ ($\sigma=s, u, x\in T^+$) is locally connected for sufficiently small $\varepsilon>0$. In §6 the proof of our Theorem will be given.

2. Preliminaries

As before let X be a colsed topological manifold with metric d and $(X, \boldsymbol{R})$ be an expanisive flow on it. This section contains some lemmas that need subsequently. Under the notations in §1, we have the following.

Lemma 2.1 ([1], *Theorem* 3). *$(X, \boldsymbol{R})$ is expansive if and only if for any $\varepsilon>0$ there exists $\alpha>0$ with the following property: if $\boldsymbol{t}=(t_i)_{i=-\infty}^{\infty}$ and $\boldsymbol{u}=(u_i)_{i=-\infty}^{\infty}$ are doubly infinite sequences of real numbers with $t_0=u_0=0$, $0<t_{i+1}-t_i\leq\alpha$, $|u_{i+1}-u_i|\leq\alpha$, $t_i\to\infty$ and $t_{-i}\to-\infty$ as $i\to\infty$, and if $x, y\in X$ satisfy $d(xt_i, yu_i)\leq\alpha$ for any $i\in\boldsymbol{Z}$, then $y=xt$ for some $|t|<\varepsilon$.*

Let $\zeta>0$ be as in Fact 1.1 and $\alpha_0>0$ be as in Lemma 2.1 for $\zeta/3$. For $0<\alpha<\min\{\alpha_0/2, \zeta/3\}$ we construct as in Fact 1.1 families $\mathcal{S}=\{S_1, \cdots, S_k\}$ and $\mathcal{T}=\{T_1, \cdots, T_k\}$ of local cross-sections of time ζ. To simplify we set the following notations.

CONVENTION For $Q\subset X$, $x\in X$ and $\delta>0$

$$B_\delta(Q) = \{x\in X;\ d(x, Q)\leq\delta\}\ ,$$
$$U_\delta(Q) = \{x\in X;\ d(x, Q)<\delta\}\ ,$$
$$S_\delta(x) = \{y\in X;\ d(x, y) = \delta\}\ ,$$

and for $Q\subset S^+$

$$B_\delta^\natural(Q) = B_\delta(Q)\cap S^+\ ,$$
$$U_\delta^\natural(Q) = U_\delta(Q)\cap S^+\ .$$

Here $B_\delta(x)$ and $U_\delta(x)$ mean $B_\delta(\{x\})$ and $U_\delta(\{x\})$ respectively. Let $\rho>0$ be as in §1 and put $D_\xi^i=S_i[-\xi, \xi]$ $(0<\xi\leq\rho)$ and P_ξ^i: $D_\xi^i\to S_i$ denote the projection along the orbits. Sometimes we write $D_\xi^i=D_\xi$ and $P_\xi^i=P_\xi$. Put $\delta_1=\min\{d(S_i, S_j);\ S_i, S_j\in\mathcal{S}, i\neq j\}$ and take $0<\delta_2<\delta_1$ such that $B_{\delta_2}^\natural(T_i)\subset S_i^*$ for $i=1, \cdots, k$, where S_i^* is the interior of S_i. Then we have

Lemma 2.2 ([6], *Theorem* 2.7). *There exists $0<c<\alpha$ such that $W_c^s(x)\cap W_c^u(x)=\{x\}$ for any $x\in T^+$.*

To prove that Proposition B is true though φ is not continuous, we prepare the following Lemmas 2.3~2.9.

Lemma 2.3 *Let $\{x_n\}\subset T^+$ converge to $x\in T^+$ as $n\to\infty$ and fix $i\in\boldsymbol{Z}$. If a_i is an accumulation point of $\{\varphi^i(x_n)\}$, then there exists $k_i\in\boldsymbol{Z}$ such that $a_i=\varphi^{k_i}(x)$, where $k_i\geq i$ if $i\geq 0$ and $k_i\leq i$ if $i<0$.*

This follows from the fact that each $T_i\in\mathcal{T}$ is closed.

Lemma 2.4 ([6], *Lemma* 2.9) *Suppose that $x_n\to x$ ($x_n\in T^+$), $y_n\to y$ ($y_n\in S^+$) as $n\to\infty$ and each $\varphi^i_{x_n}(y_n)$ is defined for $0\leq i\leq k$ ($k\leq i\leq 0$). If $\varphi^k(x_n)\to\varphi^{l_k}(x)$ as $n\to\infty$ for some integer l_k, then $\varphi^k_{x_n}(y_n)\to\varphi^{l_k}_x(y)$ as $n\to\infty$.*

Let c be as in Lemma 2.2. We find $0<\delta_3<\delta_2$, $0<\rho_1<\rho$ and $0<c_1<\min\{c, \delta_3\}$ such that

(A_1) if $d(x, y)<\delta_3$ ($x, y\in X$), then $d(xt, ys)\leq c$ for $|t|\leq 3\alpha$ and $|t-s|\leq 2\rho_1$,

(B_1) it $d(x, y)\leq c_1(x, y\in S^+)$ and $xt\in T_j(|t|\leq 3\alpha)$ for some $T_j\in\mathcal{T}$, then $yt\in D^j_{\rho_1}$.

The following is a lemma given for expansive homeomorphisms of a compact metric space by Mañé [7].

Lemma 2.5. *For any $0<\varepsilon\leq c_1/2$, there exists $0<\delta\leq\varepsilon$ such that if $d(x, y)\leq\delta$ ($x\in T^+$, $y\in S^+$) and*

$$\varepsilon\leq\max\{d(\varphi^i(x), \varphi^i_x(y));\ 0\leq i\leq n\}\leq c_1/2\,,$$

then $d(\varphi^n(x), \varphi^n_x(y))\geq\delta$.

Proof. If this is false, there exists $0<\varepsilon_0\leq c_1/2$ such that for $n\in\boldsymbol{N}$ with $1/n\leq\varepsilon_0$ there are $m_n\in\boldsymbol{N}$, $x_n\in T^+$ and $y_n\in S^+$ such that

$$d(x_n, y_n)\leq 1/n\,, \tag{1}$$

$$\varepsilon_0\leq\max\{d(\varphi^i(x_n), \varphi^i_{x_n}(y_n));\ 0\leq i\leq m_n\}\leq c_1/2\,, \tag{2}$$

$$d(\varphi^{m_n}(x_n), \varphi^{m_n}_{x_n}(y_n))<1/n\,. \tag{3}$$

By (2) we have

$$\varepsilon_0\leq d(\varphi^{l_n}(x_n), \varphi^{l_n}_{x_n}(y_n))\leq c_1/2 \tag{4}$$

for some $0\leq l_n<m_n$. Obviously $l_n\to\infty$ and $m_n-l_n\to\infty$ ($n\to\infty$). Since T^+ and S^+ are compact, $\varphi^{l_n}(x_n)\to x\in T^+$ and $\varphi^{l_n}_{x_n}(y_n)\to y\in S^+$ as $n\to\infty$ (take subsequences if necessary). By (4),

$$\varepsilon_0\leq d(x, y)\leq c_1/2\,. \tag{5}$$

Since $\{\varphi^{l_n}(x_n)\}$ converges to x, there are a subsequence $\{\varphi(\varphi^{l_{n_i}}(x_{n_i}))\}$ and $k_1>0$

such that $\varphi(\varphi^{l_{n_i}}(x_{n_i}))\to\varphi^{k_1}(x)$ as $i\to\infty$ (by Lemma 2.3). Lemma 2.4 ensures that $\varphi_*(\varphi^{l_{n_i}}_{x_{n_i}}(y_{n_i}))\to\varphi^{k_1}_x(y)$ as $i\to\infty$. While $\varphi^{k_1}(x)$ can be written as $\varphi^{k_1}(x)=xt_1$ for some t_1 with $\beta\leq t_1\leq\alpha$. Using (5), (A_1) and (B_1), we have

$$d(\varphi^j(x),\ \varphi^j_x(y))\leq c \qquad \text{for} \quad 0\leq j\leq k_1\,.$$

Obviously $\varphi(\varphi^{l_{n_i}}(x_{n_i}))=\varphi^{l_{n_i}+1}(x_{n_i})$ and $\varphi_*(\varphi^{l_{n_i}}_{x_{n_i}}(y_{n_i}))=\varphi^{l_{n_i}+1}_{x_{n_i}}(y_{n_i})$. Thus (2) and the inequality $0\leq l_n+1\leq m_n$ imply

$$d(\varphi^{k_1}(x),\ \varphi^{k_1}_x(y))\leq c_1/2\,. \tag{6}$$

Choose $k_2>0$ and a subsequence of $\{\varphi^2(\varphi^{l_{n_i}}(x_{n_i}))\}$ which converges to $\varphi^{k_2}(\varphi^{k_1}(x))$. To avoid complication let

$$\varphi(\varphi^{l_{n_i}+1}(x_{n_i}))\to\varphi^{k_2}(\varphi^{k_1}(x)) \quad (i\to\infty)\,, \tag{7}$$

then Lemma 2.4 implies that

$$\varphi_*(\varphi^{l_{n_i}+1}_{x_{n_i}}(y_{n_i}))\to\varphi^{k_2}_*(\varphi^{k_1}_x(y)) \quad (i\to\infty)\,. \tag{8}$$

From (6), (7), (8) and the fact that $\varphi^{k_2}(\varphi^{k_1}(x))=\varphi^{k_1}(x)\,t_2$ $(\beta\leq t_2\leq\alpha)$ we have

$$d(\varphi^j(x),\ \varphi^j_x(y))\leq c \qquad \text{for} \quad k_1\leq j\leq k_1+k_2\,.$$

In this fashion we have

$$d(\varphi^j(x),\ \varphi^j_x(y))\leq c \qquad \text{for} \quad j\geq 0\,.$$

Note that $\{\varphi^{l_n}(x_n)\}$ converges to x as $n\to\infty$. To show the above inequality for $j<0$, we choose $k_{-1}<0$ and a subsequence $\{\varphi^{-1}(\varphi^{l_{n_i}}(x_{n_i}))\}$ such that $\varphi^{-1}(\varphi^{l_{n_i}}(x_{n_i}))\to\varphi^{k_{-1}}(x)$ as $i\to\infty$. Since $\varphi^{k_{-1}}(x)=xt_{-1}$ for some t_{-1} with $-\alpha\leq t_{-1}\leq-\beta$, by (5), (A_1) and (B_1) we have

$$d(\varphi^j(x),\ \varphi^j_x(y))\leq c \qquad \text{for} \quad k_{-1}\leq j\leq 0\,.$$

