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FLUCTUATION OF SPECTRA IN RANDOM MEDIA II

Dedicated to Professor Nobuyuki Ikeda on his 60th birthday

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1. Introduction

Recently, we have several papers concerning eigenvalue of the Laplacian in a region with many obstacles. See Ozawa [7], [9], Figari-Orlandi-Teta [4], etc. Associated diffusion problems are studied by Chavel-Feldaman [2], Sznitman [13], etc. In the present paper the author considers the problem studied in [9] extensively. The author recommends the readers to beign to read this paper before attacking the preceding paper [9].

Let Ω be a bounded domain in R^3 with smooth boundary γ. Fix $\alpha>0$. Fix $\beta\in[1, 3)$. Let $m=1, 2, \cdots$ be a parameter. We put $n=[m^\beta]$. We remove n balls of centers $w(m)=(w_1, \cdots, w_n)\in\Omega^n$ with radius α/m from Ω and we get $\Omega_{w(m)}=\Omega\backslash\overline{n-\text{balls}}$. Remark that $\Omega_{w(m)}$ may not be connected. Let Ω^0 be a connected component of $\Omega_{w(m)}$. Let $\mu_k(\Omega^0)$ be the k-th eigenvalue of the Laplacian in Ω^0 under the Dieichlet condition on $\partial\Omega^0$. We arrange all $\mu_k(\Omega^0)$ in line (all $\Omega^0\subset\Omega_{w(m)}$, $k=1, 2, \cdots$), then we have the j-th eigenvalue $\mu_j(w(m))$ of the Laplacian in $\Omega_{w(m)}$ under the Dirichlet condition.

We consider Ω as probability space by fixing a positive continuous function V on $\bar{\Omega}$ satisfying

$$\int_\Omega V(x)\,dx = 1$$

so that $P(x\in A)=\int_A V(x)\,dx$. Let Ω^n be the product probability space. All configuration Ω^n of the centers of balls $w(m)$ can be considered as a probability space Ω^n by the statistical law stated above. Hereafter $\mu_j(w(m))$ is considered as a random variable on Ω^n.

There are several papers concerning asymptotic behaviour of $\mu_j(w(m))$ as m tends to infinity. The first step is made by Kac [6] and Huruslov-Marchenko [5], when $\beta=1$. The difficult case $\beta>1$ was first examined by the author. See [7]. Kac, Huruslov-Marchenko obtained convergence

$$\mu_j(w(m)) - \mu_j^V \to 0 \tag{1.1}$$

in probability as $m \to \infty$, where μ_j^V is the j-th eigenvalue of the Schrödinger operator $-\Delta + 4\pi\alpha\, V(x)$ in Ω under the Dirichlet condition on γ.

In the present time we have much information on the convergence (1.1). We can treat the case $\beta > 1$. We can study an error estimate of (1.1) even if $\beta > 1$. The following result is one of the results in this paper. Rather important results are Theorems 2~7. The most important result in this paper is Theorem 7 which is valid for $\beta \in [1, 3)$.

Let $|\Omega|$ be the three dimensional measure of Ω.

Theorem 1. *Assume that $V(x) \equiv |\Omega|^{-1}$. Fix j. Assume that μ_j, which is the j-th eigenvalue of the Laplacian in Ω under the Dirichlet condition on γ, is simple. Fix $\beta \in [1, 5/4)$.*
Then,

$$m^{1-(\beta/2)}(\mu_j(w(m)) - (\mu_j + 4\pi\alpha m^{\beta-1}|\Omega|^{-1}))$$

tends in distribution to Gaussian random variable Π_j of mean $E(\Pi_j) = 0$ and variance $E(\Pi_j^2)$ given by

$$(4\pi\alpha)^2 \left(\int_\Omega \varphi_j(x)^4\, dx - |\Omega|^{-1}\right)|\Omega|^{-1}.$$

Here μ_j is the normalized eigenfucntion of the Laplacian in Ω associated with μ_j.

REMARK: Figari-Orlandi-Teta [4] examined the case $\beta = 1$. Unfortunately their proof has a gap. The reason why their proof has a gap is stated in §2. The author here emphasizes that our main aim is not to correct proof of result in [4] which is a very small part of this paper. To get these result we have to develop our calculation in greater detail. These calculations include typical fundamental techniques which will be used in other problems concerning random media, random Green function. Namely calculation involving Feynman diagram (multiple product of Green's function) is developed.

Our standing point in this note is perturbative calculus (point interaction approximation of Green's function) which is first studied by [8]. Based on point interaction approximation we reduce our problem to a problem of estimating large number of multiple product of Grees's function. The reader who reads §4~7 may find various new ideas and techniques to handle various type of Feynman diagram.

The author here reemphasizes that our calculus (point interaction approximation) gives the deepest result in this direction, which might not be imagined by Kac [6] in which he gave a negative opinion on perturbative calculus.

Asymptotic formulas for the eigenvalues of the Laplacian related to this note are presented in Besson [1], Courtois [3] and Ozawa [9].

The author expresses his hearty thanks to Professor Kohei Uchiyama who pointed mistakes of the first version of this paper.

2. Rough sketch of proof of Theorem 1

Naive idea of our proof of Theorem 1 is construction of an approximate Green function of $-\Delta+ Tm^{\beta-1}(T=\text{const}>0)$ in $\Omega_{w(m)}$ under the Dirichlet condition on $\partial\Omega_{w(m)}$ by using the Green function of $-\Delta+Tm^{\beta-1}$ in Ω under the Dirichlet condition on γ.

It should be noticed that $\Omega_{w(m)}$ may not be connected. We can avoid this technical complication by introducing the following $\mathcal{O}_1(m)$ on $w(m)$ which assures that the only one connected component ω plays a role of $\Omega_{w(m)}$ and that components other than ω are neglaigible to study Theorem 1.

$\mathcal{O}_1(m)$: Let $\mathcal{F}$ be the family of all open balls of radius $m^{-\beta/3}$. Then,

$$\sup_{\mathcal{F}\ni K}(\text{the number of balls of radius } \alpha/m \text{ with the center } w_i \text{ such that ball intersect } K) \leqq (\log m)^2.$$

We see that the following holds.

Lemma 2.1. *We have*

$$P(w(m)\in\Omega^n;\ \mathcal{O}_1(m) \text{ holds})\geq 1-m^{-N}$$

for any N and any sufficiently large m depending on N.

REMARK. See the Appendix of this paper.

For a while we use the same notations as in 342p of [9].

Let ω be a maximal component of $\Omega_{w(m)}$ given by $\omega=\omega_1(w(m))$ when $g(w(m))=1$ and $\omega=\omega_k(w(m))$ when $g(w(m))\geqq 2$.

The eigenvalue problem of $-\Delta+Tm^{\beta-1}$ in ω is transformed into the eigenvalue problem of the Green operator. Put $Tm^{\beta-1}=\lambda$. Here T is a constant which will be taken as a large fixed constant in Theorems 2~7. As m tends to infinity, λ tends to infinity by this relation. Let $G(x,y)$ be the Green function of $-\Delta+\lambda$ in Ω under the Dirichlet condition on γ. We have

$$\begin{aligned} (-\Delta_x+\lambda)\, G(x,y) &= \delta(x-y)\,, \quad && x,y\in\Omega \\ G(x,y) &= 0\,, && x\in\gamma\,. \end{aligned}$$

Notice that $G(x,y)$ depends on m, β, T. Let $G(x,y;w(m))$ be the Green function of $-\Delta+\lambda$ in ω under the Dirichlet condition on $\partial\omega$. We have

$$\begin{aligned} (-\Delta_x+\lambda)\, G(x,y;w(m)) &= \delta(x-y)\,, \quad && x,y\in\omega \\ G(x,y;w(m)) &= 0\,, && x\in\partial\omega\,. \end{aligned}$$

Let $G(G_{w(m)}$, respectively) be the bounded linear operator on $L^2(\Omega)$ $(L^2(\omega)$, respectively) defined by

$$(Gf)(x) = \int_\Omega G(x,y)f(y)\,dy\,,$$
$$(G_{w(m)}g)(x) = \int_\omega G(x,y;w(m))g(y)\,dy\,,$$

respectively. Then, the eigenvalue problem of the Laplacian with respect to ω is transformed into the eigenvalue problem of $G_{w(m)}$. As a limit $m\to\infty$, we see that $\mu_j(w(m))+\lambda$ is approximated by the j-th eigenvalue of the Schrödinger operator with big potential $-\Delta+4\pi\alpha V(x)\rho m^{\beta-1}+\lambda$ in Ω under the Dirichlet condition. Here ρ is defined to be $\exp(\lambda^{1/2}\alpha/m)$ which is approximately 1 as m tends to infinity. Let A denote the Green operator of the above Schrödinger operator with big potneitial. We want to compare $G_{w(m)}$ and A. It should be remarked that the Green operator $G_{w(m)}$ and A act on different spaces $L^2(\omega)$ and $L^2(\Omega)$.

In order to relate $G_{w(m)}$ with A, we introduce the operators $H_{w(m)}$ and $\tilde{H}_{w(m)}$. The following integral kernel was introduced in [9]. We abbreviate $G(w_i, w_j)$ as G_{ij}. We put $m^*=(\log m)^2$, $\tau=4\pi\alpha\rho/m$.

$$h(x,y;w(m)) = G(x,y)-\tau\sum_{i=1}^{n} G(x,w_i)\,G(w_i,y)$$
$$+\sum_{s=1}^{m^*}(-\tau)^s\sum\nolimits_{(s)} G(x,w_{i_1})\,G_{i_1i_2}\cdots G_{i_{s-1}i_s}\,G(w_{i_s},y)\,.$$

Here the indices in $\sum_{(s)}$ run over all $i_1,\cdots,i_s$ satisfying $1\le i_1,\cdots,i_s\le n$ such that $i_\nu\neq i_\mu$ when $\nu\neq\mu$. The sum $\sum_{(s)}$ is called self-avoiding sum. A primitive form of this integral kernel was introducted in [8]. We put

$$(H_{w(m)}f)(x) = \int_\omega h(x,y;w(m))f(y)\,dy\,,\quad x\in\omega$$

and

$$(\tilde{H}_{w(m)}g)(x) = \int_\Omega h(x,y;w(m))g(y)\,dy\,,\quad x\in\Omega\,.$$

We compare $G_{w(m)}$ with $H_{w(m)}$, $H_{w(m)}$ with $\tilde{H}_{w(m)}$ and $\tilde{H}_{w(m)}$ with A.

The following Theorems 2, 3, $\cdots$, 7 are main results of this paper which are discussed in the later sections.

In Theorems 2$\sim$7 we assume that $V\equiv|\Omega|^{-1}$.

Theorem 2. *Fix $\beta\in[1,3)$. Let χ_ω be te characteristic function of ω. Fix an arbitrary $\varepsilon>0$. Then, there exists a constant T such that*

$$P(w(m)\in\Omega^n;\ w(m)\ \textit{satisfies}\ \mathcal{O}_1(m)\ \textit{and}\ (2.1))\ge 1-m^{-\varepsilon}$$

holds. Here

$$\|(H_{w(m)}-G_{w(m)})(\chi_\omega\cdot)\|_{\mathcal{L}(L^\infty(\Omega),L^2(\omega))}\le m^{5\varepsilon}\,m^{-(\beta+1)/2}\,. \tag{2.1}$$

REMARK: $\|W\|_{\mathcal{L}(X,Y)}$ denotes the norm of the bounded operator W from the Banach space X to Y. Multiplication operator by the function u is denoted by $u\cdot$.

Theorem 3. *Fix $\beta\in[1, 3)$. Fix an arbitrary $\varepsilon>0$. Then, there exists a constant T such that*

$$P(w(m)\in\Omega^n;\ w(m) \text{ satisfies } \mathcal{O}_1(m) \text{ and } (2.2))\geq 1-m^{-\varepsilon}$$

holds. Here

$$\|(\boldsymbol{H}_{w(m)}-\boldsymbol{G}_{w(m)})\,(\chi_w\cdot)\|_{\mathcal{L}(L^2(\Omega),L^2(\omega))}\leq m^{5\varepsilon}\, m^{(\beta-5)/4}\,. \tag{2.2}$$

It should be noticed that (2.2) can be replaced by

$$\|\boldsymbol{H}_{w(m)}-\boldsymbol{G}_{w(m)}\|_{\mathcal{L}(L^2(\omega),L^2(\omega))}\leq m^{5\varepsilon}\, m^{(\beta-4)/5}\,.$$

We need very delicate calculation to prove the following Theorem 4.

Theorem 4. *Fix $\beta\in[1, 3)$. Fix an arbitrary $\varepsilon>0$. Then, there exists a constant T such that*

$$P(w(m)\in\Omega^n;\ w(m) \text{ satisfies } \mathcal{O}_1(m) \text{ and } (2.3))\geq 1-m^{-\varepsilon}$$

holds. Here

$$\|\tilde{\boldsymbol{H}}_{w(m)}f\|_{2,\omega^C}\leq m^{3\varepsilon}\min(U_m\|f\|_\infty,\ \lambda^{3/4}\, U_m\|f\|_2)\,. \tag{2.3}$$

for any $f\in L^\infty(\Omega)$, where

$$U_m = m^{-(\beta+1)/2}\,.$$

IMPORTANT REMARK. It should be noticed that we can say the following:

$$P(w(m)\in\Omega^n;\ w(m) \text{ satisfies } \mathcal{O}_1(m) \text{ and } (2.4))\geq 1-m^{-\varepsilon}$$

for any $\varepsilon>0$. Here

$$\begin{aligned} &\|\chi_{w^C}\,\tilde{\boldsymbol{H}}_{w(m)}\|_{\mathcal{L}(L^2(\Omega),L^2(\Omega))}\leq m^{3\varepsilon}\,\lambda^{3/4}\, U_m\,,\\ &\|\chi_{\omega^C}\,\tilde{\boldsymbol{H}}_{w(m)}\|_{\mathcal{L}(L^\infty(\Omega),L^2(\Omega))}\leq m^{3\varepsilon}\, U_m\,. \end{aligned} \tag{2.4}$$

These are easily observed from our proof of Thoerem 4 in §7.

