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ON ULTRA WAVE FRONT SETS AND FOURIER INTEGRAL OPERATORS OF INFINITE ORDER

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Introduction. The fundamental solution of the Cauchy problem for a hyperbolic operator is given in the form of Fourier integral operator. As shown in [16] or [20] when the problem is not C^∞ well-posed, the symbol of the fundamental solution has exponential growth, that is, it is estimated not only from above but also from below by

$$C \exp[c\xi^{1/\kappa}], \qquad c>0. \tag{0.1}$$

The constant κ in (0.1) corresponds to the constant in the necessary and sufficient condition for the well-posedness in Gevrey classes given by Ivrii [5].

In the present paper we define $UWF^{(\mu)}(u)$ (ultra wave front sets) for u that belongs to the space of ultradistributions $\mathcal{S}\{\kappa\}'$ by

$$\begin{aligned}(x_0, \xi_0) \notin UWF^{(\mu)}(u) &\Leftrightarrow \\ &\forall \varepsilon>0\ \exists C;\ |(\chi u)^\wedge(\xi)| \leqq C \exp[\varepsilon\langle\xi\rangle^{1/\mu}],\end{aligned} \tag{0.2}$$

where $\chi \in \mathcal{S}\{\kappa\} \cap C_0^\infty$ and ξ belongs to a conic neighborhood of ξ_0 (see Definition 2.1). Then by using $UWF^{(\mu)}(u)$ we can state the propagation of very high singularities for the solution of not C^∞ well-posed Cauchy problem (see Theorems 3.1 and 3.2). Here, by a very high singularity of u, we mean that its local Fourier transform has an estimate like (0.1).

UWF are first defined by Wakabayashi [22] by the name "generalized wave front sets". But, his definition contains both UWF and Gevrey wave front sets and they are denoted by $WF^{\{\kappa\}}$ and $WF_{\{\kappa\}}$ respectively (see Definition 1.3.2 in [22]). He also tried to get non-trivial inner estimates for UWF, but got only a lemma ("not really satisfactory" in his words) and he gave two examples with respect to operators with constant coefficients.

In section 1 we define pseudo-differential operators and Fourier integral operators whose symbols have exponential growth and show that these operators act on the space of ultradistributions $\mathcal{S}\{\kappa\}'$. In section 2 we define the UWF of $u \in \mathcal{S}\{\kappa\}'$ and give the propagation theorem of UWF for Fourier integral operators of infinite order (Theorem 2.2). In section 3 we give exactly the

UWF of the solution of the Cauchy problem for hyperbolic operators with variable multiplicities.

1. Ultradistributions and Fourier integral operators of infinite order. Let κ satisfy $\kappa>1$. For positive constants h and ε we define a class $\mathcal{S}\{\kappa; h, \varepsilon\}$ of ultra differentiable functions by a set of functions $u(x)$ satisfying

$$|\partial_x^\alpha u(x)| \leq Ch^{-|\alpha|}\alpha!^\kappa \exp(-\varepsilon\langle x\rangle^{1/\kappa}) \tag{1.1}$$

for a positive constant C. For $u\in\mathcal{S}\{\kappa; h, \varepsilon\}$ we define a norm $||u; \mathcal{S}\{\kappa; h, \varepsilon\}||$ by

$$||u; \mathcal{S}\{\kappa; h, \varepsilon\}|| = \inf\{C \text{ of } (1.1)\}.$$

Then, $\mathcal{S}\{\kappa; h, \varepsilon\}$ is a Banach space.

DEFINITION 1.1. We define a class $\mathcal{S}\{\kappa\}$ by

$$\mathcal{S}\{\kappa\} = \operatorname*{ind\,lim}_{h\to 0,\ \varepsilon\to 0} \mathcal{S}\{\kappa; h, \varepsilon\}$$

and denote by $\mathcal{S}\{\kappa\}'$ the dual space of $\mathcal{S}\{\kappa\}$.

Lemma 1.2. *The Fourier transform $F[u]\equiv\hat{u}(\xi)$ maps $\mathcal{S}\{\kappa\}$ to $\mathcal{S}\{\kappa\}$ and hence the Fourier transform is also well-defined on $\mathcal{S}\{\kappa\}'$.*

Proof is omitted.

The class $\mathcal{S}\{\kappa\}'$ is a class of ultradistributions (see [2] and [9]), and as we shall prove later (Lemma 1.7) the class $\mathcal{S}\{\kappa\}'$ is characterized by the following: Let $u\in\mathcal{S}\{\kappa\}'$. Then, for any function $\chi(x)$ in $\mathcal{S}\{\kappa\}$ with compact support the Fourier transform $(\chi u)^\wedge(\xi)$ of χu is a measurable function and has an estimate

$$|(\chi u)^\wedge(\xi)| \leqq C_\varepsilon \exp[\varepsilon\langle\xi\rangle^{1/\kappa}]$$

for any $\varepsilon>0$.