Since $l_n\uparrow\infty$, by (2)

$$d(\varphi^{k_{-1}}(x),\ \varphi^{k_{-1}}_x(y))\leq c_1/2\,. \tag{9}$$

Take $k_{-2}<0$ n and a subsequence of $\{\varphi^{-1}(\varphi^{l_{n_i}-1}(x_{n_i})\}$ that converges to $\varphi^{k_{-2}}(\varphi^{k_{-1}}(x))$ and write $\varphi^{-1}(\varphi^{l_n-1}(x_n))\to\varphi^{k_{-2}}(\varphi^{k_{-1}}(x))$ for simplicity. Then we have

$$\varphi^{-1}_*(\varphi^{l_n-1}_{x_n}(y_n))\to\varphi^{k_{-2}}_*(\varphi^{k_{-1}}_x(y))=\varphi^{k_{-1}+k_{-2}}_x(y) \quad (i\to\infty)$$

and can write $\varphi^{k_{-2}}(\varphi^{k_{-1}}(x))=(\varphi^{k_{-1}}(x))\,t_{-2}$ for some t_{-2} with $-\alpha\leq t_{-2}\leq-\beta$. Thus from (9), an (A_1) and (B_1)

$$d(\varphi^j(x),\ \varphi^j_x(y))\leq c \qquad \text{for} \quad k_{-1}+k_{-2}\leq j\leq k_{-1}\,,$$

and on induction

$$d(\varphi^j(x), \varphi_x^j(y)) \leq c \qquad \text{for} \quad j<0\,.$$

Therefore $y=x$ by Lemma 2.2 and thus contradicting (5).

Lemma 2.6. *Let A be a connected subset of S^+. For $0<\varepsilon\leq c_1/4$, there exists $0<\delta\leq\varepsilon$ such that if $A\subset B_\delta^{\sharp}(x)$ $(x\in A\cap T^+)$, $\varphi_x^i(A)\cap S_\varepsilon^{\sharp}(\varphi^i(x))\neq\phi$ for some i with $0\leq i\leq n$ and $\bigcap_{i=0}^{n}\varphi_{\varphi^i(x)}^{-1}[B_{2\varepsilon}^{\sharp}(\varphi^i(x))]\supset A$, then $\varphi_x^n(A)\cap S_\delta^{\sharp}(\varphi^n(x))\neq\phi$.*

Proof. Take δ with $0<\delta\leq\varepsilon$ as in Lemma 2.5. Then conclusion is easily obtained.

Lemma 2.7. *Let c_1 be as above. Then for $0<r\leq c_1$ there exists $N\in\boldsymbol{N}$ such that*

$$\varphi_x^n(W_{c_1}^s(x))\subset W_r^s(\varphi^n(x))$$

and

$$\varphi_x^{-n}(W_{c_1}^u(x))\subset W_r^u(\varphi^{-n}(x))$$

for $x\in T^+$ and $n\geq N$.

Proof. We prove for the case of $W_{c_1}^s(x)$ for $x\in T^+$. If this is false, then there exists $0<r_0\leq c_1$ such that for any $n\in\boldsymbol{N}$ there are $x_n\in T^+$ and $m_n\geq n$ such that

$$\varphi_{x_n}^{m_n}(W_{c_1}^s(x_n))\not\subset W_{r_0}^s(\varphi^{m_n}(x_n))\,.$$

Then we can find $y_n\in W_{c_1}^s(x_n)$ such that for some $k_n\geq 0$

(1) $$d(\varphi^{k_n+m}(x_n), \varphi_{x_n}^{k_n+m_n}(y_n))>r_0\,.$$

If $\varphi^{k_n+m_n}(x_n)\to x\in T^+$ and $\varphi_{x_n}^{k_n+m_n}(y_n)\to y\in S^+$ as $n\to\infty$, by (1) we have

(2) $$d(x,y)\geq r_0\,.$$

Since $y_n\in W_{c_1}^s(x_n)$,

(3) $$d(\varphi^{i+k_n+m_n}(x_n), \varphi_{x_n}^{i+k_n+m_n}(y_n))\leq c_1$$

for $i\in\boldsymbol{Z}$ with $i+k_n+m_n\geq 0$. Putting $i=0$ in (3), we have

(4) $$d(x,y)\leq c_1\,.$$

Since $\{\varphi(\varphi^{k_n+m_n}(x_n))\}$ converges to some $\bar{x}\in T^+$ (take a subsequence if necessary), we can write $\varphi^{l_1}(x)=\bar{x}$ for some $l_1>0$ by Lemma 2.3 and

(5) $$\varphi^{l_1}(x)=xt\,,\quad \beta\leq t\leq\alpha\,.$$

Then Lemma 2.4 imples that $\varphi_*(\varphi_{x_n}^{k_n+m_n}(y_n))\to\varphi_x^{l_1}(y)$ as $n\to\infty$. Since $\varphi_*(\varphi_{x_n}^{k_n+m_n}$

$(y_n))=\varphi_{x_n}^{1+k_n+m_n}(y_n)$, we have by (3) that

$$d(\varphi^{1+k_n+m_n}(x_n)\,,\quad \varphi_{x_n}^{1+k_n+m_n}(y_n))\leq c_1\,,$$

from which

$$d(\varphi^{l_1}(x),\ \varphi_x^{l_1}(y))\leq c_1\,. \tag{6}$$

By (4), (5), (A_1) and (B_1)

$$d(\varphi^j(x),\ \varphi_x^j(y))\leq c \qquad \text{for} \quad 0\leq j\leq l_1\,. \tag{7}$$

As above there are $l_2>0$ and a subsequence of $\{\varphi^2(\varphi^{k_n+m_n}(x_n))\}$ which converges to $\varphi^{l_2}(x)$ as $n\to\infty$. To avoid complication let $\varphi^2(\varphi^{k_n+m_n}(x_n))\to\varphi^{l_2}(x)$ as $n\to\infty$. Then we can write

$$\varphi^2(\varphi^{k_n+m_n}(x_n)) = \varphi^{1+k_n+m_n}(x_n)\ t_2^n \quad (\beta\leq t_2^n\leq\alpha)\,.$$

Since the sequence $\{t_2^n\}$ converges to some $t\in[\beta,\alpha]$ (take a subsequence if necessary), we have

$$\varphi^{1+k_n+m_n}(x_n)\ t_2^n \to \varphi^{l_1}(x)\ t \quad (\beta\leq t\leq\alpha)\,,$$

which implies

$$\varphi^{l_2}(x) = \varphi^{l_1}(x)\ t \quad (\beta\leq t\leq\alpha)\,. \tag{8}$$

Lemma 2.4 ensures that $\varphi_*^2(\varphi_{x_n}^{k_n+m_n}(y_n))\to\varphi_x^{l_2}(y)$ as $n\to\infty$, and by (6), (8), (A_1) and (B_1) we have $d(\varphi^j(x)\ \varphi_x^j(y))\leq c$ for $l_1\leq j\leq l_2$. By (3)

$$d(\varphi^2(\varphi^{k_n+m_n}(x_n)),\ \varphi_*^{2'}(\varphi_{x_n}^{k_n+m_n}(y_n)))\leq c_1\,, \tag{9}$$

and thus inductively $d(\varphi^j(x),\ \varphi_x^j(y))\leq c$ for $j\geq 0$.

Since $m_n\geq n$ for all $n>0$, for $j<0$ there exists $m_n>0$ such that $j+k_n+m_n\geq 0$ and so $d(\varphi^j(x)\ \varphi_x^j(y))\leqq c$ for $j\leq 0$. Therefore $y\in W_c^s(x)\cap W_c^u(x)$ (i.e. $x=y$), contradicting (2).

Lemma 2.8. *For any $\varepsilon>0$ there exists $r>0$ such that $\varphi_x(B_r^\sharp(x))\subset B_\varepsilon^\sharp(\varphi(x))$ for any $x\in T^+$.*

Proof. If this is false, then there exists $\varepsilon_0>0$ such that for any $n\in \boldsymbol{N}$ there is $x_n\in T^+$ such that $y_n\in B_{1/n}^\sharp(x_n)$ and $d(\varphi(x_n),\ \varphi_{x_n}(y_n))>\varepsilon_0$. Suppose that $x_n\to x_0\in T^+$ and for some $l\geq 1$ $\varphi(x_n)\to\varphi^l(x_0)$ as $n\to\infty$. Then by Lemma 2.4 we have that $d(\varphi^{l_1}(x_0),\ \varphi_{x_0}^{l_1}(x_0))\geq\varepsilon_0$ since $y_n\to x_0$ as $n\to\infty$. But $\varphi_{x_0}^{l_1}(x_0)=\varphi^{l_1}(x_0)$, thus contradicting.

The following is easily obtained from Lemmas 2.7 and 2.8.

Lemma 2.9 ([6], *Lemma* 3.3). *For any ε with $0<\varepsilon<c_1$ there exists $e>0$*

such that

$$W^{\sigma}_{e_1}(x) \cap B^{\sharp}_{\delta}(x) = W^{\sigma}_{\varepsilon}(x) \cap B^{\sharp}_{\delta}(x) \quad (\sigma = s, u)$$

for any $x \in T^+$ *and* $0 < \delta \leq e$.

3. Proof of Proposition B

Hereafter X is a 3-dimensional closed topological manifold and d is a metric on X. Each local cross-section of families $\mathcal{S} = \{S_1, \cdots, S_k\}$ and $\mathcal{T} = \{T_1, \cdots, T_k\}$ defined in Fact 1.1 can be taken as a 2-dimensional disk. Hence there is a compatible metric (called a connected metric) on each local cross-section such that every ε-closed ball ($\varepsilon > 0$) is connected.