We make a comment on the work of Figari-Orlandi-Teta [4]. They examined a bound for $\|\boldsymbol{H}^{(*)}_{w(m)}f\|_{2,\omega^C}$ for fixed f. Here $\boldsymbol{H}^{(*)}_{w(m)}$ ($\boldsymbol{H}_{w(m)}$ in [4]) is a similar operator as $\tilde{\boldsymbol{H}}_{w(m)}$. They conclude in Theorem 1 of [4] that Theorem 1 of this paper holds for $\beta=1$. However, there is a gap in their proof (pp. 480–481 in [4]). It should be remarked that we can not derive spectral properties of the operator Y by considering Yf for any fixed f. We treat probabilistic problem. The exculsive small set in measure may depend on f. Thus, we need some expression of the form

$$P(Y)f \text{ satisfies } \cdots \text{ for any } \quad ||f||\leq 1) \to 1$$

to say about $||Y||$. However the author thinks that their paper includes very important development of study of spectral properties of random media, especially analysis concerning Theorems 5, 6 stands on their calculation.

Let $\boldsymbol{A}'$ denote the operator given by

$$\boldsymbol{A}' = \boldsymbol{G}+\sum_{s=1}^{m^*}(-4\pi\alpha\rho\, n/m)^s J_s\, \boldsymbol{G}(V\boldsymbol{G})^s .$$

Here $\rho=\exp(\lambda^{1/2}\alpha/m)$ and $J_s=(1-(1/n))\,(1-(2/n))\cdots(1-((s-1)/n))$. It is easy to see

$$(2.5) \qquad ||\boldsymbol{A}-\boldsymbol{A}'||_{\mathcal{L}(L^2(\Omega),L^2(\Omega))}\leq C(\log m)^4\, m^{-\beta}\,\lambda^{-1} .$$

Let $|||F|||$ denote the Hilbert-Schmidt norm of the operator F on $L^2(\Omega)$. We have the following

Theorem 5. *Fix* $\beta\in[1,3)$. *Fix an arbitrary* $\varepsilon>0$. *Then, there exists* $T\gg 1$ *such that*

$$P(w(m)\in\Omega^n;\ w(m) \textit{ satisfies } (2.6))\geq 1-m^{-\varepsilon}$$

holds. Here

$$(2.6) \qquad |||\tilde{\boldsymbol{H}}_{w(m)}-\boldsymbol{A}'|||\leq m^{2\varepsilon-(\beta/2)}\,\lambda^{1/2} .$$

REMARK. From (2.6) we get

$$(2.7) \qquad ||\tilde{\boldsymbol{H}}_{w(w)}-\boldsymbol{A}'||_{\mathcal{L}(L^2(\Omega),L^2(\Omega))}\leq m^{2\varepsilon-(\beta/2)}\,\lambda^{1/2} .$$

Theorem 6. *Fix* $\beta\in[1,3)$. *Fix an arbitrary* $\varepsilon>0$. *Fix* $f\in L^\infty(\Omega)$. *Then, there exists* $T\gg 1$ *such that*

$$P(w(m)\in\Omega^n;\ w(m) \textit{ satisfies } (2.8))\geq 1-m^{-\varepsilon}$$

holds. Here

$$(2.8) \qquad ||(\tilde{\boldsymbol{H}}_{w(m)}-\boldsymbol{A}')f||_{L^2(\Omega)}\leq m^{2\varepsilon-(\beta/2)}\min(\lambda^{1/2}||f||_2,\ \lambda^{-1/4}||f||_\infty) .$$

for any $f\in L^\infty(\Omega)$.

Theorem 7. ($L^\infty-L^2$ *resolvent remainder estimate*) *Fix* $\beta\in[1,3)$. *Fix an arbitrary* $\varepsilon>0$. *Then, there exists a constant* $T\gg 1$ *such that* $P(w(m)\in\Omega^n;\ w(m)$ *satisfies* $\mathcal{O}_1(m)$ *and* $(2.9))\geq 1-m^{-\varepsilon}$ *holds.*

$$(2.9) \qquad \begin{aligned}&||(\boldsymbol{G}_{w(m)}\,\chi_\omega-\chi_\omega\,\boldsymbol{A}'\,\chi_\omega)f||_{L^2(\omega)}\\ &\leq m^{6\varepsilon}(m^{-\beta/2}\,\lambda^{-1/4}+m^{-(\beta+1)/2})||f||_{\infty,\omega} \quad \textit{for any} \quad f\in L^\infty(\Omega) .\end{aligned}$$

REMARK: Note that $m^{(\beta-1)}$ (the right hand side of (2.9))$\to 0$ as $m\to\infty$, if we take $\lambda=Tm^{\beta-1}$. This means that in some sense

$$-\Delta_\omega-(-\Delta_\Omega+4\pi\alpha\rho\, m^{\beta-1}|\Omega|^{-1}) = O(m^{\beta-1-\delta})$$

for some $\delta>0$ when $\beta\in[1,3)$. This can be thought as an asymptotic remainder estimate.

3. Preliminary facts and lemmas which are used to prove Theorem 2

To prove Theorem 2 one need highly delicate calculation involving the Green function. We list fundamental lemmas which are also useful in the later section.

We put $\Phi_\theta(\lambda, x, y)=|x-y|^{-\theta}\exp(-\lambda^{1/2}|x-y|)$.

Lemma 3.1. *The term*

$$\left|\int_\Omega \Phi_\theta(\lambda, x, y)\, \Phi_\theta(\lambda, y, z)\, dy\right|$$

does not exceed $C\lambda^{-1/2}\,\Phi_0(\lambda/8, x, z)$ *when* $\theta=1$ *and* $C\Phi_1(\lambda/8, x, z)$ *when* $\theta=2$.

Lemma 3.2. *Assume that* $w(m)$ *satisfies* $\mathcal{O}_1(m)$. *If* $u\in C^\infty(\omega)\cap C^0(\bar\omega)$ *satisfies*

$$\begin{aligned}(-\Delta+\lambda)\,u(x) &= 0\,, \quad x\in\omega\\ u(x) &= 0\,, \quad x\in\gamma\cap\partial\omega\end{aligned}$$

and

$$\max_{\partial B_r\cap\partial\omega}\ |u(x)|\le M_r\,, \quad r=1, \cdots, n$$

Here B_r *is the ball of radius* α/m *with the center* w_r. *Here* M_r *is zero when* $\partial B_r\cap\partial\omega=\phi$. *Then,*

$$\|u\|_{L^2(\omega)}\le C(\alpha/m)\,\lambda^{-1/4}\Big(\sum_r^n M_r^2+\sum_{r\neq q}^n M_r\, M_q\, \Phi_0(\lambda/8, w_r, w_q)\Big)^{1/2}$$

holds for a constant C *independent of* $w(m)$.

Proof. By the Hopf maximum principle we have

$$|u(x)|\leqq C(\alpha/m)\sum_r \Phi_1(\lambda, x, w_r)\, M_r\,.$$

By Lemma 3.1 we get the desired result.

We put $\boldsymbol{Q}_{w(m)}=\boldsymbol{G}_{w(m)}-\boldsymbol{H}_{w(m)}$. Fix $f\in C^\infty(\Omega)$. Then, $u=\boldsymbol{Q}_{w(m)}(\chi_\omega f)(x)$ satisfies the assumptions in Lemma 3.2. Therefore, u is estimated if we know an upper bound for $M_r(r=1, \cdots, n)$. Notice that

$$-u(x)=\boldsymbol{H}_{w(m)}(\chi_\omega f)(x) \quad x\in\partial B_r\cap\partial\omega .$$

It is very useful to introduce the rearrangement of the Green function. For $s=0$, we put

$$(I_r^0 f)(x)=(\boldsymbol{G}f)(x)-\tau G(x, w_r)(\boldsymbol{G}f)(w_r) .$$

For $s\geqq 1$, we set $I_r^s f$ as the following term.

$$\begin{aligned}&\textstyle\sum'_{(s)} G(x, w_{i_1}) G_{i_1 i_2}\cdots G_{i_{s-1} i_s}(\boldsymbol{G}f)(w_{i_s})\\&-\tau \textstyle\sum'_{(s)} G(x, w_r) G_{r i_1} G_{i_1 i_2}\cdots G_{i_{1-1} i_s}(\boldsymbol{G}f)(w_{i_s}) .\end{aligned}$$

Here the indices $i_1, \cdots, i_s$ in $\sum'_{(s)}$ run over all $i_1, \cdots, i_s$ satisfying $i_\nu\neq i_\mu$ if $\nu\neq\mu$ and $i_\nu\neq r$ for $\nu=1, \cdots, s$. For $s\geq 2$ we set $\tilde{I}_r^s f$ as the following term.

$$\textstyle\sum''_{(s)} G(x, w_{i_1}) G_{i_1 i_2}\cdots G_{i_{s-1} i_s}(\boldsymbol{G}f)(w_{i_s}) .$$

Here the indices $i_1, \cdots, i_s$ in $\sum''_{(s)}$ run over all $i_1, \cdots, i_s$ satisfying $i_\nu\neq i_\mu$ if $\nu\neq\mu$ and $i_1\neq r$ and exactly one of i_ν $(\nu\geq 2)$ is equal to r. We have the rearrangement.

$$\begin{aligned}(\boldsymbol{H}_{w(m)} g)(x)&=\sum_{s=0}^{m^*}(-\tau)^s (I_r^s g)(x)\\&+\sum_{s=2}^{m^*}(-\tau)^s(\tilde{I}_r^s g)(x)+(-\tau)^{m^*}(Z_r^{m^*} g)(x) ,\end{aligned}$$

where

$$(Z_r^{m^*} g)(x)=\left(\textstyle\sum'_{(m^*)}+\sum''_{(m^*)}\right) G(x, w_{i_1})\cdots(\boldsymbol{G}g)(w_{i_{m^*}}) .$$

We put $I_r^s=\max\{|I_r^s f(x)|;\, x\in\partial B_r\cap\partial\omega\}$, $\tilde{I}_r^s=\max\{|\tilde{I}_r^s f(x)|;\, x\in\partial B_r\cap\partial\omega\}$ and $K_{rq}=\exp(-\lambda^{1/2}|w_r-w_q|/8)$. The following Lemma 3.3 is a fundamental tool to obtain Theorems 2,3. Hereafter we use the notations $||h||_{p,A}=||h||_{L^p(A)}$ and we abbreviate $||h||_{p,\Omega}$ as $||h||_p$.

Lemma 3.3. *Fix $f\in C^\infty(\Omega)$. Assume that $w(m)$ satisfies $\mathcal{O}_1(m)$. Then, there exists a constant C independent of m such that* (3.1) *holds.*

$$||\boldsymbol{Q}_{w(m)}(\chi_\omega f)||_{L^2(\omega)}\leq C((\log m)/m)\,\lambda^{-1/4}\Big(\sum_{k=1}^{8} R_k\Big)^{1/2} , \tag{3.1}$$

where

$$R_1=\sum_{s=0}^{m^*}\tau^{2s}\sum_{r=1}^{n}(I_r^s)^2 ,$$

$$R_2=\sum_{s=2}^{m^*}\tau^{2s}\sum_{r=1}^{n}(\tilde{I}_r^s)^2 ,$$

$$R_3=\sum_{s=0}^{m^*}\sum_{t=0}^{m^*}\tau^{s+t}\sum_{\substack{r,q\\ r\neq q}} I_r^s I_q^t K_{rq},$$

$$R_4=\sum_{s=0}^{m^*}\sum_{t=2}^{m^*}\tau^{s+t}\sum_{\substack{r,q\\ r\neq q}} I_r^s \tilde{I}_q^t K_{rq},$$

$$R_5=\sum_{s=2}^{m^*}\sum_{t=2}^{m^*}\tau^{s+t}\sum_{\substack{r,q\\ r\neq q}} \tilde{I}_r^s \tilde{I}_q^t K_{rp},$$

$$R_6=\tau^{2m^*}\sum_{r=1}^{n}(\mathcal{Z}_r f)^2 \quad (\mathcal{Z}_r f=\max_{\partial B_r\cap\partial\omega}|(Z_r^{m^*}(\chi_\omega f)(x)|),$$

$$R_7=\sum_{s=0}^{m^*}\tau^{s+m^*}\sum_{\substack{r,q\\ r\neq q}} I_r^s K_{rq}\,\mathcal{Z}_r f,$$

$$R_8=\sum_{s=2}^{m^*}\tau^{s+m^*}\sum_{\substack{r,q\\ r\neq q}} \tilde{I}_r^s K_{rq}\,\mathcal{Z}_r f.$$

Proof. Note that

$$\begin{aligned}\sum_{r=1}^{n} M_r^2 &\leq \sum_{r=1}^{n}\Big(\sum_s (I_r^s+\tilde{I}_r^s)\,\tau^s\Big)^2\\ &\leq Cm^*\sum_s \tau^{2s}\sum_{r=1}^{n}((I_r^s)^2+(\tilde{I}_r^s)^2).\end{aligned}$$

By Lemma 3.2 we get Lemma 3.3.

We put $S(x,y)=G(x,y)-(1/4\pi)\exp(-\lambda^{1/2}|x-y|)/|x-y|$. It is well known that $S(x,y)\in C^\infty(\Omega\times\Omega)$. Notice that $S(x,x)$ has singularity on $\partial\Omega$.