Let ρ and δ be real numbers satisfying $0\leqq\delta\leqq\rho\leqq 1$, $\delta<1$, $\kappa(1-\delta)\geqq 1$ and $\kappa\rho\geqq 1$.

DEFINITION 1.3 (cf. [6], [12], [17]). i) Let $w(\theta)$ be a positive and non-decreasing function in $[1, \infty)$ or a function of the type θ^m. We say that a symbol $p(x, \xi)$ belongs to a class $S_{\rho,\delta,G(\kappa)}[w]$ if $p(x, \xi)$ satisfies

$$\begin{aligned} |p_{(\beta)}^{(\alpha)}(x, \xi)| \leqq CM^{-|\alpha+\beta|}&(\alpha!^\kappa+\alpha!^{\kappa\rho}\langle\xi\rangle^{(1-\rho)|\alpha|}) \\ &\times(\beta!^\kappa+\beta!^{\kappa(1-\delta)}\langle\xi\rangle^{\delta|\beta|})\langle\xi\rangle^{-|\alpha|}w(\langle\xi\rangle) \\ &\qquad\qquad \textit{for all } x \textit{ and } \xi\,, \end{aligned}$$

where $p_{(\beta)}^{(\alpha)}=\partial_\xi^\alpha(-i\partial_x)^\beta p$. We call the above function $w(\theta)$ an order function.

ii) We say that a symbol $p(x, \xi)$ $(\in S^{-\infty})$ belongs to a class $\mathcal{R}_{G(\kappa)}$ if for any α there exists a constant C_α such that

$$|p_{(\beta)}^{(\alpha)}(x, \xi)| \leqq C_\alpha M^{-|\beta|}\langle\xi\rangle^{-|\alpha|}\beta!^\kappa \exp(-c\langle\xi\rangle^{1/\kappa})$$

hold with a positive constant c independent of α and β. We call a pseudo-differential operator with a symbol in $\mathcal{R}_{G(\kappa)}$ a regularizing operator.

REMARK 1. When $w(\theta)=\theta^m$ for a real m we denote $S_{\rho,\delta,G(\kappa)}[w]$ by $S^m_{\rho,\delta,G(\kappa)}$.

REMARK 2. When $w(\theta)=\exp(C\theta^\sigma)$ for a $\sigma>0$, the class $S_{\rho,\delta,G(\kappa)}[w]$ is a symbol class of exponential type, and this corresponds to the class investigated in [23], [14] and [1]. We also remark that the class of symbols in Gevrey classes are investigated in [10], [11], [3] and [19].

EXAMPLE. For $a(x, \xi)\in S^m_{1,0,G(\kappa)}$ the symbol $p(x, \xi)=a(x, \xi)\exp(\langle\xi\rangle^\sigma)$ belongs to $S_{1,0,G(\kappa)}[\exp(2\theta^\sigma)]$.

DEFINITION 1.4. Let $0\leqq\tau<1$. We say that a phase function $\phi(x, \xi)$ belongs to a class $\mathcal{P}_{G(\kappa)}(\tau)$ if $\phi(x, \xi)$ is real-valued and for $J(x, \xi)\equiv\phi(x, \xi)-x\cdot\xi$ the estimates

$$\sum_{|\alpha|+|\beta|\leqq 2} |J_{(\beta)}^{(\alpha)}(x, \xi)|/\langle\xi\rangle^{1-|\alpha|}\leqq\tau \tag{1.2}$$

and

$$|J_{(\beta)}^{(\alpha)}(x, \xi)|\leqq\tau M^{-(|\alpha|+|\beta|)}(\alpha!\beta!)^\kappa\langle\xi\rangle^{1-|\alpha|} \tag{1.3}$$

hold for a constant M independent of α and β. We also set

$$\mathcal{P}_{G(\kappa)}=\bigcup_{0\leqq\tau<1}\mathcal{P}_{G(\kappa)}(\tau).$$

Proposition 1.5. *Let $w(\theta)$ be an order function satisfying*

$$w(\theta)\leqq\exp[C\theta^\sigma] \tag{1.4}$$

for a constant σ with $0\leqq\sigma<1/\kappa$. For a phase function $\phi(x, \xi)\in\mathcal{P}_{G(\kappa)}$ and a symbol $p(x, \xi)\in S_{\rho,\delta,G(\kappa)}[w]$ we define a Fourier integral operator P_ϕ and a conjugate Fourier integral operator P_{ϕ^*} by

$$P_\phi u(x)=\int e^{i\phi(x,\xi)}p(x, \xi)\hat{u}(\xi)đ\xi,$$

$$P_{\phi^*}u(x)=\int e^{ix\cdot\xi}\{\int e^{-i\phi(y,\xi)}p(y, \xi)u(y)dy\}đ\xi,$$

where $đ\xi=(2\pi)^{-n}d\xi$. Then, the operators P_ϕ and P_{ϕ^*} map $\mathcal{S}\{\kappa\}$ to $\mathcal{S}\{\kappa\}$ continuously.