For the proof of Proposition B we define a new family $\mathcal{V} = \{V_1, \cdots, V_k\}$ of local cross-sections satisfying

(1) each V_i is a 2-dimensional disk,

(2) $T_i \subset V_i^* \subset V_i \subset S_i^*$ $(1 \leq i \leq k)$,

(3) $X = V^+[0, \alpha] = V^+[-\alpha, 0]$, where $V^+ = \bigcup_{i=1}^{k} V_i$,

and as before define the first return map ψ; $V^+ \to V^+$ as $\psi(x) = xt$ $(\psi^{-1}(x) = xt)$, where t is the smallest positive (largest negative) time with $xt \in V^+$.

Let $\delta_1 > 0$ be as in §2. Take δ_2 with $0 < \delta_2 < \delta_1$ such that $B_{\delta_2}(V_i) \subset S_i^*$ $(i = 1, \cdots, k)$ and assume that $\delta_0 > 0$ satisfies Fact 1.2 (replacing T_j by V_j).

For $x \in V^+$ define the η- stable set $W^s_\eta(x, \psi)$ and η- unstable set of $W^u_\eta(x, \psi)$ as follows:

$$W^s_\eta(x, \psi) = \{y \in S^+;\, d(\psi^i(x), \psi^i_x(y)) < \eta,\ i \geq 0\}\ ,$$
$$W^u_\eta(x, \psi) = \{y \in S^+;\, d(\psi^i(x), \psi^i_x(y)) < \eta,\ i \leq 0\}\ .$$

Obviously $W^\sigma_\eta(x, \psi) \subset S^+ (\sigma = s, u)$ and there exists $0 < c < \alpha$ such that $W^s_c(x, \psi) \cap W^u_c(x, \psi) = \{x\}$ for any $x \in V^+$ (see Lemma 2.2). Note that Lemmas 2.3, 2.4, 2.5 and 2.6 hold for ψ.

Let $\mathcal{C}(S^+)$ denote the set of all non-impty closed subsets of S^+, then Hausdoff metric H is defined by

$$H(A, B) = \inf \{\varepsilon > 0;\, N_\varepsilon(A) \supset B,\ N_\varepsilon(B) \supset A\} \quad (A, B \in \mathcal{C}(S^+))$$

where $N_\varepsilon(A)$ denotes the ε-neighborhood of A in S^+. Then $\mathcal{C}(S^+)$ is a compact space under H.

Lemma 3.1 (*c.f.* [2]). *Let Y be a compact connected metric space. If A is a non-empty closed subset of Y with $A \neq Y$, then every connected component in A intersects to the boundary of A in Y.*

We denote by $D_n(x)$ $(x \in V^+)$ the connected component of x in the domain of ψ_x^{-n}. Put $D_{n,\delta}(x)=D_n(x)\cap B_\delta^\sharp(x)$ and let $\Delta_{n,\delta}(x)$ be the connected component of x in $B_\delta^\sharp(x)\cap\psi_{\psi^n(x)}^{-n}[D_{n,\delta/2}(\psi^n(x))]$.

Lemma 3.2. *Let $0<\varepsilon\le c_1/4$. There exists $0<\delta\le\varepsilon$ such that if $\{x_i\}_{i\in Z}$ is a sequence in V^+ and*

(a) *if there is non-upper bound subset $\{j\}$ of $\boldsymbol{Z}$ such that*

$$\lim_{j\to\infty} x_j = x_\infty \quad \textit{and} \quad \lim_{j\to\infty}\Delta_{j,\delta}(x_j)=\Delta_\infty\,,$$

then $\Delta_\infty\subset W_\varepsilon^s(x_\infty,\psi)$,

(b) *if there is non-lower bound subset $\{j\}$ of $\boldsymbol{Z}$ such that*

$$\lim_{j\to-\infty} x_j = x_{-\infty} \quad \textit{and} \quad \lim_{j\to-\infty}\Delta_{j,\delta}(x_j)=\Delta_{-\infty}$$

then $\Delta_{-\infty}\subset W_\varepsilon^u(x_{-\infty},\psi)$.

Proof. For ε with $0<\varepsilon\le c_1/4$ we can find $0<\varepsilon'<\varepsilon$ and $\delta'>0$ such that if $d(x,y)\ge\varepsilon\,(x,y\in S^+)$ and $|s-t|<\delta'(|s|,|t|<2\alpha)$, then $d(xt,ys)\ge\varepsilon'$. Take δ with $0<\delta\le\varepsilon'$ as in Lemma 2.6. Since $\Delta_{j,\delta}(x_j)\subset B_\delta^\sharp(x_j)$, Obviously $\Delta_{j,\delta}(x_j)\to\Delta_\infty\subset B_\delta^\sharp(x_\infty)\subset B_{\varepsilon'}^\sharp(x_\infty)\;(j\to\infty)$. If $\Delta_\infty\not\subset W_\varepsilon^s(x_\infty,\psi)$, then we can find $k_0>0$ such that $\psi_{x_\infty}^{k_0}(\Delta_\infty)\not\subset B_\varepsilon^\sharp(\psi^{k_0}(x_\infty))$.

Since $x_j\to x_\infty$ and $\Delta_{j,\delta}(x_j)\to\Delta_\infty$ as $j\to\infty$, there are $0<\eta_0\le k_0$ and $l>\eta_0$ such that $\psi_{x_l}^{\eta_0}(\Delta_{l,\delta}(x_l))\not\subset B_{\varepsilon'}^\sharp(\psi^{\eta_0}(x_l))$. Hence $\psi_{x_l}^{\eta_0}(\Delta_{l,\delta}(x_l))\not\subset B_\lambda^\sharp(\psi^{\eta_0}(x_l))$ for some $\varepsilon'<\lambda<2\varepsilon'$. Thus we can find $0<\eta_1\le\eta_0$ such that

$$\begin{aligned}&\psi_{x_l}^i(\Delta_{l,\delta}(x_l))\subset B_\lambda^\sharp(\psi^i(x_l)) \quad (0\le i\le\eta_1-1)\,,\\ &\psi_{x_l}^{\eta_1}(\Delta_{l,\delta}(x_l))\not\subset B_{\varepsilon'}^\sharp(\psi^{\eta_1}(x_l))\,.\end{aligned}$$

Let A_{η_1} denote the connected component of x_l in

$$\psi_{\psi^{\eta_1}(x_l)}^{-\eta_1}\,[\psi_{x_l}^{\eta_1}(\Delta_{l,\delta}(x_l))\cap B_{\varepsilon'}^\sharp(\psi^{\eta_1}(x_l))]\,.$$

Then we have

(1) $$\psi_{x_l}^i(A_{\eta_1})\subset B_\lambda^\sharp(\psi^i(x_l)) \qquad \text{for} \quad 0\le i\le\eta_1\,.$$

Since $\psi_{x_l}^{\eta_1}(\Delta_{l,\delta}(x_l))$ is connected and $\psi_{x_l}^{\eta_1}(\Delta_{l,\delta}(x_l))\not\subset B_{\varepsilon'}^\sharp(\psi^{\eta_1}(x_l))$, from Lemma 3.1 it follows that

(2) $$\psi_{x_l}^{\eta_1}(A_{\eta_1})\cap S_{\varepsilon'}^\sharp(\psi^{\eta_1}(x_l))\neq\phi\,.$$

For $\eta_1<\eta\le l$ define A_η as the connected component of x_l in $\psi_{\psi^\eta(x_l)}^{-\eta}[\psi_{x_l}^\eta(A_{\eta-1})\cap B_{\varepsilon'}^\sharp(\psi^\eta(x_l))]$. Then we have

$$\Delta_{l,\delta}(x_l)\supset A_{\eta_1}\supset A_{\eta_1+1}\supset\cdots\supset A_l$$

and by (1)

(3) $$\psi^i_{x_l}(A_\eta)\subset B^\ddagger_\lambda(\psi^i(x_l))\subset B^\ddagger_{2\varepsilon'}(\psi^i(x_l)) \quad (0\le i\le\eta)\,.$$

Now we claim that $\psi^\eta_{x_l}(A_\eta)\cap S^\ddagger_\delta(\psi^\eta(x_l))\neq\phi$ for $\eta_1<\eta\le l$. Indeed, if $A_\eta\neq A_{\eta-1}$, then $\psi^\eta_{x_l}(A_{\eta-1})\not\subset B^\ddagger_{\varepsilon'}(\psi^\eta(x_l))$ and hence $\psi^\eta_{x_l}(A_\eta)\cap S^\ddagger_{\varepsilon'}(\psi^\eta(x_l))\neq\phi$ (by Lemma 3.1). Since $0<\delta\le\varepsilon'$, obviously $\psi^\eta_{x_l}(A_\eta)\cap S^\ddagger_\delta(\psi^\eta(x_l))\neq\phi$. For the case $A_\eta=A_{\eta-1}$ put $i_0=\min\{i\ge\eta_1;\ A_i=A_\eta\}$. Clearly $\eta_1\le i_0<\eta$. If $i_0=\eta_1$, then $\psi^{i_0}_{x_l}(A_{\eta_1})\cap S^\ddagger_{\varepsilon'}(\psi^{i_0}(x_l))\neq\phi$ by (2). If $i_0>\eta_1$, then $A_{i_0}\neq A_{i_0-1}$, and hence $\psi^{i_0}_{x_l}(A_\eta)\cap S^\ddagger_{\varepsilon'}(\psi^{i_0}(x_l))\neq\phi$. In any case we have $\psi^{i_0}_{x_l}(A_\eta)\cap S^\ddagger_{\varepsilon'}(\psi^{i_0}(x_l))\neq\phi$. Since $A_\eta\subset\Delta_{l,\delta}(x_l)\subset B^\ddagger_\delta(x_l)$, combining these facts with (3), we obtain $\psi^\eta_{x_l}(A_\eta)\cap S^\ddagger_\delta(\psi^\eta(x_l))\neq\phi$ by Lemma 2.6. Therefore the claim holds.