The following Lemmas 3.4, 3.5, 3.6 are crucial to obtain a bound for $\sum (I_r^s)^2$ etc.

Lemma 3.4. *Fix $\beta\in[1,3)$. Assume that $w(m)$ satisfies $\mathcal{O}_1(m)$. Then, there exists a constant C independent of m such that* (3.2) *and* (3.3) *hold.*

$$\max_{x\in\partial B_r\cap\partial\omega}|G(x,w_i)-G(w_r,w_i)|\leqq C(\alpha/m)\,\Phi_2(\lambda/4,w_i,w_r) \tag{3.2}$$

$$\max_{x\in\partial B_r\cap\partial\omega}|S(x,w_r)\,G(w_r,w_i)|\leqq C(\log m)^2\,\Phi_2(\lambda/4,w_i,w_r). \tag{3.3}$$

Lemma 3.5. *We can replace w_i in* (3.2), (3.3) *by any $y\in\Omega\backslash\bar{B}_r$.*

Lemma 3.6. *Fix $\xi\in(0,1]$. Under the same assumptions as in Lemma* 3.4, *the right hand sides of* (3.2) *and* (3.3) *are estimated by $Cm^{-\xi}\Phi_{1+\xi}(\lambda/4,w_i,w_r)$ and $Cm^{1-\xi}(\log m)^2\,\Phi_{1+\xi}(\lambda/4,w_i,w_r)$ respectively.*

Proof of Lemmas 3.4, 3.5, 3.6 are given in Appendix of this paper.

The term $G_{i_1i_2}\cdots G_{i_{s-1}i_s}$ is included in I_r^s. If we want to get a bound for $\sum (I_r^s)^2$, it is very helpful to estimate $E(G_{i_1i_2}\cdots G_{i_{s-1}i_s}G_{j_1j_2}\cdots G_{j_{t-1}j_t})$. Here E denotes the expectation on Ω^n. Let $G_{i_1i_2}\cdots G_{i_{s-1}i_s}$ be denoted by $G_{I(s)}$ for $s\geq 2$ and $G_{j_1j_2}\cdots G_{j_{t-1}j_t}$ be denoted by $G_{J(t)}$ for $t\geq 2$. We abbreviate $G_{I(s)}$ as G_I and

$G_{J(t)}$ as G_J, if there occurs no fear of confusion. We put $G_{IJ}=G_I\,G_J$.

DEFINITION: Assume that $I=I(s)\ni i_1, \cdots, i_s$ ($J=J(t)\ni j_1, \cdots, j_t$, respectively) is self-avoiding. If there are exactly q-couples of $(h(k), p(k))$ $(k=1, \cdots, q)$ such that $i_{h(k)}=j_{p(k)}$, we say that $(i_1, \cdots, i_s)$ and $(j_1, \cdots, j_t)$ have q-intersections. If i_μ has not a partner such that $i_\mu=j_\nu$, we say that i_μ is single.

Lemma 3.7. *Assume that $s, t\geq 2$. Assume that I and J is of q-intersections. Then,*

$$E(G_{IJ})\leq C'(C\lambda^{-1})^{s+t-(3/2)q-(1/2)} \quad (q\geq 1)$$

and

$$E(G_{IJ})\leq C'(C\lambda^{-1})^{s+t-2} \quad (q=0)$$

hold for constants C, C' independent of m, λ.

Proof. A part of the above result is given and proved in [11]. For the sake of completeness we give a proof.

We assume that $i_{h(k)}=j_{p(k)}$ for $k=1, \cdots, q$. Here $h(1), h(2), \cdots$ is a sequence satisfying $h(k)<h(k+1)$. Then, there is a permutation σ on $(1, \cdots, q)$ such that $p(\sigma(k))<p(\sigma(k+1))$, $k=1, \cdots, q-1$. For the sake of simplicity we write $p(\sigma(k))$ as $r(k)$.

We put $G_1(x, y)=G(x, y)$ and

$$G_{k+1}(x, y)=\int_\Omega G_k(x, z)\,V(z)\,G(z, y)\,dz, \quad k=1, 2, \cdots.$$

When $q\geqq 2$, we define the contracted term G_I^{co} (G_J^{co}, respectively) by

$$G_I^{co}=\prod_{k=1}^{q-1} G_{h(k+1)-h(k)}\,(w_{i_{h(k)}}, w_{i_{h(k+1)}})$$

$$G_J^{co}=\prod_{k=1}^{q-1} G_{r(k+1)-r(k)}\,(w_{j_{r(k)}}, w_{j_{r(k+1)}})\,.$$

Then, $E(G_{IJ})$ does not exceed

$$E(G_I^{co}\,G_J^{co})\times(C\lambda^{-1})^{(h(1)-1)+(s-h(q))+(r(1)-1)+(t-r(q))} \tag{3.3}$$

observing

$$\int_\Omega G(w, z)\,V(w)\,dw\leqq C\lambda^{-1}\,.$$

We want to estimate $E(G_I^{co}\,G_J^{co})$. We have

$$E((G_I^{co})^2)\leqq C'(C\lambda^{-1})^{2(h(q)-h(1))-(3/2)(q-1)} \tag{3.4}$$

by using Lemma 16 in [9]. We recall it.

For $h \geq 1$,

$$\max_{y \in \bar{\Omega}} \int_{\Omega} G_h(x, y)^2 V(x)\, dx \leqq C'(C\lambda^{-1})^{2h-(3/2)}. \tag{3.5}$$

By (3.3) and (3.4) we get the desired result for $q \geqq 2$.

The cases $q=0$, 1 are easy to treat. We get Lemma 3.7.

Lemma 3.8. *Assume that* $s, t \geqq 2$. *Assume that* I *and* J *have* q*-intersections. Moreover, we assume that one of the following conditions* (1), (2) *holds:*
(1) $i_1 \neq j_1$ *and* i_1, j_1 *are both single.*
(2) $i_1 \neq j_1$ *and* i_1 *is single and* j_1 *is not single or* i_1 *is not single and* j_1 *is single.*
Then,

$$E(G_{i_1 j_1} G_{IJ}) \leqq (C\lambda^{-1})^{s+t+1-(3/2)(q+1)-(1/2)}$$

holds.

Proof. We put $G^*_{IJ} = G_{i_1 j_1} G_{IJ}$. First we treat the case (1). Then, G^*_{IJ} has the form $G_{i_0 i_1} G_{IJ}$ with $i_0 = j_1$. Therefore, G^*_{IJ} can be thought as the form $G_{I(s+1)} G_{J(t)}$. Here $I(s+1)$ and $J(t)$ is of $(q+1)$-interssctions. Then, we have the desired result by Lemma 3.7. We treat the case (2). We assume that j_1 is single. Then, G^*_{IJ} has the from $G_{i_0 i_1} G_{IJ}$ with $i_0 = j_1$. The proof reduces to the proof of the case (1). We complete our proof of Lemma 3.8.

We give general form of Lemmas 3.7, 3.8. We consider the term

$$\Phi_{IJ} = \prod_{k=1}^{s-1} \Phi_{\theta_k}(w_{i_k}, w_{i_{k+1}}) \prod_{k=1}^{t-1} \Phi_{\vartheta_k}(w_{j_k}, w_{j_{k+1}}), \tag{3.6}$$

where $0 \leq \theta_k$, $\vartheta_k < 3/2$, $k=1, \cdots, s$. Let us consider the following sum

$$E(\sum_{I,J} \Phi_{IJ}),$$

where the indices $I(J$, respectively) run over all self-avoiding sum. If the fixed indices I and J have q-intersections, we write it as $\#(I \cap J) = q$.

Lemma 3.9. (*Easy going Lemma*) *Fix* $\beta \in [1, 3)$. *Under the above assumption on* (3.6), *we have*

$$\begin{aligned} &\sum_{q \geq 1} E(\sum_{\#(I \cap J)=q} \Phi_{IJ}) \\ &\ll \sum_{\#(I \cap J)=0} E(\Phi_{IJ}) \leq C(m^\beta / C' \lambda)^{s+t} \lambda^{\rho(s,t)}, \end{aligned}$$

when $\lambda = Tm^{\beta-1}$ *for sufficiently large fixed* T *and* $m \to \infty$, *where* $\rho(s, t) = 3 + ((\sum_{k=1}^{r-1} \theta_k + \sum_{k=1}^{t-1} \vartheta_k)/2) - (1/2)(s+t)$.

Lemma 3.10. (*Easy going Lemma II*). *Fix* $\beta \in [1, 3)$. *Under the above* *assumption on* (3.6), *we have*

$$\sum_{q\geq 1} E\Big(\sum_{\#(I\cap J)=q} \Phi_0(w_{i_1}, w_{j_1})\,\Phi_{IJ}\Big)$$
$$\ll \sum_{\#(I\cap J)=0} E(\Phi_0(w_{i_1}, w_{j_1})\,\Phi_{IJ})$$
$$\leq C(m^\beta/C'\lambda)^{s+t}\,\lambda^{p(s,t)-(3/2)}\,.$$

Proof of Lemma 3.9. First we consider the case where I and J have q-intersections ($q\geq 2$).
We assume that $i_{h(k)}=j_{p(k)}$ for $k=1,\cdots,q$. We define $r(k)$ as in the proof of Lemma 3.7. We abbreviate w_i as i and $V(w_i)\,dw_i$ as $\tilde{d}w_i$. We put

$$\Phi_{(\theta_1,\theta_2)}(i_1, i_3)=\int_\Omega \Phi_{\theta_1}(i_1, i_2)\,\Phi_{\theta_2}(i_2, i_3)\,\tilde{d}w_{i_2},\ \Phi_{(\theta_1,\theta_2,\theta_3)}(i_1, i_4)$$
$$=\int_\Omega \Phi_{(\theta_1,\theta_2)}(i_1, i_3)\,\Phi_{\theta_3}(i_3, i_4)\,\tilde{d}w_{i_3},\ \cdots\,.$$

The contracted term is defined by

$$\Phi_I^{co}=\prod_{k=1}^{q-1}\Phi_{(\theta_{h(k)},\cdots,\theta_{h(k+1)-1})}(i_{h(k)}, i_{h(k+1)})$$

$$\Phi_J^{co}=\prod_{k=1}^{q-1}\Phi_{(\vartheta_{r(1)},\cdots,\vartheta_{r(k+1)-1})}(j_{r(k)}, j_{r(k+1)})\,.$$

The inequalities

$$\max_{x\in\Omega}\left|\int_\Omega \Phi_\theta(x, y)\,\tilde{d}y\right|\leq C'(C\lambda^{-1})^{L(\theta)} \tag{3.7}$$

and

$$\max_{x\in\Omega}\left|\int_\Omega \Phi_\theta(x, y)^2\,\tilde{d}y\right|^{1/2}\leq C'(C\lambda^{-1})^{L(2\theta)/2} \tag{3.8}$$

are easy to get, where $L(x)=-(x/2)+(3/2)$. We have $E(\Phi_{IJ})\leq C'(C\lambda^{-1})^K\,E(\Phi_I^{co}\,\Phi_J^{co})$,

$$K=\sum_{k=1}^{h(1)-1}L(\theta_k)+\sum_{k=h(q)+1}^{s-1}L(\theta_k)$$
$$+\sum_{k=1}^{r(1)-1}L(\vartheta_k)+\sum_{k=r(q)+1}^{t-1}L(\vartheta_k)\,.$$

We see that $E((\Phi_I^{co})^2)^{1/2}$ does not exceed

$$\prod_{k=1}^{q-1}\mathcal{M}_k\,,$$

where

$$\mathcal{M}_k=\max_y\|\Phi_{(\theta_{h(k)},\cdots,\theta_{h(k+1)-1})}(y, \cdot)\|_{L^2(\Omega=\cdot)}\,.$$

The Hausdorff-Young inequality implies

$$\mathcal{M}_k \leq \Big(\prod_{\nu=h(k)}^{h(k+1)-2} \max_y \|\Phi_{\theta_\nu}(y,\,\cdot)\|_{L^1(\cdot)}\Big) \max_y \|\Phi_{\theta_{h(k+1)-1}}(y,\,\cdot)\|_{L^2(\cdot)} \tag{3.9}$$
$$\leq C''(C\lambda^{-1})^H ,$$

where

$$H = \sum_{\nu=h(k)}^{h(k+1)-2} L(\theta_\nu) + (L(2\theta_{h(k+1)-1})/2) .$$

Summing up these facts we get

$$E((\Phi_I^{co})^2)^{1/2} \leq C(C\lambda^{-1})^N ,$$

where

$$N = (h(q)-h(1))\,(3/2) - ((\sum_{\nu=h(1)}^{h(q)-1} \theta_\nu)/2) - (3/4)\,(q-1) .$$

Therefore, we have

$$E(\Phi_{IJ}) \leq C''(C\lambda^{-1})^F , \tag{3.10}$$

where

$$F = (3/2)\,(s+t) - 3 - (3/2)\,(q-1) - (S(\theta,\vartheta)/2) ,$$
$$S(\theta,\vartheta) = \sum_{k=1}^{s-1} \theta_k + \sum_{k=1}^{t-1} \vartheta_k .$$

The case $q=1$ is easy to treat. We get (3.10) for $q \geq 1$. For $q=0$ we have $E(\Phi_{IJ}) \leq C'(C\lambda^{-1})^P$, where

$$P = \sum_{k=1}^{s-1} L(\theta_k) + \sum_{k=1}^{t-1} L(\vartheta_k) .$$

Thus, it does not exceed

$$C''(C\lambda^{-1})^{(3/2)(s+t)-3-(S(\theta,\vartheta)/2)} .$$

Now we want to show Lemma 3.9. The second inequality in Lemma 3.9 is prvoed. We have

$$E(\sum_{\#(I\cap J)=q} \Phi_{IJ}) \leq (m^\beta)^{s+t-q}\,(3.10)$$

for $q \geq 1$. Since $\lambda^{(3/2)(q-1)}\, m^{-\beta q} \ll 1$, if $\lambda = Tm^{\beta-1}$ and $m \to \infty$, we get the desired result.