Proof. For $u(x)\in\mathcal{S}\{\kappa\}$ we denote

$$f(x)=P_\phi u(x)\equiv\int e^{i\phi(x,\xi)}p(x,\xi)\hat{u}(\xi)d\xi\,.$$

Define $L=\{1+|\nabla_\xi\phi(x,\xi)|^2\}^{-1}\{1-i\nabla_\xi\phi(x,\xi)\cdot\nabla_\xi\}$. Then, we have $Le^{i\phi(x,\xi)}=e^{i\phi(x,\xi)}$ and hence

$$f(x)=\int e^{i\phi(x,\xi)}(L^t)^N\{p(x,\xi)\hat{u}(\xi)\}d\xi\,.$$

By the induction on N we can prove

$$\begin{aligned}(1.5)\qquad |\partial_\xi^\alpha\partial_x^\beta(L^t)^N\{p(x,\xi)\hat{u}(\xi)\}|\leqq{}& CM_1^{-N}M_2^{-|\alpha+\beta|}(|\alpha|+N)!^\kappa\langle x\rangle^{-N}\\ &\times(\beta!^\kappa+\beta!^{\kappa(1-\delta)}\langle\xi\rangle^{\delta|\beta|})\exp(C_1\langle\xi\rangle^\sigma-\varepsilon\langle\xi\rangle^{1/\kappa})\end{aligned}$$

for positive constants C, M_1, M_2, C_1 and ε, since $\hat{u}(\xi)$ belongs to $\mathcal{S}\{\kappa\}$. Assume that x satisfies $C_0N^\kappa\leqq\langle x\rangle\leq C_0(N+1)^\kappa$ for a constant C_0 to be determined later. Then, using (1.5) with $\alpha=0$ and denoting $\phi_\beta(x,\xi)=e^{-i\phi(x,\xi)}\partial_x^\beta e^{i\phi(x,\xi)}$ we have

$$\begin{aligned}|\partial_x^\beta f(x)|&=\Big|\sum_{\beta'+\beta''=\beta}\binom{\beta}{\beta'}\int e^{i\phi(x,\xi)}\phi_{\beta'}(x,\xi)D_x^{\beta''}(L^t)^N\{p(x,\xi)\hat{u}(\xi)\}d\xi\Big|\\ &\leqq C\sum_{\beta'+\beta''=\beta}\binom{\beta}{\beta'}\int M_3^{-|\beta'|}\{\sum_{j=1}^{|\beta'|}(|\beta'|-j)!^\kappa\langle\xi\rangle^j\}M_1^{-N}M_2^{-|\beta''|}N!^\kappa\langle x\rangle^{-N}\\ &\qquad\times(\beta''!^\kappa+\beta''!^{\kappa(1-\delta)}\langle\xi\rangle^{\delta|\beta''|})\exp(C_1\langle\xi\rangle^\sigma-\varepsilon\langle\xi\rangle^{1/\kappa})d\xi\\ &\leqq CM_4^{-|\beta|}\beta!^\kappa M_1^{-N}N!^\kappa\langle x\rangle^{-N}\\ &\leqq CM_4^{-|\beta|}\beta!^\kappa M_1^{-N}N!^\kappa(C_0N^\kappa)^{-N}\exp(\varepsilon_1C_0^{1/\kappa}(N+1))\exp(-\varepsilon_1C_0^{1/\kappa}\langle x\rangle^{1/\kappa})\end{aligned}$$

for any positive constant ε_1. Now, take C_0 and ε_1 satisfying

$$C_0\geqq 2M_1^{-1}\,,\quad\exp(\varepsilon_1C_0^{1/\kappa})\leqq 2\,.$$

Then, $f(x)$ satisfies (1.1) with $h=M_4$ and $\varepsilon=\varepsilon_1C_0^{1/\kappa}$. Consequently, we have proved that P_ϕ maps $\mathcal{S}\{\kappa\}$ to $\mathcal{S}\{\kappa\}$ continuously. In the same way we can prove that P_{ϕ^*} maps $\mathcal{S}\{\kappa\}$ to $\mathcal{S}\{\kappa\}$ continuously. Q.E.D.

From Proposition 1.5 the following definition is well-defined

DEFINITION 1.6. Let $w(\theta)$ be an order function satisfying (1.4), that is, it satisfies

$$w(\theta)\leqq\exp(C\theta^\sigma)$$

for a constant σ with $0\leqq\sigma<1/\kappa$. Then for $\phi(x,\xi)\in\mathcal{P}_{G(\kappa)}$ and $p(x,\xi)\in S_{\rho,\delta,G(\kappa)}[w]$, the following operators

$$P_\phi : \mathcal{S}\{\kappa\}' \to \mathcal{S}\{\kappa\}',$$
$$P_{\phi^*}: \mathcal{S}\{\kappa\}' \to \mathcal{S}\{\kappa\}'$$

are defined by the principle of duality.