Since $l>\eta_1$, it follows that $\psi^l_{x_l}(A_l)\cap S^\ddagger_\delta(\psi^l(x_l))\neq\phi_l$. This contradicts the fact that $A_l\subset\Delta_{l,\delta}(x_l)$. Therefore (a) holds. In the same way (b) is proved.

Proof of Proposition B. We prove the case of $\sigma=u$. Fix $0<\varepsilon\le c_1/4$ and let $x\in T^+$. Let $0<\delta\le\varepsilon$ be as in Lemmas 2.5 and 3.2. Assume that int $W^u_\varepsilon(x)$ is not empty. If $y\in$ int $W^u_\varepsilon(x)$, then there exists $0<r\le\delta$ such that $B^\ddagger_{2r}(y)\subset$ int $W^u_\infty(x)$.

If $B_{\delta/2}(\psi^n(z))$ is not contained in the domain of $\psi^{-n}_{\psi^n(z)}$ for $z\in B^\ddagger_r(y)$ and $n\in\boldsymbol{N}$, by Lemma 2.5 and connectedness of $B^\ddagger_{\delta/2}(\psi^n(z))$ we can find $z'\in D_{n,\delta/2}(\psi^n(z))$ such that $\psi^{-n}_{\psi^n(z)}(z')\in S^\ddagger_\delta(z)\cap\Delta_{j,\delta}(z)$. We claim that there is $k>0$ such that $B^\ddagger_{\delta/2}(\psi^n(z))\subset D_n(\psi^n(z))$ for $z\in B^\ddagger_r(y)$ and $n\ge k$. For, if this is false, then for $m\in\boldsymbol{N}$ there are $z_m\in B^\ddagger_r(y)$ and $n_m\ge m$ such that $B^\ddagger_{\delta/2}(\psi^{n_m}(z_m))\not\subset D_{n_m}(\psi^{n_m}(z_m))$. Hence we can find $z'_m\in D_{n_m,\delta/2}(\psi^{n_m}(z_m))$ such that $\psi^{-n_m}_{\psi^{n_m}(z_m)}(z'_m)\in S^\ddagger_\delta(z_m)$. Let $z_m\to z_\infty\in B^\ddagger_r(y)$ and $\Delta_{n_m,\delta}(z_m)\to\Delta_\infty$ as $m\to\infty$ (take subsequences if necessary). Then we have $\Delta_\infty\subset W^s_\varepsilon(z_\infty,\psi)$ by Lemma 3.2. Obviously $W^s_\varepsilon(z_\infty,\psi)\cap W^u_\varepsilon(x)\supset\Delta_\infty\cap B^\ddagger_{2r}(y)\supset\Delta_\infty\cap B^\ddagger_r(y)\ni z_\infty$. The fact that $\Delta_{n_m,\delta}(z_m)\cap S_\delta(z_m)\ni\psi^{-n_m}_{\psi^{n_m}(z_m)}(z'_m)$ ensures that $\Delta_\infty\cap B^\ddagger_r(y)\ni z_\infty$. Hence $W^s_\varepsilon(z_\infty,\psi)\cap W^u_\varepsilon(x)$ is not one point set $\{z_\infty\}$, which contradicts the expansiveness (see Lemma 2.1).

Let $\Delta_n(z)$ denote the connected component of z ($z\in V^+$) in $\psi^{-n}_{\psi^n(z)}(B^\ddagger_{\delta/2}(\psi^n(z)))$. Then for any $0<\eta\le r$ and $k'\ge k$ there exists $n\ge k'$ such that $B^\ddagger_\eta(z)\supset\Delta_n(z)$ ($z\in B^\ddagger_r(y)$). Indeed, if there is $0<\eta\le r$ so that for $n\ge k$ there exists $z_n\in B^\ddagger_r(y)$ such that $B^\ddagger_\eta(z_n)\not\supset\Delta_n(z_n)$, then we have $\Delta_n(z_n)\cap S^\ddagger_\eta(z_n)\neq\phi$ by Lemma 3.1, which implies $\Delta_{n,\delta}(z_n)\cap S^\ddagger_\eta(z_n)\neq\phi$ since $\eta<\delta$.

If $z_n\to z_\infty\in B^\ddagger_r(y)$ and $\Delta_{n,\delta}(z_n)\to\Delta_\infty\in\mathcal{C}(S^+)$ as $n\to\infty$. Then we have $\Delta_\infty\cap S^\ddagger_\eta(z_\infty)\neq\phi$, and by Lemma 3.2, $\Delta_\infty\subset W^s_\varepsilon(z_\infty,\psi)$. Obviously $B^\ddagger_\eta(z_\infty)\subset B^\ddagger_{2r}(y)\subset W^u_\varepsilon(x)$, from which

$$W^s_\varepsilon(z_\infty,\psi)\cap W^u_\varepsilon(x)\supset\Delta_\infty\cap B^\ddagger_\eta(z_\infty)\ni z_\infty\,.$$

By expansiveness (see Lemma 2.1) we can conclude

(1) $$W^s_\varepsilon(z_\infty,\psi)\cap W^u_\varepsilon(x)=\{z_\infty\}\,.$$

Therefore $\Delta_\infty\cap B^\ddagger_\eta(z_\infty)=\{z_\infty\}$, which contradicts the fact that $\Delta_\infty\cap S^\ddagger_\eta(z_\infty)\neq\phi$.

We have shown that for any $0<\eta\leq r$ there exists $n\geq k'$ such that $B_\eta^\sharp(z)\supset\Delta_n(z)$ for $z\in B_r^\sharp(y)$. Thus $\psi_z^n(\Delta_n(z))=B_{\delta/2}^\sharp(\psi^n(z))$. Since $S\in\mathcal{S}$ has interior points, the cardinal number of $B_r^\sharp(y)$, Card $B_r^\sharp(y)$, is infinite, which ensures that there exist m-distinct points $z_1, \cdots, z_m$ in $B_r^\sharp(y)$ for $m>0$. Since η is arbitrary, we can choose $0<\eta\leq r$ such that $B_\eta^\sharp(z_i)$ $(i=1, \cdots m)$ are mutually disjoint. Using Lemmas 2.5 and 3.2 we can easily check that there is $n\geq k$ such that $B_{\delta/2}^\sharp(\psi^n(z_i))$ $\not\ni\psi^n(z_j)$ for i, j with $i\neq j$. Hence $B_{\delta/5}^\sharp(z_i)$ $(i=1, \cdots, m)$ are mutually disjoint. This contradicts the compactness of S^+ since m is any positive number.

4. Proof of Proposition A

Let $\mathcal{V}=\{V_1, \cdots, V_k\}$ and ψ: $V^+\to V^+$ be as in §3. In this section Proposition A will be proved. For the proof we need the following

Lemma 4.1. *For $\varepsilon>0$ there is $0<\mu<\varepsilon$ such that if $\{x_i\}\subset V^+$ converges to $x_\infty\in V^+$ and $\{B_i\}\subset C(S^+)$ converges to $B_\infty\in C(S^+)$ and if $B_i\subset W_\mu^\sigma(x_i, \psi)$ for any $i\geq 1$, then $B_\infty\subset W_\varepsilon^\sigma(x_\infty, \psi)$ $(\sigma=s, u)$.*

Proof. Let ρ_1 be as in §2 and δ_0, δ_2 be as in §3. For $\varepsilon>0$ there are $0<\rho_\varepsilon<\rho_1$, $0<\delta_\varepsilon<\delta_0$ and $0<\mu<\min\{\varepsilon, \delta_\varepsilon\}$ such that

(A_ε) $d(x, y)\leq\delta_\varepsilon(x, y\in X)$ impies $d(xt, ys)\leq\varepsilon$ for $|t|\leq 3\alpha$ and $|t-s|\leq\rho_\varepsilon$.

(B_ε) if $d(x, y)\leq\mu$ $(x, y\in S^+)$ and there is $V_j\in\mathcal{V}$ with $xt\in B_{\delta_2}^\sharp(V_i)$ for $|t|\leq 3\alpha$, then $yt\in D_{\rho_\varepsilon}^j$.

We give the proof for the case of $\sigma=s$ and then the proof of the case $\sigma=u$ is done in the same way. Since $B_i\to B_\infty$, for $z\in B_\infty$ we can find $y_i\in B_i$ with $y_i\to z$ $(i\to\infty)$, and

$$d(\psi^n(x_i), \psi_{x_i}^n(y_i))\leq\mu \quad (n\geq 0)\,. \tag{1}$$

holds because $B_i\subset W_\mu^s(x_i, \psi)$. Since $d(x_i, y_i)\leq\mu$ for i, we have $d(x_\infty, z)\leq\mu$. Replace φ by ψ and use Lemma 2.3. Then there is $l_1\geq 1$ such that $\psi(x_i)\to$ $\psi^{l_1}(x_\infty)$ as $i\to\infty$ (take a subsequence if necessary), and so we write $\psi^{l_1}(x_\infty)=x_\infty t$, for some t with $\beta\leq t\leq\alpha$. Applying Lemma 2.4 for ψ we have

$$d(\psi^{l_1}(x_\infty), \psi_{x_\infty}^{l_1}(z))\leq\mu\,. \tag{2}$$

Note that $d(x_\infty, z)\leq\mu$. Then from (A_ε), (B_ε) we have $d(\psi^j(x_\infty), \psi_{x_\infty}^j(z))\leq\varepsilon$ for $0\leq j\leq l_1$.

Since $\psi(x_i)\to\psi^{l_1}(x_\infty)$, there is $l_2\geq 1$ such that $\psi^2(x_i)$ converges to $\psi^{l_2}(\psi^{l_1}(x_\infty))$ as $i\to\infty$ (take a subsequence if necessary). Thus we have $d(\psi^j(x_\infty), \psi_{x_\infty}^j(z))\leq\varepsilon$ for $l_1\leq j\leq l_1+l_2$ by (A_ε) and (B_ε) and so $d(\psi^j(x_\infty), \psi_{x_\infty}^j(z))\leq\varepsilon$ $(0\leq j\leq l_1+l_2)$. In this fashion we see that the above inequality holds for all $j\geq 0$. Hence $z\in W_\varepsilon^s$

(x_∞, ψ) and therefore $B_\infty \subset W^s_\varepsilon(x_\infty, \psi)$.