Proof of Lemma 3.10. If $q \gtrsim 1$, then

$$E(\Phi_0(w_{i_1}, w_{j_1})\, \Phi_{IJ}) \leq CE(\Phi_{IJ}) .$$

Therefore,

$$E(\sum_{\#(I\cap J)=q}\Phi_0\,\Phi_{IJ})\leq(m^{\beta})^{s+t-q}\,(3.10)\,.$$

On the other hand,

$$\sum_{\#(I\cap J)=0}E(\Phi_0\,\Phi_{IJ})\sim(m^{\beta})^{s+t}(C\lambda)^{-3/2}\,C''(C\lambda^{-1})^{p}\,.$$

Thus, we get the desired result.

As a corollary of Lemma 3.10 we get the following:

Lemma 3.11. *Consider the following sum over the indices $I(s), J(t)$. Then, there exists T in λ such that*

$$\begin{aligned}&P(\sum_{I,J}\Phi_0(\lambda, w_{i_1}, w_{j_1})\,G_{IJ}\leq m^{2\varepsilon}(m^{\beta}/C\lambda)^{s+t}\,\lambda^{1/2})\\ &\geq 1-m^{-\varepsilon}\end{aligned}$$

holds for any fixed $\varepsilon>0$.

Lemma 3.12. *Fix $\theta>-2$. We assume that $w(m)$ satisfies $\mathcal{O}_1(m)$. Then, there exists a constant C independent of m such that*

$$\sup_{x\in\omega}\sum_{r}\Phi_{1-\theta}(\lambda, w_r, x)\leqq C(\log m)^2(\lambda^{-(2+\theta)/2}\,m^{\beta}+m^{1-\theta}) \tag{3.11}$$

holds. When $\theta=1$, we can replace ω in (3.11) by Ω.

Proof. See [11; Lemmas 1,2].

4. Facts which imply Theorem 2

We state propositions about estimates of $\sum(I_r^s)^2$, $\sum I_r^s\,I_q^s\,K_{rq}$, etc. Throughout this section we assume that $\beta\in[1,3)$. Propositions 4.1~4.11 are proved in the section 5. The very important remark is given below Propostion 4.11.

We put $\chi_\omega f=f_\omega$.

Proposition 4.1. *Fix $f\in C^\infty(\Omega)$. Assume that $w(m)$ satisfies $\mathcal{O}_1(m)$. Fix $\varepsilon>0$. Then, there exists a constant C independent of f, m such that*

$$\sum_r (I_r^o)^2\leqq C\,m^{\varepsilon}(m^{-1}+m^{\beta-2}\,\lambda^{-1})\,||f||_{2,\omega}^2$$

holds when we take sufficiently large T.

Proposition 4.2. *Fix $f\in C^\infty(\Omega)$. Fix $s\geq 1$. Fix $\varepsilon>0$. Then, there exists $T\gg 1$ and $C>0$ such that*

$$P(\sum_r (I_r^s)^2\leq m^{2\varepsilon}(m^{\beta}/C\lambda)^{2s}\,q_{m,\lambda}||Gf_\omega||_\infty^2)\geqq 1-m^{-\varepsilon}$$

holds, where

$$q_{m,\lambda}=\lambda^2 m^{-1}+m^{\beta-2}\lambda+m^{-\beta-1}\lambda^{7/2}.$$

Proposition 4.3. *We fix $f\in C^\infty(\Omega)$. Fix $\varepsilon>0$. Put*

$$\sum_{\substack{r,q\\ r\neq q}} I_r^o I_q^o K_{rq}=\mathcal{L}_1.$$

Then, there exists $T\gg 1$ and $C>0$ such that

$$P(\mathcal{L}_1\leq C\, m^{2\varepsilon+2\beta-2}\lambda^{-5/2}\,||f||^2_{\infty,\omega})\geq 1-m^{-\varepsilon}$$

holds.

Proposition 4.4. *We fix $f\in C^\infty(\Omega)$. We put*

$$\sum_{\substack{r,q\\ r\neq q}} I_r^o I_q^s K_{rq}=\mathcal{L}_2$$

for $s\geq 1$. Fix any $\varepsilon>0$. Then, there exists $T\gg 1$ and $C>0$ such that

$$P(\mathcal{L}_2\leq m^{2\varepsilon}(m^\beta/C\lambda)^s\, t_{m,\lambda}\,||Gf_\omega||_\infty ||f||_{\infty,\omega})\geq 1-m^{-\varepsilon}$$

holds, where

$$t_{m,\lambda}=m^{2\beta-2}\lambda^{-3/2}+m^{\beta-2}.$$

Proposition 4.5. *We fix $f\in C^\infty(\Omega)$. We put*

$$\sum_{\substack{r,q\\ r\neq q}} I_r^s I_q^t K_{rq}=\mathcal{L}_3$$

for $s, t\geq 1$. Fix any $\varepsilon>0$. Then, there exists $T\gg 1$ and $C>0$ such that

$$P(\mathcal{L}_3\leq m^{4\varepsilon}(m^\beta/C\lambda)^{s+t}\, u_{m,\lambda}\,||Gf_\omega||^2_\infty)\geq 1-m^{-\varepsilon}$$

holds, where

$$u_{m,\lambda}=m^{2\beta-2}\lambda^{-1/2}+m^{\beta-2}\lambda+m^{-2}\lambda^{5/2}+m^{-1}\lambda^2.$$

Proposition 4.6. *We fix $f\in C^\infty(\Omega)$. Fix $\varepsilon>0$. Then, there exists $T\gg 1$ and $C>0$ such that*

$$P(\sum_r (\tilde{I}_r^s)^2\leq m^{2\varepsilon}(m^\beta/C'\lambda)^{2s}\, m^{-\beta}\lambda^3\,||Gf_\omega||^2_\infty)\geq 1-m^{-\varepsilon}$$

holds.

Proposition 4.7. *We fix $f\in C^\infty(\Omega)$. Fix $\varepsilon>0$. Then, there exist $T\gg 1$ and $C>0$ such that*

$$P(|\sum_{\substack{r,q\\ r\neq q}} \tilde{I}_r^s \tilde{I}_q^t K_{rq}|\leq m^{2\varepsilon}(m^\beta/C'\lambda)^{s+t}\, s_{m,\lambda}\,||Gf_\omega||^2_\infty)\geq 1-m^{-\varepsilon}$$

holds for $s, t\geq 2$, where

$$s_{m,\lambda}=\lambda^4 m^{1-2\beta}+\lambda^{9/2}m^{-2\beta}+\lambda^{3/2}+\lambda^3 m^{-\beta}.$$

Proposition 4.8. *We fix $f\in C^\infty(\Omega)$. Fix $\varepsilon>0$. Then, there exist $T\gg 1$ and $C>0$ such that*

$$P(|\sum_{\substack{r,q\\ r\neq q}} I_r^o \tilde{I}_q^t K_{rq}| \leq m^{2\varepsilon}(m^\beta/C'\lambda)^t m^{-1}\lambda^{-1/2}(m^\beta+\lambda^{3/2})$$
$$\times \|f\|_{\infty,\omega}\|Gf_\omega\|_\infty)\geq 1-m^{-\varepsilon}$$

holds.

Proposition 4.9. *We fix $f\in C^\infty(\Omega)$. Fix $\varepsilon>0$. Then, there exists $T\gg 1$ and $C>0$ such that*

$$P(|\sum_{\substack{r,q\\ r\neq q}} I_r^s \tilde{I}_q^t K_{rq}| \leq m^{2\varepsilon}(m^\beta/C'\lambda)^{s+t} r_{m,\lambda} \|Gf_\omega\|_\infty^2)\geq 1-m^{-\varepsilon}$$

holds for $s\geq 1$, $t\geq 2$, where

$$r_{m,\lambda}=\lambda^{1/2}m^{\beta-1}+\lambda^3 m^{-\beta}+\lambda^{7/2}m^{-\beta-1}.$$

Proposition 4.10. *We fix $f\in C^\infty(\Omega)$. If we take sufficiently large T, then*

$$P(\tau^{m^*}\mathcal{Z}_r f\leq 2^{-(\log m)^2}\|f\|_{\infty,\omega})\geq 1-m^{-\varepsilon}$$

holds for any $\varepsilon>0$.

As a consequence of Proposition 4.10 we get the following.

Proposition 4. 11. *If we take sufficiently large T, then*

$$P(|R_6|+|R_7|+|R_8|\leq C' 2^{-(\log m)^2}\|f\|_{\infty,\omega}^2)\geq 1-m^{-\varepsilon}$$

holds for any $\varepsilon>0$.

IMPORTANT REMARK. In each Propositions 4.1～4.11 we said that for fixed f

(4.1) $$P(\text{an event for the } f \text{ holds})\geq 1-m^{-\varepsilon}.$$

Observing the proofs of the above Propositions 4.1～4.11 we know that

$$P(\text{the same event in (4.1) holds for any } f\in X\geq 1-m^{-\varepsilon}$$

is valid for some set X. We can take X in Propositions 4.1, 4.5, 4.6, 4.7, 4.9 as $L^2(\Omega)$. We can take X in Propostions 4.1～4.11 as $L^\infty(\Omega)$. The reason why we can take X is easy to explain. The reader may find that f appeares as in the form $Gf(w_i)$ and so on.

Proof of Theorem 2. Note that $(\log m)^2=m^{2(\log\log m)/\log m}\leq m^\varepsilon$ for any $\varepsilon>0$ as $m\to\infty$. Notice that s, t in Propositions 4.1～4.11 run over $[1, (\log m)^2]$. Therefore, we get

$$P(\|(H_{w(m)}-G_{w(m)})(\chi_\omega\cdot)\|_{\mathcal{L}(L^\infty(\Omega),L^2(\Omega))}\leq m^{5\varepsilon}m^{-(\beta+1)/2})\geq 1-m^{-\varepsilon}$$

from Lemma 3.3, Propositions 4.1~4.11 and the important remark below Proposition 4.11. Here we used $\|Gf_\omega\|_\infty \leq C\lambda^{-1}\|f\|_{\infty,\omega}$.

5. Proof of Propositions 4.1~4.11

The following situation frequently happen in this section. We assume that $w(m)$ satisfies $\mathcal{O}_1(m)$ and we take an expectation of a random variable $\mathcal{R}$ over Ω^n. Then, we get an inequality of the type

$$P(|\mathcal{R}| \leq E(|\mathcal{R}|)\, m^\varepsilon) \geq 1 - m^{-\varepsilon}.$$

This abuse of calculation which does not touch with conditional probability and conditional expectation with respect to $\mathcal{O}_1(m) \subset \Omega^n$ can be justifed observing Lemma 2.1.

Proof of Proposition 4.1. Fix $f \in C^\infty(\Omega)$. Put $f_\omega = \chi_\omega f$. Recall that $I_r^o(\chi_\omega f)(x) = J_r^1 + J_r^2$, where

$$J_r^1 = Gf_\omega(x) - Gf_\omega(w_r)$$
$$J_r^2 = -\tau S(x, w_r)\,(Gf_\omega)(w_r).$$

Let B_r^* be the ball of radius $2\alpha/m$ with the center w_r. Put $D_r = (B_r^*)^C \cap \omega$ and $\Phi_2(\lambda/8, w_r, y) = \Phi(r, y)$. Then,

$$\begin{aligned}(5.1)\qquad & \sum_r \max_{x \in \partial B_r} (J_r^1)^2 \\ & \leq 4 \sum_r \max_{x \in \bar{B}_r} |G(\chi_{B_r^*} f)(x)|^2 \\ & + C' m^{-2} \sum_r \Big(\int_{D_r} \Phi(r, y)\, |f_\omega(y)|\, dy\Big)^2\end{aligned}$$

observing $|G(x, y) - G(w_r, y)| \leq C m^{-1} \Phi(r, y)$, $y \in D_r$. We have

$$\begin{aligned}(5.2)_r\qquad & \max_{x \in \bar{B}_r} \Big|\int_{B_r^*} G(x, y) f_\omega(y)\, dy\Big|^2 \\ & \leq C m^{-1} \Big|\int_\Omega f_\omega(y)^2 \chi_{B_r^*}(y)\, dy\Big|.\end{aligned}$$

Here we have used

$$\max_{x \in \bar{B}_r} \int_{B_r^*} G(x, y)^2\, dy \leq C m^{-1}.$$

By $\mathcal{O}_1(m)$ we have

$$\sum_r \chi_{B_r^*}(y) \leq C (\log m)^2.$$

Therefore,

$$(5.3)\qquad \sum_r (5.2)_r \leq C m^{-1} (\log m)^2 \|f_\omega\|_2^2.$$

We want to estimate the second term in the right hand side of (5.1). It does not exceed

$$C m^{-2} \sum_r T_\xi \int_{D_r} \Phi_{1+\xi}(\lambda/8, w_r, y) f_\omega(y)^2 \, dy , \tag{5.4}$$

$$T_\xi = \int_{D_r} \Phi_{3-\xi}(\lambda/8, w_r, y) \, dy < C_\xi < +\infty$$

for any $\xi>0$. Notice that $|y-w_r|\geq 2\alpha/m$ for $y\in D_r$. Therefore

$$(5.4)\leq C m^{-2}(\max_{y\in\omega} \sum_r \Phi_{1+\xi}(\lambda/8, w_r, y)) \, ||f||^2_{2,\omega} . \tag{5.5}$$

By the above facts and Lemma 3.9 we get

$$(5.1)\leq C m^{-2+\varepsilon}(\lambda^{-1} m^\beta + m) \, ||f||^2_{2,\omega} \tag{5.6}$$

for any $\varepsilon>0$.