EXAMPLE. For $a(x, \xi) \in S^m_{1,0,G(\kappa)}$ $(\kappa<2)$ we consider a symbol $p(x, \xi) = a(x, \xi) \exp(c\langle\xi\rangle^{1/2})$ with $c>0$. Then, it belongs to $S_{1,0,G(\kappa)}[\exp(2c\theta^{1/2})]$ and for $1<\kappa<2$ the following maps are well-defined:

$$P_\phi : \mathcal{S}\{\kappa\}' \to \mathcal{S}\{\kappa\}',$$
$$P_{\phi^*}: \mathcal{S}\{\kappa\}' \to \mathcal{S}\{\kappa\}',$$

where ϕ is a phase function in $\mathcal{P}_{G(\kappa)}$.

Lemma 1.7. *For $u \in \mathcal{S}\{\kappa\}'$ and $\chi \in \mathcal{S}\{\kappa\} \cap C_0^\infty$ the Fourier transform $(\chi u)^\wedge(\xi)$ of χu is a measurable function and has an estimate*

$$|(\chi u)^\wedge(\xi)| \leqq C_\varepsilon \exp(\varepsilon\langle\xi\rangle^{1/\kappa})$$

for any $\varepsilon>0$.

Proof. We may assume that $u \in \mathcal{S}\{\kappa\}'$ has a compact support and prove that, for any fixed ε, $\exp(-\varepsilon\langle\xi\rangle^{1/\kappa})\hat{u}$ is a functional on L^1 and has the following estimate

$$|\langle \exp(-\varepsilon\langle\xi\rangle^{1/\kappa})\hat{u}, \psi\rangle| \leqq C\|\psi\|_{L^1} \tag{1.6}$$

for $\psi \in L^1$. Then, we find that $\exp(-\varepsilon\langle\xi\rangle^{1/\kappa})\hat{u}$ belongs to L^∞ and we have an estimate

$$|\hat{u}(\xi)| \leqq C_\varepsilon \exp(\varepsilon\langle\xi\rangle^{1/\kappa})$$

for any ε. Denote by $\tilde{\psi}(x)$ the inverse Fourier transform of $\psi(\xi)$ and take a function $\chi(x)$ in $\mathcal{S}\{\kappa; h, 1\} \cap C_0^\infty(R^n)$ with $h=\varepsilon^\kappa\kappa^{-\kappa}/2$ such that $\chi(x)=1$ on the support of u. Then, we have for $\psi \in \mathcal{S}\{\kappa; h, 1\}$

$$\begin{aligned}\langle \exp(-\varepsilon\langle\xi\rangle^{1/\kappa})\hat{u}, \psi\rangle &= \langle \hat{u}, \exp(-\varepsilon\langle\xi\rangle^{1/\kappa})\psi\rangle \\ &= \langle u, \exp(-\varepsilon\langle D\rangle^{1/\kappa})\tilde{\psi}\rangle \\ &= \langle u, \chi(x)\exp(-\varepsilon\langle D\rangle^{1/\kappa})\tilde{\psi}\rangle .\end{aligned}$$

Here, we have used Proposition 1.5 for well-definedness of the third and fourth members of the above equation. Hence, by the definition and the fact that $u \in \mathcal{S}\{\kappa\}'$ we have

$$|\langle \exp(-\varepsilon\langle\xi\rangle^{1/\kappa})\hat{u}, \psi\rangle| \leqq C\|\chi(x)\exp(-\varepsilon\langle D\rangle^{1/\kappa})\tilde{\psi}; \mathcal{S}\{\kappa; h, 1\}\| . \tag{1.7}$$

Write

$$\chi(x)\exp(-\varepsilon\langle D\rangle^{1/\kappa})\tilde{\psi}(x)=\int e^{ix\cdot\xi}\chi(x)\exp(-\varepsilon\langle\xi\rangle^{1/\kappa})\psi(\xi)d\!\!{}^-\xi\,.$$

Then, from $h=\varepsilon^{\kappa}\kappa^{-\kappa}/2$, we have

$$|\partial_x^{\alpha}(\chi(x)\exp(-\varepsilon\langle D\rangle^{1/\kappa})\tilde{\psi})|\leqq Ch^{-|\alpha|}\alpha!^{\kappa}\exp(-\langle x\rangle^{1/\kappa})\|\psi\|_{L^1}$$

and hence

$$\|\chi(x)\exp(-\varepsilon\langle D\rangle^{1/\kappa})\tilde{\psi};\,\mathcal{S}\{\kappa;h,1\}\|\leqq C\|\psi\|_{L^1}\,.$$

This and (1.7) yields (1.6) for $\psi\in\mathcal{S}\{\kappa;h,1\}$. Finally, using the limiting process we have (1.6) for any $\psi\in L^1(R^n)$. Q.E.D.