The proof of the following lemma is very similar to that of Lemma 4.1 and so we omit the proof.

Lemma 4.2. *For $\varepsilon>0$ there is $0<\mu<\varepsilon$ such that if $\{x_i\}\subset V^+$ converges to $x_\infty\in V^+$ and $\{B_i\}\subset\mathcal{C}(S^+)$ converges to $B_\infty\in\mathcal{C}(S^+)$ and if $\psi^n_{x_i}(B_i)\subset B^\sharp_\mu(\psi^n(x_i))$ for $0\le n\le i$ $(-i\le n\le 0)$, then $B_\infty\subset W^s_\varepsilon(x_\infty, \psi)$ $(B_\infty\subset W^u_\varepsilon(x_\infty, \psi))$, where $i\in\boldsymbol{N}$.*

REMARK 4.3. The above Lemmas 4.1 and 4.2 hold for the first return map φ: $T^+\to T^+$.

We are ready to prove Proposition A. Let c_1 be as in §2. Since $C^\sigma_\varepsilon(x)\subset C^\sigma_{\varepsilon'}(x)$ $(x\in T^+)$ if $0<\varepsilon<\varepsilon'$, we may prove the proposition for $0<\varepsilon\le c_1/8$.

We first give the proof for $\sigma=s$. Take $0<\mu<\varepsilon$ as in Lemma 4.2. We can find $0<\delta\le\mu$ as in Lemma 2.5, wihch is our requirement.

Indeed, take and fix $x\in T^+$. For simplicity write $x(j)=\varphi^j(x)$ $(j\ge 0)$. Since T^+ is compact, we have $x(j)\to x_\infty\in T^+$ as $j\to\infty$. From Proposition B it follows that int $W^u_{2\varepsilon}(x_\infty)=\phi$. For $0<\eta\le\delta/2$ there is $m_\eta>0$ such that

$$\varphi^{-m_\eta}_{x_\infty}(B^\sharp_{\eta/2}(x_\infty))\not\subset B^\sharp_{2\mu}(\varphi^{-m_\eta}(x_\infty)) . \tag{1}$$

We may assume that the number m_η is the smallest one satisfying (1). Since $x(j)\to x_\infty$, we choose a large number $j_\eta\ge m_\eta$ such that $d(x(j_\eta), x_\infty)\le\eta/2$ and

$$\operatorname{diam} \varphi^{-m_\eta}_{x_\infty}[B^\sharp_\eta(x(j_\eta)]\ge 2\mu . \tag{2}$$

Since $T^+\subset V^+$ and x_∞ is an interior point in V^+, for $\eta>0$ small enough we can find a positive integer l_η such that $m_\eta\le l_\eta<j_\eta$ and $\varphi^{-m_\eta}_{x_\infty}[B^\sharp_\eta(x(j_\eta)))]=\psi^{-l_\eta}_{x(j_\eta)}[B^\sharp_\eta(x(j_\eta))]$. From (2)

$$\operatorname{diam} \psi^{-l_\eta}_{x(j_\eta)}[B^\sharp_\eta(x(j_\eta))]\ge 2\mu . \tag{3}$$

Let $j'_\eta\ge j_\eta$ be an integer such that $x(j_\eta)=\psi^{j'_\eta}(x)$. Then (3) can be rewriteten as follows: we have

$$\operatorname{diam} \psi^{-l_\eta}_{x(j_\eta)}[B^\sharp_\eta(\psi^{j'_\eta}(x)]\ge 2\mu , \tag{4}$$

from which there exists $0<n_\eta\le l_\eta$ such that for $0\le i<n_\eta$

$$\psi^{-i}_{x(j_\eta)}[B^\sharp_\eta(\psi^{j'_\eta}(x))]\subset B^\sharp_\mu(\psi^{j'_\eta-i}(x)) , \tag{5}$$

$$\psi^{-n_\eta}_{x(j_\eta)}[(B^\sharp_\eta(\psi^{j'_\eta}(x))]\not\subset B^\sharp_\mu(\psi^{j'_\eta-n_\eta}(x)) . \tag{6}$$

Denote by $\Delta_{n_\eta}(\psi^{j'_\eta-n_\eta}(x))$ the connected component of $\psi^{j'_\eta-n_\eta}(x)$ in the subset

$$B^\sharp_\mu(\psi^{j'_\eta-n_\eta}(x))\cap\psi^{-1}_{\psi^{j'_\eta-n_\eta+1}(x)}[B_\mu(\psi^{j'_\eta-n_\eta+1}(x))]\cdots$$
$$\cdots\cap\psi^{-n_\eta+1}_{\psi^{j'_\eta-1}(x)}[B^\sharp_\mu(\psi^{j'_\eta-1}(x))]\cap\psi^{-n_\eta}_{\psi^{j'_\eta}(x)}[B^\sharp_{\delta/2}(\psi^{j'_\eta}(x))] ,$$

and denote by $C(\psi^{j'_\eta-n_\eta}(x))$ the connected component of $\psi^{j'_\eta-n_\eta}(x)$ in the subset

$$B^{\sharp}_{\mu}(\psi^{j'_\eta-n_\eta}(x))\cap\psi^{-n_\eta}_{\psi^{j'_\eta}(x)}[B^{\sharp}_{\eta}(\psi^{j'_\eta}(x))]\,.$$

Since $\eta\leq\delta/2$, by (5) we have

$$\Delta_{n_\eta}(\psi^{j'_\eta-n_\eta}(x))\supset C(\psi^{j'_\eta-n_\eta}(x))\,. \tag{7}$$

From (6) and Lemma 3.1

$$C(\psi^{j'_\eta-n_\eta}(x))\cap S^{\sharp}_{\mu}(\psi^{j'_\eta-n_\eta}(x))\neq\phi\,.$$

Since $B^{\sharp}_{\eta}(\psi^{j'_\eta}(x))$ is connected, by (7)

$$\Delta_{n_\eta}(\psi^{j'_\eta-n_\eta}(x))\cap S_{\mu}(\psi^{j'_\eta-n_\eta}(x))\neq\phi\,. \tag{8}$$

Put $\Delta(0)=\Delta_{n_\eta}(\psi^{j'_\eta-n_\eta}(x))$ and for $k>0$ let $\Delta(k)$ be the connected component of $\psi^{j'_\eta-n_\eta-k}(x)$ in the subset

$$\psi^{-1}_{\psi^{j'_\eta-n_\eta-k+1}(x)}[\Delta(k-1)]\cap B^{\sharp}_{\mu}(\psi^{j'_\eta-n_\eta-k}(x))\,.$$

Then we have

$$\begin{aligned}\psi_x^{j_\eta-n_\eta+1}(\Delta(j'_\eta-n_\eta))&\subset\psi^{i}_{\psi^{j'_\eta-n_\eta}(x)}(\Delta(0))\\&\subset B^{\sharp}_{\mu}(\psi^{j'_\eta-n_\eta+1}(x))\end{aligned}$$

for $0\leq i\leq n_\eta-1$ and so

$$\psi_x^{i}(\Delta(j'_\eta-n_\eta))\subset B^{\sharp}_{\mu}(\psi^{i}(x))\quad(0\leq i\leq j'_\eta-1) \tag{9}$$

and

$$\psi_x^{j'_\eta}(\Delta(j'-n_\eta))\subset B_{\delta/2}(\psi^{j'_\eta}(x))\,. \tag{10}$$

To see the existence of $0\leq i\leq j'_\eta-n_\eta$ such that

$$\psi_x^{i}(\Delta(j'_\eta-n_\eta))\cap S^{\sharp}_{\mu}(\psi^{i}(x))\neq\phi\,, \tag{11}$$

suppose that this relation is false (i.e. $\psi_x^{i}(\Delta(j'_\eta-n_\eta))\cap S^{\sharp}_{\mu}(\psi^{i}(x))=\phi$ $(0\leq i\leq j'_\eta-n_\eta)$). Then we have $\Delta(j'_\eta-n_\eta)\subset U^{\sharp}_{\varepsilon}(x)$. Since $\psi^{-1}_{\psi(x)}(\Delta(j'_\eta-n_\eta-1))\backslash B^{\sharp}_{\mu}(x)\neq\phi$ implies $\Delta(j'_\eta-n_\eta)\cap S^{\sharp}_{\mu}(x)\neq\phi$ by Lemma 3.1, this is inconsistent with the assumption. Thus $\psi^{-1}_{\psi(x)}(\Delta(j'_\eta-n_\eta-1))\subset B^{\sharp}_{\mu}(x)$ and $\Delta(j'_\eta-n_\eta)=\psi^{-1}_{\psi(x)}(\Delta(j'_\eta-n_\eta-1))$. This shows that $\psi_x(\Delta(j'_\eta-n_\eta))=\Delta(j'_\eta-n_\eta-1)$. To obtain the conclusion we use induction on i. Suppose that there is $0\leq i\leq j'_\eta-n_\eta$ with

$$\psi_x^{i}(\Delta(j'_\eta-n_\eta))=\Delta(j'_\eta-n_\eta-i)\,. \tag{12}$$

By Lemma 3.1 $\psi^{-1}_{\psi^{i+1}(x)}(\Delta(j'_\eta-n_\eta-i-1))\backslash B^{\sharp}_{\mu}(\psi^{i+1}(x))\neq\phi$ implies $\Delta(j'_\eta-n_\eta-i)\cap S^{\sharp}_{\mu}(\psi^{i}(x))\neq\phi$. $\psi_x^{i}(\Delta(j'_\eta-n_\eta))\cap S^{\sharp}_{\mu}(\psi^{i}(x))=\phi$ by hypothesis, thus contradicting our assumption. Therefore $\psi^{-1}_{\psi^{i+1}(x)}(\Delta(j'_\eta-n_\eta-i-1))\subset B^{\sharp}_{\mu}(\psi^{i+1}(x))$ and so

$$\Delta(j'_\eta-n_\eta-i) = \psi^{-1}_{\psi^{i+1}(x)}(\Delta(j'_\eta-n_\eta-i-1)) .$$

From (12)

$$\psi^i_x(\Delta(j'_\eta-n_\eta)) = \psi^{-1}_{\psi^{i+1}(x)}(\Delta(j'_\eta-n_\eta-i-1)) ,$$

and hence

$$\psi^{i+1}_x(\Delta(j'_\eta-n_\eta)) = \Delta(j'_\eta-n_\eta-i-1) .$$

Since $\psi^i_x(\Delta(j'_\eta-n_\eta))=\Delta(j'_\eta-n_\eta-i)$ $(0\leq i\leq j'_\eta-n_\eta)$, we have

$$\psi^{j'_\eta-n_\eta}_x(\Delta(j'_\eta-n_\eta)) = \Delta(0) = \Delta_{n_\eta}(\psi^{j'_\eta-n_\eta}(x)) .$$

Therefore our assumption is inconsistent with (8).