Second we have

$$\sum_r \max_{x\in\partial B_r} (J_r^2)^2 \leq C m^{-2}(\log m)^4 \sum_r U_r , \tag{5.7}$$

where

$$U_r = \max_{x\in\partial B_r} S(x, w_r)^2 (G(\chi_{B_r^*} f_\omega))(w_r)^2 + \left(\int_{D_r} \Phi(r, y) f_\omega(y) \, dy\right)^2$$

by using Lemmas 3.4, 3.5. Since $|x-w_r|=\alpha/m$, we have $|S(x, w_r)|\leq C(m/\alpha)$ by the property of the Green function $G(x, y)$. We get a similar estimate for (5.7). We complete our proof of Proposition 4.1.

Proof of Proposition 4.2. (1) The case $s=1$: First we treat the case $s=1$. We introduce delicate techniques of estimating $\sum (I_r^1)^2$. Namely, we use both T_{ri}, S_{ri} in estimation. Hereafter we put

$$\mathcal{F} = ||Gf_\omega||_\infty .$$

And we put

$$T_{ri} = (\log m)^2 m^{-1} \Phi(r, w_i) \mathcal{F}$$
$$S_{ri} = (\log m)^2 m^{-(1/2)+\xi} \Phi_{(3/2)-\xi}(\lambda/8, w_r, w_i) \mathcal{F}$$

($\xi>0$). Then,

$$|I_r^1| \leq \sum_{\substack{i=1 \\ i\neq r}}^{n} \mathrm{Min}\,(T_{ri}, S_{ri})$$

by Lemma 3.6. Therefore

$$\sum_r (I_r^1)^2 \leq \sum_{\substack{r,i \\ r\neq i}} S_{ri}^2 + \sum_{\substack{\neq r i, j\neq i \\ r\neq j}} T_{ri} T_{rj} .$$

We have

$$P\Big(\sum_{r,i} S_{ri}^2 \leq m^{\varepsilon} \sum_{r,i} E(S_{ri}^2)\Big) \geq 1 - m^{-\varepsilon}. \tag{5.8}$$

We know that

$$E(S_{ri}^2) \leq C m^{-(1-2\xi)} (\log m)^4 \mathcal{F}^2,$$
$$P\Big(\sum_{r,i,j} T_{ri} T_{rj} \leq m^{\varepsilon} \sum_{r,i,j} E(T_{ri} T_{rj})\Big)$$
$$\geq 1 - m^{-\varepsilon}.$$

We know that

$$\sum_{r,i,j} E(T_{ri} T_{rj}) \leq C (\log m)^4 m^{3\beta-2} \lambda^{-1} \mathcal{F}^2$$

observing $i \neq j$. We complete our proof of Proposition 4.2 in the case $s=1$.
(2) The case $s \geq 2$: We put

$$T_{rI} = T_{ri_1} G_{I(s)}, \quad S_{rI} = S_{ri_1} G_{I(s)}.$$

We have

$$\sum_r (I_r^s)^2 \leq K_1 + K_2 + K_3,$$

where

$$K_1 = \sum_{\substack{r,I,J \\ i_1 = j_1}} S_{ri_1}^2 G_{I(s)} G_{J(s)}$$
$$K_2 = \sum_{r,I,J} T_{ri_1} T_{rj_1} G_{I(s)} G_{J(s)},$$

where the indices in K_2 run over r, I, J satisfying conditions $i_1 \neq j_1$ and (1) or (2) in Lemma 3.8 with respect to I, J. And

$$K_3 = \sum_{r,I,J} S_{ri_1} S_{rj_1} G_{IJ},$$

where the indices in K_3 run over all r, I, J satisfying $i_1 \neq j_1$ and r, I, J which do not satisfy (1) and (2) in Lemma 3.8.

We have

$$\int_{\Omega} S_{ri_1}^2 V(w_r)\, dw_r \leq C (\log m)^4 m^{-(1-2\xi)} \mathcal{F}^2.$$

Thus, $E(K_1) \leq C m^{\beta} (\log m)^4 m^{-(1-2\xi)} \big(\sum_{I,J} E(G_{IJ})\big) \mathcal{F}^2$. Every term in K_1 has the property that its index satisfies $i_1 = j_1$. Thus,

$$\sum_{I,J} E(G_{IJ}) \leq \sum_{q \geq 1} (C\lambda^{-1})^{2s-(3/2)q-(1/2)} (m^{\beta})^{2s-q}.$$

Note that $\beta \in [1, 3)$. If we take sufficiently large T, then the series converges and we get

$$\sum_{I,J} E(G_{IJ}) \leq C''(m^\beta)^{2s-1}(C\lambda^{-1})^{2s-2}$$

holds.

Summing up these facts we get

$$P(K_1 \leq C' m^{\beta-1+2\varepsilon}(m^\beta/C\lambda)^{2s} \lambda^2 m^{-\beta} \mathcal{F}^2) \geq 1-m^{-\varepsilon}. \tag{5.9}$$

Next we want to show

$$P(K_2 \leq m^{2\varepsilon+\beta-2} \lambda(m^\beta/C\lambda)^{2s} \mathcal{F}^2) \geq 1-m^{-\varepsilon}.$$

By Lemma 3.1 we have

$$\int_\Omega T_{ri_1} T_{rj_1} V(w_r)\, dw_r \leq C(\log m)^4 m^{-2} \Phi_1(\lambda/8, w_i, w_j) \mathcal{F}^2 .$$

Therefore,

$$E(K_2) \leq C(\log m)^4 m^{\beta-2} \big(\sum_{I,J} \Phi_1 G_{I(s)} G_{J(s)}\big) \mathcal{F}^2 .$$

We see that the term $\Phi_1 G_I G_J$ has a similar form as $G_{i_1j_1} G_{IJ}$. Thus, $E(\Phi_1 G_I G_J) \leq (C\lambda^{-1})^{2s+1-(3/2)(q+1)-(1/2)}$ holds. Here $q \geq 0$. Thus, $E(K_2) \leq C'm^{\beta-2}$ $(\log m)^4 \lambda(m^\beta/C\lambda)^{2s} \mathcal{F}^2$.

We want to show

$$P(K_3 \leq C(m^\beta/C'\lambda)^{2s} m^{2\varepsilon-\beta-1} \lambda^{7/2} \mathcal{F}^2) \geq 1-m^{-\varepsilon}. \tag{5.10}$$

We have $E(S_{ri_1} S_{rj_1}) \leq C(\log m)^4 m^{-(1-2\xi)} \mathcal{F}^2$. Therefore,

$$E(K_3) \leq (\log m)^4 m^{-(1-2\xi)+\beta} \big(\sum_{I,J} E(G_{IJ})\big) \mathcal{F}^2. \tag{5.11}$$

Here the indices I and J in K_3 have at least 2-intersections. Then, we have

$$\sum_{I,J} E(G_{IJ}) \text{ in } (5.11) \leq C'(m^\beta)^{2s-2} (C\lambda^{-1})^{2s-(7/2)} ,$$

if we take sufficiently large T. We get (5.10).

Summing up these facts we get Proposition 4.2.

Proof of Propositoin 4.3. Assume that $w(m)$ satisfies $\mathcal{O}_1(m)$. Then, we see that I_r^ϱ does not exceed

$$\begin{aligned} C' |G(\chi_{B_r^*} f_\omega)(x)|_{\,|\,x\in \bar{B}_r} &+ m^{-1}(\log m)^2 \int_{D_r} \varphi(r, y) |f(y)| dy \\ &\leq C(m^{-2}+m^{-1}(\log m)^2 \lambda^{-1/2}) ||f||_\infty \end{aligned} \tag{5.12}$$

observing

$$\int_\Omega \varphi(r, y)\, dy \leq C\lambda^{-1/2} .$$

Therefore $\mathcal{L}_1$ does not exceed $C(m^{-4}+m^{-2}\lambda^{-1}(\log m)^4)\sum_{\substack{r,q\\ r\neq q}} K_{rq}\,\|f\|_\infty^2$. By lemma 3.9, we have

$$\sum_{\substack{r,q\\ r\neq q}} K_{rq}\leq C\, m^\beta(\log m)^2(\lambda^{-3/2}m^\beta+1)\,.$$

Therefore we get Proposition 4.3.

Proof of Proposition 4.4. We have to get a bound for

$$\sum_{\substack{r,q\\ r\neq q}} I_q^s K_{rq} \tag{5.13}$$

to estimate $\mathcal{L}_2$. Since $|I_q^s|\leq \mathcal{F} m^{-1}(\log m)^2\,\Phi_2(q, i_1)\,G_I$, we have

$$(5.13)=\sum_{\substack{r,q\\ r\in I}}+\sum_{\substack{r,q\\ r\notin I}}\,.$$

We have $E(K_{rq}\,\Phi_2(q, i_1)\,G_I)\leq \tilde{C}(C\lambda^{-1})^{s+1}$ when $r\notin I$, and $\leq C'(C\lambda^{-1})^{s-(1/2)}$ when $i_1=r$ and $\leq C'(C\lambda^{-1})^{s-(1/2)}$ when $i_\nu=r$ for $\nu\geq 2$. The third inequality (5.13) is proved by the following way. We have

$$\left|\int K_{rq}\,\Phi_2(q, i_1)\,V(w_q)\,\tilde{d}w_q\right|\leq \lambda^{-1/2}\,\Phi_0(\lambda/20, r, i_1)$$

and

$$\begin{aligned}&\left|\int \Phi_0(r, i_1)\,G_{i_{\nu-1}r}\,G_{ri_{\nu+1}}\,V(w_r)\,\tilde{d}w_r\right|\\ &\quad\leq C\left(\int \Phi_0(r, i_1)^2\,\tilde{d}w_r\right)^{1/5}\Phi_{1/2}(\lambda/20, i_{\nu-1}, i_{\nu+1})\,.\end{aligned}$$

Thus,

$$\begin{aligned}&E(K_{rq}\,\Phi_2(q, i_1)\,G_I)\\ &\quad\leq C\lambda^{-1/2}\,E(\Phi_0(r, i_1)\,G_I)\\ &\quad\leq C\lambda^{-5/4}\,E(G_{i_1i_2}\cdots\Phi_{1/2}(\lambda/20, i_{\nu-1}, i_{\nu+1})\cdots G_{i_{s-1}i_s})\\ &\quad\leq C'(C\lambda^{-1})^{s-(1/2)}\,.\end{aligned}$$

Summing up these facts we get

$$E((5.13))\leq C\mathcal{F} m^{-1}(\log m)^2\,(m^\beta/C'\lambda)^s\,(m^{2\beta}\,\lambda^{-1}+m^\beta\,\lambda^{1/2})\,.$$

Therefore, we get the desired Proposition 4.4.

Proof of Proposition 4.5. We assume that $w(m)$ satisfies $\mathcal{O}_1(m)$. We have

$$|I_r^s|\leq C(\log m)^2\,\mathcal{F}\sum_I m^{-\theta}\,\Phi_{I+\theta}(r, i_1)\,G_{I(s)}\,,$$

where $\Phi_\delta(r, i_1)=\Phi_\delta(\lambda/8, w_r, w_{i_1})$. We have

$$|\mathcal{L}_3|\leq\sum_{k=1}^4\mathcal{L}^{(k)}\,,$$

where

$$\mathcal{L}^{(1)} = \sum_{\substack{r,q,I,J \\ r\notin J, q\notin I}} m^{-2\theta}(\log m)^4 \mathcal{F}^2 \Phi_{1+\theta}(r, i_1)\, \Phi_{1+\theta}(q, j_1)\, G_{IJ}\, K_{rq}$$

$$\mathcal{L}^{(2)} = \sum_{\substack{r,q,I,J \\ r\notin J, q\in I}} \text{(the same term as above)}$$

$$\mathcal{L}^{(3)} = \sum_{\substack{r,q,I,J \\ r\in J, q\notin I}} \text{(the same term as above)}$$

$$\mathcal{L}^{(4)} = \sum_{\substack{r,q,I,J \\ r\in J, q\in I}} \text{(the same term as above)}\,.$$

It should be remarked that we can take distinct θ for $\mathcal{L}^{(k)}$ ($k=1, \cdots, 4$).

Part 1=Estimation of $\mathcal{L}^{(1)}$. Hereafter we put $\tilde{d}w_r = V(w_r)\, dw_r$. For the indices of terms in $\mathcal{L}^{(1)}$ we take $\theta=1$. We have

$$\left|\iint \Phi_2(r, i_1)\, \Phi_2(q, j_1)\, K_{rq}\, \tilde{d}w_r\, \tilde{d}w_q\right| \leq C\lambda^{-1}\, \Phi_0(\lambda/20, i_1, j_1)\,.$$

By Lemma 3.11 we get

$$P(|\mathcal{L}^{(1)}| \leq m^{4\varepsilon}(m^{\beta}/C'\lambda)^{s+t}\, m^{2\beta-2}\, \lambda^{-1/2}\, \|Gf_{\omega}\|_{\infty}^2) \geq 1-m^{-\varepsilon}\,.$$

Part 2=Estimation of $\mathcal{L}^{(4)}$. Terms in $\mathcal{L}^{(4)}$ up to $m^{-2\theta}(\log m)^4 \mathcal{F}^2$ factor are represented as

$$K_{rq}\, \Phi_{1+\theta}(r, i_1)\, \Phi_{1+\theta}(q, j_1)\, G_I\, G_J\, |_{i_p=q, j_{p'}=r}\,, \tag{5.14}$$

where $I\backslash i_p, J\backslash j_{p'}$ is self-avoiding indices, respeatively.