From Lemma 1.7 we get the following Lemma 1.8, which states that the pseudo-differential operator with a symbol in $\mathcal{R}_{G(\kappa)}$ is a regularizing operator.

Lemma 1.8. *For $u\in\mathcal{S}\{\kappa\}'$ with compact support and $r(x,\xi)\in\mathcal{R}_{G(\kappa)}$ we have*

$$r(X,D_x)u\in\mathcal{B}\{\kappa\}\,.$$

Here, $f(x)\in\mathcal{B}\{\kappa\}$ means that there exists a constant C such that

$$|\partial_x^{\alpha}f(x)|\leqq CM^{-|\alpha|}\alpha!^{\kappa}\qquad \textit{for any } x\,.$$

In the following section we also need

Lemma 1.9. *Let $r(x,\xi)$ satisfies*

$$|r_{(\beta)}^{(\alpha)}(x,\xi)|\leqq CM^{-|\alpha+\beta|}\alpha!^{\kappa}\times(\beta!^{\kappa}+\beta!^{\kappa(1-\delta)}\langle\xi\rangle^{\delta|\beta|})\exp(-c_0\langle x\rangle^{1/\kappa}-c_0\langle\xi\rangle^{1/\kappa})\tag{1.8}$$

for a positive constant c_0. Then, for $u\in\mathcal{S}\{\kappa\}'$, $r(X,D_x)u$ is well-defined and belongs to $\mathcal{B}\{\kappa\}$.

We can prove the lemma as Proposition 1.5 and Lemma 1.7. The detailes are omitted.

2. Ultra wave front set

DEFINITION 2.1. Let κ and μ satisfy $\kappa\leqq\mu$. For $u\in\mathcal{S}\{\kappa\}'$ we define a *UWF* (ultra wave front set) of u as follows: We say that a point (x_0,ξ_0) in $T^*R^n\backslash\{0\}$ does not belong to $UWF^{(\mu)}(u)$ if there exist a function $\chi(x)$ in $\mathcal{S}\{\kappa\}\cap C_0^{\infty}$ with $\chi(x_0)\neq0$, a conic neighborhood Γ of ξ_0, and for any positive constant ε there exists a constant C such that

$$|(\chi u)^{\wedge}(\xi)|\leqq C\exp[\varepsilon\langle\xi\rangle^{1/\mu}]\qquad \textit{for}\quad \xi\in\Gamma\,.\tag{2.1}$$

REMARK 1. As stated in Introduction this definition is the same as that of Wakabayashi. (See Definition 1.3.2 in [22]).

REMARK 2. Let $u \in \mathcal{S}\{\kappa\}'$ and let $\kappa \leqq \mu$. Then, $(x_0, \xi) \notin UWF^{(\kappa)}(u)$ for all ξ is equivalent to that $\chi u \in \mathcal{S}\{\mu\}'$ for some $\chi \in \mathcal{S}\{\kappa\}$ with $\chi(x_0) \neq 0$. (See Lemma 1.3.3 of [22]). Especially, from Lemma 1.7 we have $UWF^{(\kappa)}(u) = \phi$ for $u \in \mathcal{S}\{\kappa\}$.

Theorem 2.2. *Let $\kappa < \mu$ and let $\phi(x, \xi) \in \mathcal{P}_{G(\kappa)}$ and $p(x, \xi) \in S_{\rho,\delta,G(\kappa)}[\exp(c\theta^{\sigma})]$ for some σ with $\sigma < 1/\mu$. Assume that $\phi(x, \xi)$ is positively homogeneous for large $|\xi|$. Then, for $u \in \mathcal{S}\{\kappa\}'$ and $(y_0, \eta_0) \in T^*R^n \setminus \{0\}$ with $|\eta_0| \gg 1$, $(y_0, \eta_0) \notin UWF^{(\mu)}(u)$ yields*

$$(x_0, \xi_0) \notin UWF^{(\mu)}(P_{\phi}u), \tag{2.2}$$

where

$$\xi_0 = \nabla_x \phi(x_0, \eta_0), \quad y_0 = \nabla_\xi \phi(x_0, \eta_0). \tag{2.3}$$

This theorem corresponds to the theorem for the propagation of Gevrey wave front sets investigated in Theorem 4 in [18].