From (9), (10), (11) and Lemma 2.5 it follows that

$$\Delta(j'_\eta-n_\eta)\cap S_\delta(x) \neq \phi . \tag{13}$$

Since $\Delta(j'_\eta-n_\eta)\subset\Delta_{j'_\eta}(x)$ and $\Delta_{j'_\eta}(x)\cap S_\delta(x)\neq\phi$ and since $\Delta_{j'_\eta}(x)\to\Delta_\infty\in\mathcal{C}(S^+)$ as $\eta\to 0$, we have $\Delta_\infty\cap S_\delta(x)\neq\phi$. Notice that Δ_∞ is connected because each $\Delta_{j'_\eta}$ is so. Since

$$\psi^i_x(\Delta_{j'_\eta}(x))\subset B^{\sharp}_{\mu}(\psi^i(x)) \qquad \text{for} \quad 0\leq i\leq j'_\eta ,$$

we have $\Delta_\infty\subset W^s_\varepsilon(x_\infty, \psi)$ by Lemma 4.2. If $C^s_\varepsilon(x, \psi)$ and $C^s_\varepsilon(x)$ denote the connected component of x in $W^s_\varepsilon(x, \psi)$ and $W^s_\varepsilon(x)$ respectively, then we have $\Delta_\infty\subset C^s_\varepsilon(x, \psi)$. Thus $C^s_\varepsilon(x, \psi)\cap S^{\sharp}_\delta(x)\neq\phi$. Since $W^s_\varepsilon(x, \psi)\subset W^s_\varepsilon(x, \varphi)$ for $x\in T^+$, $C^s_\varepsilon(x, \psi)\subset C^s_\varepsilon(x)$ and therefore $C^s_\varepsilon(x)\cap S^{\sharp}_\delta(x)\neq\phi$.

The proof of $\sigma=u$ is done in the same fashion and so we omit it.

REMARK 4.4. Let $x\in V^+$ and denote by $C^\sigma_\varepsilon(x, \psi)$ the connected component of x in $W^\sigma_\varepsilon(x, \psi)$ $(\sigma=s, u)$. From the proof of Proposition A the following is concluded: for $\varepsilon>0$ there is $0<\delta\leq\varepsilon$ such that $C^\sigma_\varepsilon(x, \psi)\cap S_\delta(x)\neq\phi$ for $x\in V^+$ $(\sigma=s, u)$.

5. Local connectedness of $C^\sigma_\varepsilon(x)$

Let c_1 be as in §2 and let $0<\varepsilon_1<c_1/4$ be as in Lemma 4.1 for c_1. As before $\mathcal{S}$ and $\mathcal{T}$ denote families of local cross-sections.

Proposition C. *$C^\sigma_\varepsilon(x)$ $(\sigma=s, u)$ are locally connected for all $0<\varepsilon\leq\varepsilon_1$ and $x\in T^+$.*

This was proved in K. Hiraide [5] for homeomorphisms. However the technique of [5] is adapted for the first return map φ: $T^+\to T^+$. For completeness we give a proof.

Fix $x \in T$ $(T \in \mathcal{T})$ and let $\delta > 0$ be as in Proposition A for $0 < \varepsilon \leq \varepsilon_1$. To obtain the conclusion for $\sigma = s$, assume that $C_\varepsilon^s(x)$ is not locally connected. Then we see that there are $y \in C_\varepsilon^s(x)$ and $0 < r \leq \delta/2$ such that the connected component of y in $C_\varepsilon^s(x) \cap B_r^\sharp(y)$ does not contain $C_\varepsilon^s(x) \cap B_\lambda^\sharp(y)$ for all $\lambda > 0$. Denote by $\mathcal{K}$ the set of all connected component in $C_\varepsilon^s(x) \cap B_r^\sharp(y)$. Since $C_\varepsilon^s(x)$ is connected and $C_\varepsilon^s(x) \cap B_r^\sharp(y) \subsetneqq C_\varepsilon^s(x)$, we have by Lemma 3.1 that $K \cap S_r^\sharp(y) \neq \phi$ for all $K \in \mathcal{K}$.

Fix $0 < t < r$ and put $\mathcal{K}_t = \{K \in \mathcal{K} : K \cap B_t^\sharp(y) \neq \phi\}$. Then it is easily checked that $\mathcal{K}_t$ is an infinite set. Hence there is a sequence $\{K_i\}_{i \in N}$ in $\mathcal{K}_t$ with $K_i \cap K_j = \phi$ for $i \neq j$ such that $K_i \to K_\infty \in \mathcal{C}(C_\varepsilon^s(x) \cap B_r^\sharp(y))$ as $i \to \infty$. Since each K_i is connected, so is K_∞. Hence K_∞ is contained in a connected component in $C_\varepsilon^s(x) \cap B_r^\sharp(y)$. Therefore we may assume that $K_i \cap K_\infty = \phi$ for all $i \in \boldsymbol{N}$.

Since S $(S \in \mathcal{S}$ and $T^* \subset S)$ is a disk, we have that $A = B_r^\sharp(y)/U_t^\sharp(y)$ is an annulus bounded by circles $S_r^\sharp(y)$ and $S_t^\sharp(y)$. Since $K_i \cap S_r^\sharp(y) \neq \phi$, we take $a_i \in K_i \cap S_r^\sharp(y)$. Denote by L_i the connected component of a_i in $A \cap K_i$. Since K_i is connected and $B_t^\sharp(y) \cap K_i \neq \phi$, there is $b_i \in L_i \cap S_t^\sharp(y) \neq \phi$ by Lemma 3.1. Since $K_i \cap K_j = \phi$ for $i \neq j$, we have that $L_i \cap L_j = \phi$, $a_i \neq a_j$ and $b_i \neq b_j$. By compactness we may assume that $a_i \to a_\infty \in S_r^\sharp(y)$, $b_i \to b_\infty \in S_t^\sharp(y)$ and $L_i \to L_\infty \in \mathcal{C}(A)$ as $i \to \infty$. Then $a_\infty, b_\infty \in L_\infty$. Since $L_i \subset K_i$, it follows that $L_\infty \subset K_\infty$. Since $K_i \cap K_\infty = \phi$, we have that $L_i \cap L_\infty = \phi$, $a_i \neq a_\infty$ and $b_i \neq b_\infty$. Therefore by taking a subsequence of $\{a_i\}_{i \in N}$ if necessary, we can choose the arcs $a_i a_\infty$ in $S_r^\sharp(y)$ from a_i to a_∞ such that

$$a_i a_\infty \supsetneqq a_2 a_\infty \supsetneqq \cdots \supsetneqq a_i a_\infty \supsetneqq \cdots . \tag{1}$$

In the same way, choose the arcs $b_i b_\infty$ in $S_t^\sharp(y)$ from b_i to b_∞ such that

$$b_1 b_\infty \supsetneqq b_2 b_\infty \supsetneqq \cdots \supsetneqq b_i b_\infty \supsetneqq \cdots . \tag{2}$$

Since L_i, L_{i+1} and L_∞ are connected and mutually disjoint, it is checked that the orientation of $a_i a_\infty$ from a_i to a_∞ coincides with that of $b_i b_\infty$ from b_i to b_∞. Indeed, we can take mutually disjoint connected neighborhoods N_i, N_{i+1} and N_∞ of L_i, L_{i+1} and L_∞ in A respectively. Then there is an arc A_i in N_i from a_i to b_i such that $A_i \cap S_r^\sharp(y) = \{a_i\}$ and $A_i \cap S_t^\sharp(y) = \{b_i\}$, and there is an arc A_∞ in N_∞ from a_∞ to b_∞ such that $A_\infty \cap S_r^\sharp(y) = \{a_\infty\}$ and $A_\infty \cap S_t^\sharp(y) = \{b_\infty\}$. Since $N_i \cap N_\infty = \phi$, obviously $A_i \cap A_\infty = \phi$. Hence $A \backslash \{A_i \cup A_\infty\}$ is decomposed into two connected components U_1 and U_2. Since $a_{i+1} \in U_1 \cup U_2$ we may assume that $a_{i+1} \in U_1$. If the orientation of $a_i a_\infty$ differs from that of $b_i b_\infty$, then $b_{i+1} \in U_2$ by (1) and (2). In this case, every arc in N_{i+1} from a_{i+1} to b_{i+1} must intersect A_i or A_∞. This contradicts the fact that N_i, N_{i+1} and N_∞ are mutually disjoint. Therefore the orientation of $a_i a_\infty$ must coincide with that of $b_i b_\infty$.