We have three cases $Z(1)$, $Z(2)$, $Z(3)$.

$Z(1)$: $i_1 \neq q, j_1 \neq r$

$Z(2)$: $i_1 = q, j_1 \neq r$ or $i_1 \neq q, j_1 = r$

$Z(3)$: $i_1 = q, j_1 = r$.

First we consider the case $Z(1)$. In this case the following (ξ, ξ')-*technique* is very useful. We assume that $p<s$, $p'<t$. Other cases can be treated by similar way as follows. Apply the Hölder inequality

$$\begin{aligned} &\left|\iint f(r, q)\, g(r, q)\, \tilde{d}w_r\, \tilde{d}w_q\right| \\ &\quad \leq \left(\iint f(r, q)^{\xi}\, \tilde{d}w_r\, \tilde{d}w_q\right)^{1/\xi} \left(\iint g(r, q)^{\xi'}\, \tilde{d}w_r\, \tilde{d}w_q\right)^{1/\xi'} \end{aligned} \tag{5.15}$$

to $f(r, q) = K_{rq}\, \Phi_{1+\theta}(r, i_1)\, \Phi_{1+\theta}(q, j_1)$, $g(r, q) = G_{i_{p-1}q}\, G_{qi_{p+1}}\, G_{j_{p'-1}r}\, G_{rj_{p'+1}}$. Here (ξ, ξ') is a pair satisfying

$$\xi^{-1}+(\xi')^{-1} = 1,\ \xi'<3,\ 3/2<\xi,\ (1+\theta)\,\xi<3\,. \tag{5.16}$$

Note that we can take (ξ, ξ') when $\theta<1$. Then, the factors of the right hand side of (5.15) are estimated by $\lambda^{-(3/\xi)+(1+\theta)}\, \Phi_0(\lambda/20, i_1, j_1)$ and $C\, \Phi_{1-\rho}(\lambda/20, i_{p-1}, i_{p+1})\, \Phi_{1-\rho}(\lambda/20, j_{p'-1}, j_{p'+1})$, where $\rho=(3/\xi')-1$.

From now on we use the abbreviation

$$G_I[k, s\,|\,p] \quad \text{for} \quad G_{i_k i_{k+1}} \cdots \Phi_{1-\rho}(i_{p-1}, i_{p+1}) \cdots G_{i_{1-s} i_s} .$$

We also use $G_J[k, t\,|\,p']$. By (ξ, ξ')-technique we have

$$E((5.14)) \leq C' E(\Phi_0(\lambda/20, i_1, j_1)\, G_I[1, s\,|\,p]\, G_J[1, t\,|\,p']) .$$

By the easy going Lemma II we get

$$\begin{aligned} (5.17) \qquad & E(\text{the sum of all terms of the type } Z(1) \\ & \quad \text{in } \ \mathcal{L}^{(4)} \leq C\, m^{\varepsilon} (m^{\beta}/C'\lambda)^{s+t}\, (\mathcal{F}^2 m^{-2}\, \lambda^{5/2}) \end{aligned}$$

putting $\theta \to 1$ and $\xi \to 3/2$. By SUM $Z(1)$ we denote the sumof all terms of the type $Z(1)$.

Next we consider the case $Z(3)$. The term $K_{rq}\, \Phi_{1+\theta}(r, q)^2\, G_{q i_2}\, G_{r j_2}$ appears. We take θ such that it satisfies $\theta < 1/2$ for these indices in $Z(3)$. We have

$$\left| \iint K_{rq}\, \Phi_{1+\theta}(r, q)^2\, G_{q i_2}\, G_{r j_2}\, d\tilde{w}_r\, d\tilde{w}_q \right| \leq C \lambda^{-(1-\theta)}\, \Phi_0(\lambda/20, i_2, j_2) .$$

Thus, $E((5.14))$ does not exceed

$$C \lambda^{-(1-\theta)}\, E(\Phi_0(\lambda/20, i_1, j_1)\, G_{i_2 i_3} \cdots G_{i_{s-1} i_s}\, G_{j_2 j_3} \cdots) .$$

By the easy going Lemma II we have for any $\xi > 0$

$$(5.18) \qquad E(\text{SUM } Z(3) \text{ in } \mathcal{L}^{(4)}) \leq C m^{\varepsilon} (m^{\beta}/C'\lambda)^{s+t}\, \lambda^{-(1-\theta)}\, \lambda^{5/2} m^{-2\theta}\, \mathcal{F}^2 .$$

Note that we can take θ as close as $1/2$.

Finally we examine the case $Z(2)$. Without loss of generalities we study the terms

$$\begin{aligned} & K_{rq}\, \Phi_{1+\theta}(r, q)\, \Phi_{1+\theta}(q, j_1)\, G_{q i_2}\, G_{i_2 i_3} \cdots G_{i_{s-1} i_s} \\ & \times G_{j_1 j_2} \cdots G_{j_{p'-1} r}\, G_{r j_{p'+1}} \cdots G_{j_{t-1} j^t} . \end{aligned}$$

We have $|K_{rq}| \leq C$. We use the (ξ, ξ')-technique for the pair $(\Phi_{1+\theta}(r, q), G_{j_{p'-1} r}$ $G_{r j_{p'+1}}$ with (5.16) on measure $d\tilde{w}_r$ and we see that $E((5.14))$ does not exceed

$$E(\Phi_{1+\theta}(q, j_1)\, G_{q i_2} \cdots G_{i_{s-1} i_s}\, G_J[1, t\,|\,p']) .$$

By the easy going Lemma I we get

$$(5.19) \qquad E(\text{SUM } Z(2) \quad \text{in} \quad \mathcal{L}^{(4)}) \leq C\, m^{2\varepsilon} (m^{\beta}/C'\lambda)^{s+t}\, \mathcal{F}^2 m^{-2}\, \lambda^{5/2}$$

for any $\varepsilon > 0$ letting $\theta \to 1$ and $\xi \to 3/2$.

Summing up the above facts, we get

$$E(\text{SUM } \mathcal{L}^{(4)}) \leq C\, m^{2\varepsilon} (m^{\beta}/C'\lambda)^{s+t}\, \mathcal{F}^2 (m^{-2}\, \lambda^{5/2} + m^{-1}\, \lambda^2) .$$

Part 3=Estimation of $\mathcal{L}^{(2)}$, $\mathcal{L}^{(3)}$. We study $\mathcal{L}^{(2)}$. Terms in $\mathcal{L}^{(2)}$ up to

$m^{-2\theta}(\log m)^4 \mathcal{F}^2$ factor are represented by

$$K_{rq}\,\Phi_{1+\theta}(r, i_1)\,\Phi_{1+\theta}(q, j_1)\,G_I\,G_J\,|_{i_p=q}\,,$$

where $I\backslash i_p$ and J is self-avoiding, respectively.

We have two cases $W(1)$, $W(2)$.

$W(1)$: $i_1 \neq q$

$W(2)$: $i_1 = q$.

We now consider the case $W(1)$. We see that

$$\left|\int K_{rq}\,\Phi_{1+\theta}(r, i_1)\,\tilde{d}w_r\right| \leq C\lambda^{-(1-(\theta/2))}\,\Phi_0(\lambda/20, q, i_1)\,.$$

Then, we use (ξ, ξ')-technique for the pair $(\Phi_0(\lambda/20, q, i_1)\,\Phi_{1+\theta}(q, j_1), G_{i_{p-1}q}\,G_{qi_{p+1}})$ with (5.16) on measure $\tilde{d}w_q$, which implies

$$E((5.14)) \leq E(\Phi_0(\lambda/20, i_1, j_1)\,G_I[1, p\,|\,s]\,G_J)\,. \tag{5.20}$$

By the easy going Lemma II we get

$$\begin{aligned} &E(\text{SUM } W(1) \text{ in } \mathcal{L}^{(2)}) \\ &\quad\leq C\,m^{2\varepsilon}(m^\beta/C'\lambda)^{s+t}\,\mathcal{F}^2\,\lambda^{(1+\theta)/2}\,m^{\beta-(1+\theta)} \end{aligned} \tag{5.21}$$

for any $\varepsilon > 0$, $\theta < 1$. We can put θ as close as 1.

The case $W(2)$ is easily investigated. We have $|K_{rq}| \leq C$ and

$$\left|\int \Phi_{1+\theta}(r, q)\,\tilde{d}w_r\right| \leq C\lambda^{-(1-(\theta/2))}\,.$$

Then,

$$\begin{aligned} E((5.14)) &\leq C\lambda^{-(1-(\theta/2))}E(\Phi_{1+\theta}(q, j_1)\,G_{qi_2}\cdots G_{i_{s-1}i_s}G_J) \\ &\leq C''\lambda^{-(3/2)+\theta}\,E(\Phi_0(\lambda/20, i_2, j_1)\,G_J\,G_{i_2i_3}\cdots)\,. \end{aligned}$$

Therefore, we get

$$E(\text{SUM } W(2) \text{ in } \mathcal{L}^{(2)}) \leq C\,m^{2\varepsilon}(m^\beta/C'\lambda)^{s+t}\,\mathcal{F}^2\,\lambda^\theta\,m^{\beta-(1-\theta)}\,. \tag{5.22}$$

We can let $\theta \to 1$.

Summing up (5.21) and (5.22) we get the following.

$$E(\text{SUM } \mathcal{L}^{(2)}) \leq C\,m^{2\varepsilon}(m^\beta/C'\lambda)^{s+t}\,\mathcal{F}^2\,\lambda m^{\beta-2} \tag{5.23}$$

for any $\varepsilon > 0$.

We combine the above results in Part 1, 2, 3 and we get Proposition 4.5.

Proof of Proposition 4.6 is given below our proof of Proposition 4.7.

Proof of Proposition 4.7. Since there is universal constant $C > 0$ such that $|x - w_{i_1}| \geq C\,|w_r - w_{i_1}|$ for $x \in \partial B_r \cap \partial\omega$, we can get

$$\max_{\partial B_r \cap \partial\omega} |G(x, w_{i_1})| \leq C_\theta\,m^\theta\,\Phi_{1-\theta}(\lambda/8, r, i_1)\,.$$

Thus, we have

$$|\tilde{I}_r^s| \leq C \mathcal{F} m^\theta (\log m)^2 \sum_I \Phi_{1-\theta}(r, i_1)\, G_I\,.$$

We have

$$|\sum \tilde{I}_r^s \tilde{I}_q^t K_{rq}| \leq C(\mathcal{N}^{(1)}+\mathcal{N}^{(2)}+\mathcal{N}^{(3)}+\mathcal{N}^{(4)})\,,$$

where

$$\mathcal{N}^{(1)} = \sum_{\substack{r,q,I,J\\ r\notin J,q\notin I}} m^{2\theta}(\log m)^2 \mathcal{F}^2 \Phi_{1-\theta}(r, i_1)\, G_{qj_1}\, G_I\, G_J$$

$$\mathcal{N}^{(2)} = \sum_{\substack{r,q,I,J\\ r\notin J,q\in I}} \text{(the same term as above)}$$

$$\mathcal{N}^{(3)} = \sum_{\substack{r,q,I,J\\ r\in J,q\notin I}} \text{(the same term as above)}$$

$$\mathcal{N}^{(4)} = \sum_{\substack{r,q,I,J\\ r\in J,q\in I}} \text{(the same term as above)}\,.$$

Part 1=Estimation of $\mathcal{N}^{(1)}$. In this case $\theta=0$. Terms in $\mathcal{N}^{(1)}$ up to factor $m^{2\theta}(\log m)^2 \mathcal{F}^2$ are represented as

$$K_{rq}\, G_{ri_1}\cdots G_{i_{p-1}r}\, G_{ri_{p+1}}\cdots G_{i_{s-1}i_s} \times G_{qj_1}\cdots G_{j_{p'-1}q}\, G_{qj_{p'+1}}\cdots G_{j_{t-1}j_t}\,. \tag{5.24}$$

We use (ξ, ξ')-technique for the pair $(K_{rq}\, G_{ri_1}\, G_{qj_1}, G_{i_{p-1}r}\, G_{ri_{p+1}}\, G_{j_{p'-1}q}\, G_{qj_{p'+1}})$ with (5.16) on the measure $\tilde{d}w_r\, \tilde{d}w_q$. As a result we have

$$E((5.24)) \leq C\lambda^{-(3-\xi)/\xi} E(\Phi_0(\lambda/20, i_1, j_1)\, G_I[1, s\,|\,p]\, G_J[1, t\,|\,p'])\,.$$

We put $\xi>3/2$ as close as $3/2$.

By the easy going Lemma II we get

$$E(\mathrm{SUM}\ \mathcal{N}^{(1)}) \leq Cm^\varepsilon (m^\beta/C'\lambda)^{s+t} \mathcal{F}^2 \lambda^{3/2}\,. \tag{5.25}$$

Part 2=Estimation of $\mathcal{N}^{(4)}$. Terms in $\mathcal{N}^{(4)}$ up to factor $m^{2\theta}(\log m)^2 \mathcal{F}^2$ are represented as

$$K_{rq}\, \Phi_{1-\theta}(r, i_1)\, \Phi_{1-\theta}(q, j_1)\, G_I\, G_J \tag{5.26}$$

with $i_p=r$, $i_\nu=q$, $j_{p'}=q$, $j_\mu=r$. There are three cases.

(Y(1)): $i_2 = r, j_2 = q$
(Y(2)): $i_2 \neq r, j_2 = q$ or $i_2 = r, j_2 \neq q$
(Y(3)): $i_2 \neq r, j_2 \neq q$.