Proof. Assume $(y_0, \eta_0) \notin UWF^{(\mu)}(u)$. Then, from the definition we can take a neighborhood V_2 of y_0 and a conic neighborhood Γ_2 of η_0 such that for any ε and $\chi \in \mathcal{S}\{\kappa\}$ with supp $\chi \subset V_2$ an inequality

$$|(\chi u)^{\wedge}(\eta)| \leqq C_{\varepsilon} \exp[\varepsilon \langle \eta \rangle^{1/\mu}] \qquad for \quad \eta \in \Gamma_2 \tag{2.4}$$

holds. Next, using (2.3) we take neighborhoods V_1 and V_2' of x_0 and y_0, and conic neighborhoods Γ_1 and Γ_2' of ξ_0 and η_0 satisfying

$$V_2' \Subset V_2, \quad \Gamma_2' \cap S_\eta^{n-1} \Subset \Gamma_2 \cap S_\eta^{n-1}$$

and

$$\begin{cases} \text{i)} \ \ \nabla_\xi \phi(x, \eta) \in V_2' & \text{for} \quad x \in V_1, \ \eta \in \Gamma_2', \\ \text{ii)} \ \ \nabla_x \phi^{-1}(x, \xi) \in \Gamma_2' & \text{for} \quad x \in V_1, \ \xi \in \Gamma_1, \end{cases} \tag{2.5}$$

where $\eta = \nabla_x \phi^{-1}(x, \xi)$ is the inverse function of $\xi = \nabla_x \phi(x, \eta)$. Let $\chi_1(x)$ and $\chi_2(x)$ be functions in $\mathcal{S}\{\kappa\}$ and $\psi_1(\xi)$ and $\psi_2(\xi)$ be symbols in $S^0_{1,0,G(\kappa)}$ satisfying

$$\text{supp}\, \chi_1 \subset V_1, \tag{2.6}$$

$$\text{supp}\, \chi_2 \subset V_2, \quad \chi_2(y) = 1 \qquad for \quad y \in V_2', \tag{2.7}$$

$$\text{supp}\, \psi_1 \subset \Gamma_1, \quad \psi_1(\xi) = 1 \qquad for \quad \xi \in \Gamma_1^0 \tag{2.8}$$

with some conic neighborhood Γ_1^0 of ξ_0, and

$$\text{(2.9)} \qquad \operatorname{supp} \psi_2 \subset \Gamma_2 , \quad \psi_2(\eta) = 1 \qquad \textit{for} \quad \eta \in \Gamma_2' .$$

Now, write $\chi_1(x) P_\phi u$ as

$$\text{(2.10)} \quad \begin{aligned} \chi_1 P_\phi u &= \chi_1 P_\phi \psi_2(D) \chi_2 u + \chi_1 P_\phi \psi_2(D)(1-\chi_2) u + \chi_1 P_\phi (1-\psi_2(D)) u \\ &\equiv f_1(x) + f_2(x) + f_3(x) . \end{aligned}$$

From (2.5) and (2.8)–(2.9) we can show that $\sigma(\psi_1(D)\chi_1 P_\phi (1-\psi_2(D))$ satisfies (1.8) and hence from Lemma 1.9 we have

$$\psi_1(D) f_3 = \psi_1(D) \chi_1 P_\phi (1-\psi_2(D)) u \in \mathcal{B}\{\kappa\}$$

and

$$\text{(2.11)} \qquad |\hat{f}_3(\xi)| \leqq C \qquad \textit{for} \quad \xi \in \Gamma_1^0 .$$

Similarly, from (2.5)–(2.7) we obtain that $\sigma(\chi_1 P_\phi \psi_2(D)(1-\chi_2))$ satisfies (1.8) and hence we get

$$f_2(x) = \chi_1 P_\phi \psi_2(D)(1-\chi_2) u \in \mathcal{B}\{\kappa\} .$$

This yields

$$\text{(2.12)} \qquad |\hat{f}_2(\xi)| \leqq C \qquad \textit{for all} \quad \xi .$$

Next, we consider $f_1(x)$. Let τ be a constant satisfying (1.2)–(1.3) and write

$$\text{(2.13)} \quad \begin{aligned} \hat{f}_1(\xi) &= \iint e^{i(-x\cdot\xi+\phi(x,\eta))} \chi_1(x) p(x, \eta) \psi_2(\eta) (\chi_2 u)^\wedge(\eta) đ\eta\, dx \\ &= \iint_{|\xi-\eta|\leqq\lambda\langle\eta\rangle} e^{i(-x\cdot\xi+\phi(x,\eta))} \chi_1(x) p(x, \eta) \psi_2(\eta) (\chi_2 u)^\wedge(\eta) đ\eta\, dx \\ &\quad + \iint_{|\xi-\eta|\geqq\lambda\langle\eta\rangle} e^{i(-x\cdot\xi+\phi(x,\eta))} \chi_1(x) p(x, \eta) \psi_2(\eta) (\chi_2 u)^\wedge(\eta) đ\eta\, dx \\ &\equiv I_1 + I_2 \end{aligned}$$

with $\lambda=(1+\tau)/2$. Since the absolute value of the integrand of I_1 is estimated by

$$\begin{aligned} C \exp[c\langle\eta\rangle^\sigma + \varepsilon\langle\eta\rangle^{1/\mu}] &\leqq C' \exp[2\varepsilon\langle\eta\rangle^{1/\mu}] \\ &\leqq C' \exp[2\varepsilon\{2/(1-\tau)\}^{1/\mu}\langle\xi\rangle^{1/\mu}] , \end{aligned}$$

we have

$$\text{(2.14)} \qquad |I_1| \leqq C'' \exp[2\varepsilon\{2/(1-\tau)\}^{1/\mu}\langle\xi\rangle^{1/\mu}] .$$