For $i \geq 2$, take $z_i \in L_i$ such that $d(y, z_i) = t + (r-t)/2$, since $L_i \subset K_i \subset C_\varepsilon^s(x)$, obviously $z_i \in C_\varepsilon^s(x) \cap C_\varepsilon^u(z_i, \psi)$. Hence $C_\varepsilon^s(x) \cap C_\varepsilon^u(z_i, \psi) = \{z_i\}$ by expansiveness. Since $z_i \in L_i$ and $L_{i-1} \cup L_{i+1} \subset C_\varepsilon^s(x)$, we have that $(L_{i-1} \cup L_{i+1}) \cap (C_\varepsilon^u(z_i, \psi) \cup L_i) = \phi$. Hence we can take a connected neighborhood N_{i-1} of L_{i-1} in A and a connected neighborhood N_{i+1} of L_{i+1} in A such that $N_{i-1} \cap N_{i+1} = \phi$ and $(N_{i-1} \cup N_{i+1}) \cap (C_\varepsilon^u(z_i, \psi) \cup L_i) = \phi$. Then there is an arc A_{i-1} in N_{i-1} from a_{i-1} to b_{i-1} such that $A_{i-1} \cap S_r^\sharp(y) = \{a_{i-1}\}$ and $A_{i-1} \cap S_t^\sharp(y) = \{b_{i-1}\}$, and there is an arc A_{i+1} in N_{i+1} from a_{i+1} to b_{i+1} such that $A_{i+1} \cap S_r^\sharp(y) = \{a_{i+1}\}$ and $A_{i+1} \cap S_t^\sharp(y) = \{b_{i+1}\}$. Obviously $(A_{i-1} \cup A_{i+1}) \cap (C_\varepsilon^u(z_i, \psi) \cup L_i) = \phi$. Denote by $a_{i-1}\, a_{i+1}$ the subarc in $a_{i-1}\, a_\infty$ from a_{i-1} to a_{i+1} and by $b_{i-1}\, b_{i+1}$ the subarc in $b_{i-1}\, b_\infty$ from b_{i-1} to b_{i+1}. Then we have

$$\Gamma = A_{i-1} \cup A_{i+1} \cup a_{i-1}\, a_{i+1} \cup b_{i-1}\, b_{i+1}$$

is a simple closed curve in A. From the relation betwee the orientations of $a_{i-1}\, a_\infty$ and $b_{i-1}\, b_\infty$, it follows that Γ bounds a disk D in A. Then we see that z_i is an interior point of D. Since $r \leq \delta/2$, we have $C_\varepsilon^u(z_i, \psi) \cap S_r^u(y) \neq \phi$ (see $C_\varepsilon^s(x, \psi) \cap S_\delta(x) \neq \phi$ in the proof of Proposition A). By the connectedness of $C_\varepsilon^u(z_i, \psi)$ we have $\Gamma \cap C_\varepsilon^u(z_i, \psi) \neq \phi$. Since $(A_{i-1} \cup A_{i+1}) \cap C_\varepsilon^u(z_i, \psi) = \phi$, it is clear that

$$C_\varepsilon^u(z_i, \psi) \cap a_{i-1}\, a_{i+1} \neq \phi \quad \text{or} \quad C_\varepsilon^u(z_i, \psi) \cap b_{i-1}\, b_{i+1} \neq \phi\,.$$

Without loss of generality we have

$$w_i \in C_\varepsilon^u(z_i, \psi) \cap a_{i-1}\, a_{i+1} \neq \phi\,.$$

Since $\operatorname{diam}(a_i\, a_\infty) \to 0$ as $i \to \infty$, we see that $w_i \to a_\infty$ as $i \to \infty$. Since $L_i \to L_\infty$, we may assume that $z_i \to z_\infty \in L_\infty$ as $i \to \infty$. That $d(y, z_\infty) = t + (r-t)/2$ and $w_i \in C_\varepsilon^u(z_i, \psi)$ ensures $a_\infty \in W_{c_1}^u(z_\infty, \psi)$ (see Lemma 4.1). Since $a_\infty, z_\infty \in L_\infty \subset K_\infty \subset C_\varepsilon^s(x)$, we obtain by expansiveness that $a_\infty = z_\infty$. This contradicts the facts that $a_\infty \in S_r^\sharp(y)$ and $d(y, z_\infty) = t + (r-t)/2$. Therefore $C_\varepsilon^s(x)$ is locally connected. In the same way, the conclusion for $\sigma = u$ is obtained.

REMARK 5.1. Proposition C is true for $C_\varepsilon^\sigma(x, \psi)$ $(x \in V^+)$.

6. Proof of Theorem

In this section our Theorem will be proved. Let α_0 and c_1 be as in §2 respectively. Let $0 < \varepsilon_1 \leq \min\{\alpha_0/2, c_1/4\}$ be as in §5.

Lemma 6.1. *Let $0 < \varepsilon \leq \varepsilon_1$ and A and B be non-empty subsets of T^+. If $W_\eta^s(x) \cap W_\varepsilon^u(y) \neq \phi$ for any $(x, y) \in A \times B$, then $W_\varepsilon^s(x) \cap W_\varepsilon^u(y)$ consists of exactly one point $[x, y]$ and in fact $[\ ,\ ]: A \times B \to S^+$ is a continuous map.*

Proof. Take $z_1, z_2 \in W_\varepsilon^s(x) \cap W_\varepsilon^u(y)$. Then $d(\varphi^m(x), \varphi_x^m(z_i)) \leq \varepsilon\ (i=1, 2)$ for any $m \geq 0$, and so $d(\varphi_x^m(z_1), \varphi_x^m(z_2)) \leq 2\varepsilon \leq \alpha_0$. Since $\varphi^j(z_1) = \varphi^{j-1}(z_1)\, t_{j-1}$ and $\varphi^j(z_2) = \varphi^{j-1}(z_2)\, t'_{j-1} (\beta \leq t_j,\ t'_{j-1} \leq \alpha)$ by definition, there exist $\{a_i\}$ and $\{b_i\}$

($i=1, 2$) such that

$$\varphi_x^m(z_1)=z_1(\sum_{i=0}^{m}(t_i+a_i)), \quad |a_i|\leq\rho$$

and

$$\varphi_x^m(z_2)=z_2(\sum_{i=0}^{m}(t'_i+b_i)), \quad |b_i|\leq\rho\,.$$

We can easily caluculate

$$|\sum_{i=0}^{m}(t_i+a_i)-\sum_{i=0}^{m-1}(t_i+a_i)|=|t_m+a_m|\leq\alpha+\rho<\alpha_0$$

and (*)

$$|\sum_{i=0}^{m}(t'_i+b_i)-\sum_{i=0}^{m-1}(t'_i+b_i)|=|t'_m+b_m|\leq\alpha+\rho<\alpha_0\,.$$

Since $\varphi^j(z_1)=\varphi^{j+1}(z_1)\,t_j$ and $\varphi^j(z_2)=\varphi^{j+1}(z_2)\,t'_j$ $(-\alpha\leq t_j, t'_j\leq-\beta)$, for $m<0$ as above we can write

$$\varphi_x^m(z_1)=z_1(\sum_{i=-1}^{m}(t_i+a_i)), \quad |a_i|\leq\rho$$

and

$$\varphi_x^m(z_2)=z_2(\sum_{i=-1}^{m}(t'_i+b_i)), \quad |b_i|\leq\rho\,.$$

For this case (*) holds. By Lemma 2.1 here we have that $z_1=z_2\,t$ for some $|t|<\zeta/3$, from which $z_1=z_2$.

To show that $[\ ,\ ]$: $A\times B\to S^+$ is continuous, assume that a sequence $\{(x_i, y_i)\}_{i\in N}$ in $A\times B$ converges to $(x, y)\in A\times B$ and put $z_i=[x_i, y_i]$. Then there are a subsequence $\{z_j\}$ of $\{z_i\}$ and $z_\infty\in S^+$ such that $z_j\to z_\infty$ as $j\to\infty$. Since $z_j\in W_\varepsilon^s(x_j)$, it follows from Remark 4.3 that $z_\infty\in W_c^s(x)$. In the same way we have $z_\infty\in W_{c_1}^u(y)$. Since $c_1<\alpha\leq\alpha_0/2$, we see that $W_{c_1}^s(x)\cap W_{c_1}^u(y)$ consists of one point. Hence $W_{c_1}^s(x)\cap W_{c_1}^u(y)\supset W_\varepsilon^s(x)\cap W_\varepsilon^u(y)=\{[x, y]\}$, and therefore $z_\infty=[x, y]$. Continuity of $[\ ,\ ]$ was proved.

Lemma 6.2. *$C_\varepsilon^\sigma(x)$ $(0<\varepsilon\leq\varepsilon_1)$ is arcwise connected and locally arcwise connected ($\sigma=s, u$).*

Proof. Combining Proposition C with Theorem 5.9 of [3], we see that $C^\sigma(x)$ is a peano space. Then Theorem 6.29 of [3] completes the proof.

Lemma 6.3. *Let $0<\varepsilon\leq\varepsilon_1$. For each pair (y, z) of distinct points in $C_\varepsilon^\sigma(x)$ ($\sigma=s, u$) there exists an arc from y to z in $C_\varepsilon^\sigma(x)$. Furthermore such an arc is unique.*

Proof. The first statement follows from Lemma 6.2. We prove the second statement for $\sigma=s$. To do this, we assume that there are two arcs from y to z in $C^s_\varepsilon(x)$. Then we can find a simple closed curve Γ in $C^s_\varepsilon(x)$. Choose r with $0<r\leq\varepsilon/2$ by Lemma 2.8 such that $\varphi_x(B^\sharp_r(x))\subset B^\sharp_\varepsilon(\varphi(x))$ for all $x\in T^+$. Let c_1 be as in §2. Then we can find $N\in \boldsymbol{N}$ such that $\varphi^n_x(W^s_{c_1}(x))\subset W^s_r(\varphi^n(x))$ for all $n\geq N$. Since $\Gamma\subset C^s_\varepsilon(x)\subset W^s_{c_1}(x)$, we have $\varphi^n_x(\Gamma)\subset W^s_r(\varphi^n(x))\subset B^\sharp_r(\varphi^n(x))$ for all $n\geq N$. Since $B^\sharp_r(\varphi^N(x))$ is a disk and $\varphi^N_x(\Gamma)$ is a simple closed curve in $B^\sharp_r(\varphi^N(x))$, we see that $\varphi^N_x(\Gamma)$ bounds a disk D in $B^\sharp_r(\varphi^N(x))$.