We consider the case $(Y(1))$. In this case the term

$$K_{rq}\, \Phi_{1-\theta}(r, i_1)\, G_{i_1r}\, G_{ri_3} \times \Phi_{1-\theta}(q, j_1)\, G_{j_1q}\, G_{qj_3}$$

appears. The case (Y(1)) is divided into three cases (Y(4)), (Y(5)) and (Y(6)).

(Y(4)): $i_1 = q, j_1 = r$

(Y(5)): $i_1 = q, j_1 \neq r$ or $i_1 \neq q, j_1 = r$
(Y(6)): $i_1 \neq q, j_1 \neq r$.

For the case (Y(4)) we take $\theta < 1/2$. We have

$$\left| \iint \Phi_{1-\theta}(r, q)^2 G_{rq}^2 G_{rj_3} G_{qj_3} \tilde{d}w_r \tilde{d}w_q \right| \le C\lambda^{-1/2} \Phi_0(\lambda/20, i_3, j_3).$$

Thus, $E((5.26)) \le C\lambda^{-1/2} E(\Phi_0(i_3, j_3) G_{i_3 i_4} \cdots G_{j_3 j_4} \cdots)$. By the easy going Lemma II we have

$$E(\mathrm{SUM}\,(\mathrm{Y}(4))) \le C\, m^{2\theta} (\log m)^2 (m^\beta / C'\lambda)^{s+t} \mathcal{F}^2 \lambda^4 m^{-2\beta}. \tag{5.27}$$

It should be noticed that the distinct number of indices in SUM (Y(4)) is at most $(m^\beta)^{s+t-2}$. We put $\theta > 1/2$ as close as 1/2 in (5.27).

We study the case (Y(5)). For these indices we put $\theta > 0$. The term

$$\begin{aligned} &\Phi_{1-\theta}(r, q) G_{qr} G_{ri_3} G_{i_3 i_4} \cdots \\ &\quad \times \Phi_{1-\theta}(q, j_1) G_{j_1 q} G_{qj_3} \cdots G_{j_{p'-i} j_{p'}} G_{j_{p'} j_{p'+1}} \cdots \end{aligned} \tag{5.28}$$

with $j_{q'} = r$ appears. The (ξ, ξ')-technique for the pair $(\Phi_{1-\theta}(r, q) G_{qr} G_{ri_3} \Phi_{1-\theta}(q, j_1), G_{j_1 q} G_{qj_3} G_{i_{p'-1} r} G_{r j_{p'+1}})$ is used and we get

$$\begin{aligned} &E((5.28)) \\ &\le E(\Phi_0(j_1, i_3) \Phi_{1-\rho}(j_1, j_3) G_J[3, t \mid p'] G_{i_3 i_4} G_{i_4 i_5} \cdots). \end{aligned}$$

By Lemma 3.10 we get

$$E(\mathrm{SUM}\ \mathrm{Y}(5)) \le C\, m^{2\theta} (\log m)^2 (m^\beta / C'\lambda)^{s+t} \mathcal{F}^2 \lambda^{9/2} m^{-2\beta}. \tag{5.29}$$

We can let $\theta \to 0$.

We study ((Y(6)). The following inequality is very useful.

$$\left| \int f g h \,\mathrm{dm} \right| \le \|f\|_{L^\xi(dm)} \|g\|_{L^{\xi'}(dm)} \|h\|_{L^{\xi''}(dm)} \tag{5.30}$$

for (ξ, ξ', ξ'') satisfying

$$\xi^{-1} + (\xi')^{-1} + (\xi'')^{-1} = 1, \quad \xi, \xi', \xi'' > 0.$$

We examine the terms

$$K_{rq} \Phi_{1-\theta}(r, i_1) \Phi_{1-\theta}(q, j_1) G_{i_1 r} G_{ri_3} \cdots G_{j_1 q} G_{qj_3} \cdots \tag{5.31}$$

with $i_p = q$, $p \ge 3$, $j_{p'} = r$, $p' \ge 3$.

Assume that $p \ge 4$, $p' \ge 4$. Then, we use inequality

(5.32) for $dm = \tilde{d}w_r \tilde{d}w_q$,

$$\begin{aligned} f &= K_{rq} \Phi_{1-\theta}(r, i_1) \Phi_{1-\theta}(q, j_1) \\ g &= G_{i_{p-1} q} G_{q i_{p+1}} G_{i_{p'-1} r} G_{r i_{p'+1}} \\ h &= G_{i_1 r} G_{r i_3} G_{j_1 q} G_{q j_3}, \end{aligned}$$

where $\theta>0$ and (ξ, ξ', ξ'') satisfying

$$\xi>3,\ (1-\theta)\,\xi<3,\ \xi'=\xi''<3\,. \tag{5.33}$$

As a result we get

$$E((5.26))\leq CE(\Phi_0(i_1, j_1)\,\Phi_{1-\rho}(i_1, i_3)\,\Phi_{1-\rho}(j_1, j_3) \times G_I[3, s\,|\,p]\,G_J[3, t\,|\,p'])\,.$$

By the easy going Lemma II we obtain

$$E(\mathrm{SUM}\,(\mathrm{Y}(6))\text{ with }p, p'\geq 4\text{ in }\mathcal{N}^{(4)}) \leq C'(m^{\beta}/C'\lambda)^{s+t}\,\mathcal{F}^2\,\lambda^{9/2}\,m^{-2\beta+2\varepsilon}$$

for any $\varepsilon>0$ letting $\theta\to 0$, $\xi'\to 3$.

We continue to study ((Y(6)). Assume that $p=3, p'=3$. Then, we use (ξ, ξ')-technique for the pair $(G^2_{rq}\,\Phi_{1-\theta}(r, i_1)\,\Phi_{1-\theta}(q, j_1), G_{i_1r}\,G_{rj_4}\,G_{j_1q}\,G_{qi_4})$ with (ξ, ξ') satisfying (5.16) and we get

$$E((5.26)) \leq E(\Phi_0(i_1, j_1)\,\Phi_{1-\rho}(i_1, j_4)\,\Phi_{1-\rho}(j_1, i_4) \times G_{i_4i_5}\cdots\times G_{j_4j_5}\cdots)\,.$$

By the easy going Lemma II we get

$$E(\mathrm{SUM}\,(\mathrm{Y}(6))\text{ with }p, p'=3\text{ in }\mathcal{N}^{(4)}) \leq C(m^{\beta}/C'\lambda)^{s+t}\,\mathcal{F}^2\,\lambda^{9/2}\,m^{-2\beta+2\varepsilon}$$

for any $\varepsilon>0$. We have similar estimates for other cases. Thus,

$$E(\mathrm{SUM}\,(\mathrm{Y}(6)))\leq C'(m^{\beta}/C'\lambda)^{s+t}\,\mathcal{F}^2\,\lambda^{9/2}\,m^{-2\beta+2\varepsilon}\,.$$

Summing up the above facts we get

$$E(\mathrm{SUM}\,(\mathrm{Y}(1))) \leq C\,m^{2\varepsilon}(m^{\beta}/C'\lambda)^{s+t}(\lambda^4\,m^{1-2\beta}+\lambda^{9/2}\,m^{-2\beta})\,\mathcal{F}^2 \tag{5.34}$$

for any $\varepsilon>0$.

We make a comment on the terms of the type (Y(2)). Without loss of generalities we assume that $i_2=r, j_2\neq q$. We examine the terms

$$K_{rq}\,\Phi_{1-\theta}(r, i_1)\,G_{i_1r}\,G_{ri_3} \quad \Phi_{1-\theta}(q, j_1)\,G_{j_1j_2}\cdots G_{j_{p'-1}q}\,G_{qj_{p'+1}}\cdots$$

with $i_\nu=q, j_\mu=r$. In any case we can apply (ξ, ξ'), (ξ, ξ', ξ'')-techniques and we get the same bound for $E(\mathrm{SUM}\,(\mathrm{Y}(2)))$ as $E(\mathrm{SUM}\,(\mathrm{Y}(1)))$.

We study the terms of type (Y(3)). We can use the techniques developed above and we get a same bound for $E(\mathrm{SUM}\,(\mathrm{Y}(3))$ as $E(\mathrm{SUM}\,(\mathrm{Y}(1)))$. Therefore, (5.35) $$E(\mathrm{SUM}\ \mathcal{N}^{(4)}) \leq C\, m^{2\varepsilon}\, \mathcal{F}^2 (m^\beta/C'\lambda)^{s+t} (\lambda^4\, m^{1-2\beta} + \lambda^{9/2}\, m^{-2\beta}).$$

Part 3=Estimation of $\mathcal{N}^{(2)}$. Terms in $\mathcal{N}^{(2)}$ up to factor $(\log m)^2\, m^{2\theta}\, \mathcal{F}^2$ are represented as (5.36)$=K_{rq}\, \Phi_{1-\theta}(r, i_1)\, \Phi_{1-\theta}(q, j_1)\, G_{IJ}$ with $i_p=r$, $i_\nu=q$, $j_\mu=q$. We have two cases.

(X(1)): $i_1 = q$
(X(2)): $\nu \geq 2$, $i_\nu = q$.

First we consider the case (X(1)). We have (X(1))=(X(3))+(X(4)), where

(X(3)): $i_2 = r$
(X(4)): $i_p = r$ for $p \geq 3$.

For the case ((X(3)) we put $\theta > 0$ as $\theta \sim 0$. We have

$$\left| \int K_{rq}\, G_{ri_3}\, \Phi_{1-\theta}(r, q)\, G_{rq}\, \tilde{d}w_r \right| \leq C \lambda^{-\theta/2}\, \Phi_0(\lambda/20, q, i_3).$$

Then, we use the Schwartz inequality for the pair $(\Phi_0(\lambda/20, q, i_3)\, G_{qj}, G_{j_{p'-1}q}\, G_{qj_{p'+1}})$ and we get

$$\begin{aligned} &E((5.36)) \\ &\quad \leq C \lambda^{-(\theta/2)-(1/4)}\, E(\Phi_0(i_3, j_1)\, G_{i_3 i_4} \cdots G_{i_{s-1} i_s} \\ &\quad \times G_{j_1 j_2} \cdots \Phi_{1/2}(j_{p'-1}, j_{p'+1}) \cdots G_{j_{t-1} j_t}). \end{aligned}$$

By the easy going Lemma II we get

$$E(\mathrm{SUM}\ (\mathrm{X}(3))) \leq C\, m^{2\varepsilon} (m^\beta/C'\lambda)^{s+t}\, \mathcal{F}^2\, \lambda^3 m^{-\beta}$$

for any $\varepsilon > 0$ observing the fact that the number of distinct indices is at most $(m^\beta)^{s+t-1}$.

For the case (X(4)) we put $\theta > 0$. We use the (ξ, ξ', ξ'')-technique for the triple

$$\begin{aligned} &(\Phi_{1-\theta}(r, q)^{1/3} G_{qi_2}\, G_{qj_1}, \Phi_{1-\theta}(r, q)^{1/3} G_{i_{p-1} r}\, G_{ri_{p+1}}, \\ &\quad \Phi_{1-\theta}(r, q)^{1/3}\, G_{j_{p'-1} q}\, G_{qj_{p'+1}}) \end{aligned}$$

with (ξ, ξ', ξ'') satisfying (5.30) on the measure $\tilde{d}w_r\, \tilde{d}w_q$ and we get

$$E((5.36)) \leq C \lambda^{-1+\varepsilon'}\, E(\Phi_{1-\rho}(i_2, j_1)\, G_I[2, p \,|\, s]\, G_J[1, p' \,|\, t])$$

for any $\varepsilon' > 0$ letting $\xi \to 3$, $\xi' = \xi'' \to 3$, $\rho \to 0$. Thus,

$$E(\mathrm{SUM}\ (\mathrm{X}(4))) \leq C\, m^{2\varepsilon} (m^\beta/C'\lambda)^{s+t}\, \mathcal{F}^2\, \lambda^3 m^{-\beta}$$

for any $\varepsilon > 0$. Therefore $E(\mathrm{SUM}\ (\mathrm{X}(1)))$ has the same bound.

Next we study (X(2)). We examine the case $q \notin (i_{p-1}, i_{p+1})$. We have

$$\begin{aligned} &\left| \int K_{rq}\, \Phi_{1-\theta}(r, i_1)\, G_{i_{p-1} r}\, G_{ri_{p+1}}\, \tilde{d}w_r \right| \\ &\quad \leq \lambda^{-1/4}\, \Phi_0(q, i_1)\, \Phi_{1/2}(i_{p-1}, i_{p+1}) \end{aligned}$$

by the Schwartz inequality. Then, we use (ξ, ξ', ξ'')-technique for the triple $(\Phi_0(q, i_1)\, \Phi_{1-\theta}(q, j_1),\, G_{i_{\nu-1}q}\, G_{qi_{\nu+1}},\, G_{j_{p'-1}q}\, G_{j_{p'+1}})$ on the measure $\tilde{d}w_q$. Thus,

$$\begin{aligned} &E((5.36)) \\ &\quad \le C\lambda^{-1/4} E(\Phi_0(i_1, j_1)\, G_{i_1 i_2} \cdots \Phi_{1-\rho}(i_{\nu-1}, i_{\nu+1}) \cdots \Phi_{1/2}(i_{p-1}, i_{p+1}) \\ &\quad \cdots \times G_J[1, p' \mid t]) . \end{aligned}$$

By the easy going Lemma II we get the same bound for $E(\mathrm{SUM}\ (\mathrm{X}(2)),\ q \in (i_{p-1}, i_{p+1}))$ as $E(\mathrm{SUM}\,(\mathrm{X}(1)))$. For the case $q \in (i_{p-1}, i_{p+1})$ we get the same result.