Let $L = -i|-\xi+\nabla_x\phi(x, \eta)|^{-2}(-\xi+\nabla_x\phi(x, \eta))\cdot\nabla_x$. Then, we have $L \exp[i(-x\cdot\xi+\phi(x, \eta))] = \exp[i(-x\cdot\xi+\phi(x, \eta))]$. Hence, using the integration by parts and $|-\xi+\nabla_x\phi(x, \eta)| \geqq C(\langle\xi\rangle+\langle\eta\rangle)$ on the support of the integrand of I_2 we can obtain

(2.15) $$|I_2| \leqq C\,.$$

Combining (2.10)–(2.15) we obtain

$$|(\chi_1 P_\phi u)^\wedge(\xi)| \leqq C \exp [2\varepsilon \{2/(1-\tau)\}^{1/\mu} \langle\xi\rangle^{1/\mu}] \qquad for \quad \xi \in \Gamma^0\,.$$

Since we can take ε arbitrary, we obtain (2.2). Q.E.D.

3. Propagation of ultra wave front sets. The propagation of Gevrey wave front sets are investigated in [8], [13] and [15] for the solutions of not C^∞ well-posed Cauchy problem of hyperbolic operators. In this section, we give the propagation of the *UWF* for the solutions of the following two degenerate hyperbolic operators in $[s,\ T] \times R_x^1$:

$$L = D_t^2 - t^{2j} D_x^2 + ait^k D_x$$

and

$$L = D_t^2 - g(x)^{2j} D_x^2 + aiD_x\,,$$

where $D_t = -i\partial_t$ and $D_x = -i\partial_x$. First, we consider the former degenerate hyperbolic operator

(3.1) $$L = D_t^2 - t^{2j} D_x^2 + ait^k D_x \qquad in \quad [s,\ T] \times R_x^1\,,$$

where a is a real constant. Then, Shinkai [16] proves that the fundamental solution $E(t,s)$ for the Cauchy problem

(3.2) $$Lu(t) = 0\,, \quad u(s) = 0\,, \quad \partial_t u(s) = u_0\,,$$

when $s<0<t$, is constructed in the form

(3.3) $$E(t,\ s) = \sum_{m,n=1}^2 E_{m,n,\phi_{m,n}}(t,s\ )\,,$$

where $\phi_{m,n}(t,\ s) \equiv \phi_{m,n}(t,\ s;\ \xi)$ are phase functions defined by

$$\phi_{m,n}(t,\ s;\ \xi) = x\xi + \{(-1)^m t^{j+1} + (-1)^n s^{j+1}\}\, \xi/(j+1)\,.$$

In (3.3) the symbols $e_{m,n}(t,\ s;\ \xi)$ of $E_{m,n,\phi_{m,n}}(t,\ s)$ satisfy

(3.4) $$e_{m,n}(t,\ s;\ \xi) = a_{m,n} \exp [C_{m,n} \xi^\sigma] \xi^{-1}(1+o(1))\,, \qquad \xi \to +\infty\,,$$

where

$$\sigma = (j-k-1)/(2j-k)\,.$$

So, in (3.4), if $\mathrm{Re}\, C_{m,n} > 0$, then $E_{m,n,\phi_{m,n}}(t,\ s)$ is a Fourier integral operator of infinite order. Using the fundamental solution in (3.3) we have the following theorem

Theorem 3.1 ([16]). *Assume* $k<j-1$. *Let* $u(t,\ x)$ *be the solution of* (3.2)

for (3.1) *with* $u_0(x)=\delta(x)$ (*Dirac function*). *Let* $\Gamma_{m,n}$ *be the trajectory associated to* $\phi_{m,n}$ *for* $t>0$. *Then we get*

$$UWF^{(1/\sigma)}(u(t)) = \cup_P \Gamma_{m,n}, \tag{3.5}$$

where $P=\{(m, n);\ \mathrm{Re}\, C_{m,n}>0\}$.

REMARK. The result (3.5) shows that if $k<j-1$, then (3.2) for (3.1) is not C^∞ well-posed and is $\gamma^{(\kappa)}$-well-posed for $1<\kappa<(2j-k)/(j-k-1)$ (for the $\gamma^{(\kappa)}$-well-posedness see also [5]).