Now we claim that $\varphi^i_{\varphi^N(x)}(D)\subset B^\sharp_r(\varphi^{N+i}(x))$ for all $i\geq 0$. Indeed, $D\subset B^\sharp_\varepsilon(\varphi^N(x))$ by the choice of r. Since $\varphi^{N+1}_x(\Gamma)\subset B^\sharp_r(\varphi^{N+1}_x(x))$ and $\varphi^{N+1}(\Gamma)$ is the boundary of $\varphi_{\varphi^N(x)}(D)$, it follows that $\varphi_{\varphi^N(x)}(D)\subset B^\sharp_r(\varphi^{N+1}(x))$. In the same way, we obtain $\varphi^i_{\varphi^N(x)}(D)\subset B^\sharp_r(\varphi^{N+i}(x))$ for all $i\geq 2$. Therefore the above claim holds and so $D\subset W^s_r(\varphi^N(x))$, thus contradicting Proposition B (since $0<r\leq\varepsilon\leq c_1/4$). Therefore an arc from y to z in $C^s_\varepsilon(x)$ is unique. In the same way the conclusionfor $\sigma=u$ is obtained.

Let ψ: $V^+\to V^+$ be the first return map defined in §3 and $C^\sigma_\varepsilon(x,\psi)$ denote the connected component of x in $W^\sigma_\varepsilon(x,\psi)$ as before. Notice that $C^\sigma_\varepsilon(x,\psi)\subset C^\sigma_\varepsilon(x)$ for $x\in T^+(\sigma=s,u)$ (since $W^\sigma_\varepsilon(x,\psi)\subset W^\sigma_\varepsilon(x)$).

REMARK 6.4. Lemmas 6.2 and 6.3 hold for the first return map ψ.

Let y and z be distinct elements of $C^\sigma_\varepsilon(x)$ ($C^\sigma_\varepsilon(x,\psi)$). Since there is an arc from y to z in $C^\sigma_\varepsilon(x)$ ($C^\sigma_\varepsilon(x,\psi)$) and such an arc is unique by Lemma 6.3, we denote it by $\sigma_\varepsilon(y,z;x)$ ($\sigma_\varepsilon(y,z;x,\psi)$). Remark that $C^\sigma_\varepsilon(x)\subset C^\sigma_{\varepsilon_1}(x)$. Then we see easily that $\sigma_\varepsilon(y,z;x)=\sigma_{\varepsilon_1}(y,z;x)$. Hence we omit ε and write $\sigma(y,z;x)=\sigma_\varepsilon(y,z;x)$. We denote by $IC^\sigma_\varepsilon(x)$ the union of all open arcs in $C^\sigma_\varepsilon(x)$ and define

$$BC^\sigma_\varepsilon(x)=C^\sigma_\varepsilon(x)\backslash(IC^\sigma_\varepsilon(x)\cup\{x\})\,.$$

x belongs to $IC^\sigma_\varepsilon(x)$. For ψ we define $IC^\sigma_\varepsilon(x,\psi)$ and $BC^\sigma_\varepsilon(x,\psi)$ in the same fashion as above. Then for $0<\varepsilon\leq\varepsilon_1$ it holds that $BC^\sigma_\varepsilon(x)\neq\phi$ and

$$C^\sigma_\varepsilon(x)=\bigcup_{b\in BC^\sigma_\varepsilon(x)}\sigma(x,b;x)\,.$$

If A be an arc in $C^\sigma_\varepsilon(x)$ and if x is an end point of A, then there exists $b\in BC^\sigma_\varepsilon(x)$ such that $A\subset\sigma(x,b;x)$.

Let a,b and c be elements of $C^\sigma_\varepsilon(x)$ such that $a\neq b$ and $a\neq c$. When $\sigma(a,b;x)\cap\sigma(a,c;x)\supsetneqq\{a\}$, we write $\sigma(a,b;x)\sim\sigma(a,c;x)$. In this case, we see by Lemma 6.3 that $\sigma(a,b;x)\cap\sigma(a,c;x)$ is a subarc of both $\sigma(a,b;x)$ and $\sigma(a,c;x)$. From this fact it follows that "$\sim$" is an equivalence relation on $\{\sigma(x,b;x); b\in BC^\sigma_\varepsilon(x)\}$. We define

$$P^\sigma_\varepsilon(x)=\sharp[\{\sigma(x,b;x)\colon b\in BC^\sigma_\varepsilon(x)\}/\sim]$$

and define in the same fashion

$$P_\varepsilon^\sigma(x, \psi) = \#[\{\sigma(x, b;\, x, \psi);\ b\in BC_\varepsilon^\sigma(x, \psi)/\sim]\,,$$

where $\#[\cdot]$ denotes the cardinal number of $\cdot$. Under the these notations we have $P_\varepsilon^\sigma(x)=P_{\varepsilon_1}^\sigma(x)$ $(x\in T^+)$ and $P_\varepsilon^\sigma(x, \psi)=P_{\varepsilon_1}^\sigma(x, \psi)$ $(x\in V^+)$. Since $P_\varepsilon^\sigma(x)$ is independent of $\varepsilon(0<\varepsilon\leq\varepsilon_1)$, we omit ε and write $P^\sigma(x)=P_\varepsilon^\sigma(x)$.

Put $\mathrm{Sing}^\sigma(\varphi)=\{x\in T^+\colon P^\sigma(x)\geq 3\}$ and $\mathrm{Sing}^\sigma(\psi)=\{x\in V^+;\ P^\sigma(x, \psi)\geq 3\}$. Then we have that $\mathrm{Sing}^\sigma(\varphi)$ is a finite set for $\sigma=s, u$ and that if $P^\sigma(x)\geq 3$ $(P^\sigma(x, \psi)\geq 3)$ for $\sigma=s$ or u, then $x\in\mathrm{Per}(\varphi)$ $(\mathrm{Per}(\psi))$, where $\mathrm{Per}(\varphi)$ and $\mathrm{Per}(\psi)$ are the sets of all periodic points of φ and ψ respectively. Hence if $P^\sigma(x)$ $(P^\sigma(x, \psi))$ is infinite, then $x\in\mathrm{Per}(\varphi)$ $(\mathrm{Per}(\psi))$. Thus Lemma 6.3 ensures that $P^\sigma(x)$ $(P^\sigma(x, \psi))$ is finite for $x\in T^+(V^+)$ (c.f. [5], Lemma 4.10).

Let $0<\varepsilon\leq\varepsilon_1$, $x\in T^+$ and $y\in C_\varepsilon^\sigma(x)\setminus\{x\}$ $(\sigma=s, u)$. We say that y is a *branch point* of $C_\varepsilon^\sigma(x)$ if there are distinct element a_1, a_2 of $BC_\varepsilon^\sigma(x)$ such that $\sigma(x, a_1;\, x)\cap\sigma(x, a_2;\, x)=\sigma(x, y;\, x)$. In this case, we remark that $\sigma(x, y;\, x)\subsetneqq\sigma(x, a_i;\, x)$ $(i=1, 2)$. If y is a branch point of $C_\varepsilon^\sigma(x)$, then $y\in\mathrm{Sing}^\sigma(\varphi)$.

Lemma 6.5. *There exists sufficiently small $\varepsilon_2>0$ such that for $0<\varepsilon\leq\varepsilon_2$, $C_\varepsilon^\sigma(x)$ has at most one branch point $(\sigma=s, u)$. If $P^\sigma(x)\geq 3$, then $C_\varepsilon^\sigma(x)$ has no branch points.*

Using Lemma 6.5 we can show that $P^\sigma(x)\geq 2$ for $x\in T^+(\sigma=s, u)$. Moreover we have the following

Lemma 6.6. *For any $\varepsilon>0$ there exists $0<\delta\leq\varepsilon$ such that*

$$S_\delta^\sharp(x)\cap\sigma(x, a;\, x)\neq\phi \quad (\sigma=s, u)$$

for all $x\in T^+$ and all $a\in BC_\varepsilon^\sigma(x)$.

Let $\varepsilon>0$ be sufficiently small and let $0<\delta\leq\varepsilon$ be as in Lemma 6.6. By Lemma 6.5, for every $x\in T^+$ we can choose $0<\varepsilon(x)<\delta/2$ such that $C_\varepsilon^\sigma(x)\cap B_{\varepsilon(x)}^\sharp(x)$ has no branch points $(\sigma=s, u)$ of $C_\varepsilon^\sigma(x)$ and then define

$$S_{\varepsilon(x)}^\sigma(x) = \{a\in S_{\varepsilon(x)}^\sharp(x)\cap C_\varepsilon^\sigma(x)\colon \sigma(x, a;\, x)\setminus\{a\}\subset U_{\varepsilon(x)}^\sharp(x)\}\,.$$

Here we remark that $S_{\varepsilon(x)}^\sharp(x)$ is a circle for every $x\in T^+$. Obviously $\#[S_{\varepsilon(x)}^\sigma(x)]=P^\sigma(x)$ for all $x\in T^+$ and $\sigma=s, u$. The following ensures the existence of transversal singular foliations on a neighborhood of each point of T^+.

Lemma 6.7. *For every $x\in T^+$, $S_{\varepsilon(x)}^\sigma(x)$ is a finite set with at least two elements $(\sigma=s, u)$. If $I_1^s, I_2^s, \cdots, I_l^s$ denote all open arcs in which $D_{\varepsilon(x)}^s(x)$ cut $S_{\varepsilon(x)}^\sharp(x)$, then each element of $S_{\varepsilon(x)}^u(x)$ is contained in some I_i^s and distinct two elements of $S_{\varepsilon(x)}^u(x)$ is not contained in same I_i^s where $i=1, 2, \cdots, l$.*

By Lemma 6.7 we have $P^s(x)=P^u(x)$ for $x\in T^+$.

Lemma 6.8. *There exists $\eta>0$ such that for every $x\in T^+$ there is $0<\delta<\varepsilon(x)$ such that if*

$$y\in B_\delta(x)\setminus \bigcup_{a\in S^\sigma_{\varepsilon(x)}(x)} \sigma(x, a; x)$$

then $C^\sigma_\eta(y, \psi)$ is an arc ($\sigma=s, u$).

Using Lemmas 6.1, 6.3, 6.7 and 6.8 we can construct a singular foliated neighborhood U_x and transversal singular foliations on U_x for each $x\in T^+$. The details of the construction is described in K. Hiraide [5] and so we omit it.

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