Next, we consider a degenerate hyperbolic operator with respect to the space variable:

$$L = D_t^2-g(x)^{2j}D_x^2+aiD_x \tag{3.6}$$

with a positive constant a, where j is an even number and $g(x)$ is an function in $\mathcal{B}\{\kappa\}$ satisfying $g(x)=x$ for $|x|\leqq 1$, $g(x)\geqq 1$ for $x>1$ and $g(x)\leqq -1$ for $x<-1$. It is well-known that the Cauchy problem (3.2) for (3.6) is not C^∞ well-posed (see [5], [21] and [4]). Assume

$$2j/(2j-1)\leqq\kappa\leqq 2j/(j+1).$$

Let $\phi_\pm(t, s; x, \xi)$ be the phase functions corresponding to the characteristic roots $\pm g(x)^j\xi$ of (3.6). Then, the fundamental solution of the Cauchy problem (3.2) for (3.6) is constructed in the form

$$E(t, s) = E_{+,\phi_+}(t, s)+E_{-,\phi_-}(t, s)+(\textit{regularizing operator}) \tag{3.7}$$

and the symbols $e_\pm(t, s; x, \xi)$ of the Fourier integral operators $E_{\pm,\phi_\pm}(t, s)$ can be written in the form

$$e_\pm(t, s; x, \xi) = \exp[f_\pm(t, s; x, \xi)]e'_\pm(t, s; x, \xi) \tag{3.8}$$

with symbols $f_\pm(t, s; x, \xi)$ in $S^{1/2}_{1-\delta,\delta,G(\kappa)}$ and elliptic symbols $e'_\pm(t, s; x, \xi)$ in $S^0_{1-\delta,\delta,G(\kappa)}$. Here, $\delta=1/(2j)$. Moreover, when $s<t$, the symbols $f_\pm(t, s; x, \xi)$ of (3.8) satisfy

$$\mathrm{Re} f_+(t, s; x, \xi)\geqq C(t-s)\langle\xi\rangle^{1/2}/(|x|^j\langle\xi\rangle^{1/2}+1), \tag{3.9}$$

$$\mathrm{Re} f_-(t, s; x, \xi)\leqq -C(t-s)\langle\xi\rangle^{1/2}/(|x|^j\langle\xi\rangle^{1/2}+1) \tag{3.10}$$

for a positive constant C. Hence, $E_{+,\phi_+}(t, s)$ is a Fourier integral operator with infinite order. For a conic set V in T^*R^1 we set $\Gamma(t, s; V)=\cup_\pm\{(x, \xi);\ (x, \xi)$ is a point at t of the bicharacteristic strip of $\pm g(x)^j\xi$ emanating from (y, η) in $V\}$. Then, using the fundamental solution (3.7) we have

Theorem 3.2 ([20]). *Let* $u(t)$ *be the solution of the Cauchy problem* (3.2) *of the operator* (3.6) *for* u_0 *in* $\mathcal{S}\{\kappa\}'$ *with compact support. Then, when* μ *satisfies* $\kappa<\mu<2$ *we have*

$$UWF^{(\mu)}(u(t)) = \Gamma(t, s; UWF^{(\mu)}(u_0))$$

and when $\mu\geqq 2$ *we have*

$$UWF^{(\mu)}(u(t))\subset\Gamma(t, s; UWF^{(\mu)}(u_0))\cup T_0^*R,$$

especially, we have

$$UWF^{(\mu)}(u(t))\setminus T_0^*R = \Gamma(t, s; UWF^{(\mu)}(u_0)\setminus T_0^*R),$$

where $T_0^*R=\{(0, \xi); \xi\in R\setminus\{0\}\}$. *In particular, when* $u_0=\delta(x)$ (*Dirac function*) *we have*

$$(0, \pm 1)\in UWF^{(2)}(u(t)).$$

For the construction of the fundamental solution (3.7) we use finite order Fourier integral operators with complex phase functions $\phi_\pm(t, s; x, \xi)-if_\pm(t, s; x, \xi)$ as in [7] instead of Fourier integral operators of exponential order. Then, we can give the estimate (3.10) from below.

REMARK. In the above we assumed $a>0$. But, if we assume $a<0$ we can also constructe the fundamental solution $E(t, s)$ for (3.6) in the same form (3.7) with (3.9)–(3.10) replaced by

$$\begin{cases} \operatorname{Re} f_-(t, s; x, \xi)\geqq C(t-s)\langle\xi\rangle^{1/2}/(|x|^j\langle\xi\rangle^{1/2}+1), \\ \operatorname{Re} f_+(t, s; x, \xi)\leqq -C(t-s)\langle\xi\rangle^{1/2}/(|x|^j\langle\xi\rangle^{1/2}+1). \end{cases}$$

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