Summing up the above facts we get

$$E(\mathrm{SUM}\ \mathcal{N}^{(2)}) \le C\, m^{2\varepsilon}\, \mathcal{F}^2 (m^\beta / C'\lambda)^{s+t}\, \lambda^3\, m^{-\beta} . \tag{5.37}$$

Summing up (5.25), (5.35) and (5.37) we get

$$E(\sum_k \mathcal{N}^{(k)}) \le m^{2\varepsilon} (m^\beta / C'\lambda)^{s+t}\, \mathcal{F}^2\, P(\lambda, m) ,$$

where

$$P(\lambda, m) = \lambda^4 m^{1-2\beta} + \lambda^{9/2} m^{-2\beta} + \lambda^{3/2} + \lambda^3 m^{-\beta} .$$

We proved Proposition 4.7.

Proof of Proposition 4.6. We have

$$(\tilde{I}_r^s)^2 \le \mathcal{F}^2\, m^{2\theta} \sum_{I,J} \Phi_{1-\theta}(r, i_1)\, \Phi_{1-\theta}(r, j_1)\, G_I\, G_J .$$

We examine the term

$$\begin{aligned} &\Phi_{1-\theta}(r, i_1)\, G_{i_1 i_2} \cdots G_{i_{p-1} r}\, G_{r i_{p+1}} \cdots G_{i_{s-1} i_s} \\ &\quad \times \Phi_{1-\theta}(r, j_1)\, G_{j_1 j_2} \cdots G_{j_{p'-1} r}\, G_{r j_{p'+1}} \cdots G_{j_{s-1} j_s} . \end{aligned}$$

By using the (ξ, ξ', ξ'')-technique with (5.30) for the triple

$$(\Phi_{1-\theta}(r, i_1)\, \Phi_{1-\theta}(r, j_1),\, G_{i_{p-1} r}\, G_{r i_{p+1}},\, G_{j_{p'-1} r}\, G_{r j_{p'+1}})$$

we get

$$\begin{aligned} &E(\Phi_{1-\theta}(r, i_1)\, \Phi_{1-\theta}(r, j_1)\, G_{IJ}) \\ &\quad \le CE(\Phi_{1-\rho}(i_1, j_1)\, G_I[1, p \mid s]\, G_J[1, p' \mid s]) . \end{aligned}$$

when $i_1 \ne j_1$. When $i_1 = j_1$, we take $\theta > 1/2$ as close as $1/2$. Then,

$$E(\Phi_{1-\theta}(r, i_1)^2\, G_{IJ}) \le CE(G_I[1, p \mid s]\, G_J[1, p' \mid s]) .$$

Thus, by the easy going Lemma II we get

$$E(\sum_r (\tilde{I}_r^s)^2) \le m^{2\varepsilon}\, \mathcal{F}^2 (m^\beta / C'\lambda)^{2s} (\lambda^3 m^{-\beta} + \lambda^4 m^{-2\beta+1})$$

for any $\varepsilon > 0$ letting $\theta \to 0$ and $\xi \to 3$, $\rho \to 0$, $\theta \to 1/2$.

Proof of Proposition 4.8. We know (5.12). Thus, we have to get a bound for $\sum \tilde{I}_q^s K_{rq}$ to estimate $\sum I_r^o \tilde{I}_q^s K_{rq}$. We see that

$$|\tilde{I}_q^s| \leq \mathcal{F} m^\theta \sum_I \Phi_{1-\theta}(q, i_1) G_{i_1 i_2} \cdots G_{i_{p-1} q} G_{q i_{p+1}} \cdots$$

The term

$$K_{rq} \Phi_{1-\theta}(q, i_1) G_I \tag{5.38}$$

is classified as (i), (ii).

(i): $r \in I$

(ii): $r \notin I$.

First we consider the case (i). We put $\theta=0$. Then, by the Schwartz inequality for the pair $(K_{rq} \Phi_{1-\theta}(q, i_1), G_{i_{p-1}q} G_{qi_{p+1}})$ on the measure $\tilde{d}w_q$ implies

$$\begin{aligned} &E((5.38)) \\ &\leq C\lambda^{-1/4} E(\Phi_0(r, i_1) G_{i_1 i_2} \cdots \Phi_{1/2}(i_{p-1}, i_{p+1}) \cdots) \\ &\leq C'(C\lambda^{-1})^s . \end{aligned} \tag{5.39}$$

Next we consider the case (ii). Using $|\Phi_0(r, i_1)| \leq C$ we get

$$E((5.38)) \leq C'(C\lambda^{-1})^{s-(3/2)} . \tag{5.40}$$

Summing up (5.12), (5.39), (5.40) we get

$$\begin{aligned} &E(\sum I_r^o \tilde{I}_q^s K_{rq}) \\ &\leq C m^{2\varepsilon} (m^\beta / C'\lambda)^s (m^\beta + \lambda^{3/2}) (m^{-1} (\log m)^2 \lambda^{-1/2}) \mathcal{F} \|f\|_\infty . \end{aligned}$$

We get Proposition 4.8.

Proof of Proposition 4.9. We have

$$|\sum I_r^s I_q^t K_{rq}| \leq C(\mathcal{S}^{(1)} + \mathcal{S}^{(2)} + \mathcal{S}^{(3)} + \mathcal{S}^{(4)}) ,$$

where

$$\begin{aligned} \mathcal{S}^{(1)} &= \sum_{\substack{r,q,I,J \\ r\notin J, q\notin I}} (\log m)^4 m^{\theta'-\theta} \mathcal{F}^2 K_{rq} \Phi_{1+\theta}(r, i_1) \Phi_{1-\theta'}(q, j_1) G_{IJ} \\ \mathcal{S}^{(2)} &= \sum_{\substack{r,q,I,J \\ r\notin J, q\in I}} (\text{the same term as above}) \\ \mathcal{S}^{(3)} &= \sum_{\substack{r,q,I,J \\ r\in J, q\notin I}} (\text{the same term as above}) \\ \mathcal{S}^{(4)} &= \sum_{\substack{r,q,I,J \\ r\in J, q\in I}} (\text{the same term as above}) . \end{aligned}$$

Part 1=Estimation of $\mathcal{S}^{(1)}$. Terms in $\mathcal{S}^{(1)}$ up to factor $(\log m)^4 m^{\theta'-\theta} \mathcal{F}^2$ are represented as

$$K_{rq} \Phi_{1+\theta}(r, i_1) G_I \Phi_{1-\theta'}(q, j_1) G_{j_1 j_2} \cdots G_{j_{p'-1} q} G_{q j_{p'+1}} \cdots . \tag{5.41}$$

For these indices we take $\theta < 1$ and $\theta'=0$. We use (ξ, ξ')-technique for the pair

$(K_{rq}^{1/\xi}\,\Phi_{1+\theta}(r,i_1)\,\Phi_{1-\theta'}(q,j_1),K_{rq}^{1/\xi'}\,G_{j_{p'-1}q}\,G_{qj_{p'+1}})$ with (5.16) on the measure $\tilde{d}w_r\,\tilde{d}w_q$ and we get

$$E((5.41))\leq C\lambda^{-1+\varepsilon'}\,E(\Phi_0(i_1,j_1)\,G_I\,G_J[1,p'\,|\,t])$$

for any $\varepsilon>0$ letting $\theta\to 1$, $\xi\to 3/2$, $\rho\to 0$. By the easy going Lemma II we get

$$E(\text{SUM}\,\mathcal{S}^{(1)})\leq m^{2\varepsilon}(m^\beta/C'\lambda)^{s+t}\,\mathcal{F}^2\,m^{\beta-1}\,\lambda^{1/2}\,.$$

Part 2=Estimation of $\mathcal{S}^{(4)}$. Terms in $\mathcal{S}^{(4)}$ up to $(\log m)^4\,m^{\theta'-\theta}\,\mathcal{F}^2$ are represented as

$$K_{rq}\,\Phi_{1+\theta}(r,i_1)\,\Phi_{1-\theta'}(q,j_1)\,G_I\,G_J \tag{5.42}$$

with $i_\nu=q$, $j_{p'}=q$, $j_\mu=r$. There are six cases.

(D(1)): $i_1=q,\ j_1=r,\ j_2=q$
(D(2)): $i_1=q,\ j_1=r,\ j_\nu=q\quad(\nu\geq 3)$
(D(3)): $i_1=q,\ j_{p'}=r\quad(p'\geq 2)$
(D(4)): $i_1\neq q,\ j_1=r,\ j_2=q$
(D(5)): $i_1\neq q,\ j_1=r,\ j_2\neq q$
(D(6)): $i_1\neq q,\ j_1\neq r.$

First we consider the case (D(1)). We put $\theta=0$, $\theta'>0$ in this case. Then, the term

$$K_{rq}\,\Phi_{1+\theta}(r,q)\,G_{qi_2}\cdots\Phi_{1-\theta'}(q,r)\,G_{rq}\,G_{qj_3}\cdots$$

is estimated. We have

$$\left|\int K_{rq}\,\Phi_{1+\theta}(r,q)\,\Phi_{1-\theta'}(q,r)\,G_{rq}\,\tilde{d}w_r\right|\leq C\,.$$

Thus,

$$E((5.42))\leq E(G_{qi_2}\cdots G_{qj_3}\cdots)\leq C'(C\lambda^{-1})^{s+t-3}\,.$$

Therefore,

$$E(\text{SUM (D(1))})\leq m^{2\varepsilon}(m^\beta/C'\lambda)^{s+t}\,\mathcal{F}^2\,\lambda^3\,m^{-\beta} \tag{5.43}$$

for any $\varepsilon>0$.

We consider the case (D(2)). We put $\theta=0$, $\theta'>0$ for these indices. Integration by the measure $\tilde{d}w_r$ implies

$$E((5.42))\leq E(G_{qi_2}\cdots\Phi_0(q,j_2)\cdots G_{j_{p'-1}q}\,G_{qj_{p'+1}}\cdots)\,. \tag{5.44}$$

We use (ξ,ξ')-technique for (5.44) and we get

$$E((5.42))\leq\lambda^{-1/2}\,E(\Phi_0(i_2,j_3)\,G_J[2,p'\,|\,t]\,G_{i_2i_3}\cdots)\,.$$

Therefore, we get

$$E(\text{SUM ((D(2)))})\leq m^{2\varepsilon}(m^\beta/C'\lambda)^{s+t}\,\mathcal{F}^2\,\lambda^3 m^{-\beta} \tag{5.45}$$

for any $\varepsilon>0$.

We study the case ((D(3)). We take $\theta=1$ and $\theta'>0$ for these indices. By (ξ, ξ')-technique for the pair $(\Phi_{1+\theta}(r, q), G_{j_{\mu-1}r}\, G_{rj_{\mu+1}})$ on the measure $\tilde{d}w_r$ and we get

$$\mathrm{E}((5.42))\leq E(\Phi_{1-\theta'}(q, j_1)\, G_J[1, \nu\,|\, t]\, G_{qi_2}\, G_{i_2 i_3}\cdots)\,.$$

Note that $j_{p'}=q$ for $p'\geq 2, p'\neq\nu$. We use (ξ, ξ')-technique with (5.16) on the measure $\tilde{d}w_q$ and we see that E((5.42)) does not exceed

$$E(\Phi_0(i_2, j_1)\, G_{i_2 i_3}\cdots G_J[1, \nu, p'\,|\, t])\,,$$

where $G_J[1, \nu, p'\,|\,t]$ is given by

$$G_{j_1 j_2}\cdots\Phi_{1-\tilde{\rho}}(j_{\nu-1}, j_{\nu+1})\cdots\Phi_{1-\rho}(j_{p'-1}, j_{p'+1})\cdots G_{j_{t-1}j_t}\,,$$

if $p'>\nu+1$ and

$$G_{j_1 j_2}\cdots\Phi_{1-\tilde{\rho}}(j_{\nu-1}, j_{\nu+2})\cdots G_{j_{t-1}j_t}\,,$$

if $p'=\nu+1, \cdots$. Here ρ and $\tilde{\rho}\to 0$ as $\xi\to 3/2$, $\theta'\to 0$. Therefore,

$$E(\mathrm{SUM}\,(\mathrm{D}(3)))\leq m^{2\varepsilon}(m^\beta/C'\lambda)^{s+t}\,\mathcal{F}^2\,\lambda^{7/2}m^{-\beta-1}$$

for any $\varepsilon>0$.

We consider the case (D(4)). We examine the term

$$\begin{aligned}&K_{rq}\,\Phi_{1+\theta}(r, i_1)\cdots G_{i_{\nu-1}q}\, G_{qi_{\nu+1}}\cdots\\ &\times\Phi_{1-\theta'}(q, r)\, G_{rq}\, G_{qj_3}\cdots\,.\end{aligned}$$

We have

$$\left|\int K_{rq}\,\Phi_{1+\theta}(r, i_1)\,\Phi_{1-\theta'}(q, r)\, G_{qr}\,\tilde{d}w_r\right|\leq C\,\Phi_0(q, i_1)\,.$$

Therefore,

$$\mathrm{E}((5.42))\leq E(G_{i_1 i_2}\cdots G_{i_{\nu-1}q}\, G_{qi_{\nu+1}}\cdots\Phi_0(i_1, q)\, G_{qj_3}\cdots)\,.$$

Therefore, we get

$$E(\mathrm{SUM}\,(\mathrm{D}(4)))\leq m^{2\varepsilon}(m^\beta/C'\lambda)^{s+t}\,\mathcal{F}^2\,\lambda^3 m^{-\beta}$$

for any $\varepsilon>0$.

Let us examine the case ((D(5)). We have

$$\begin{aligned}&\left|\int K_{rq}\,\Phi_{1+\theta}(r, i_1)\,\Phi_{1-\theta'}(q, r)\, G_{rj_2}\,\tilde{d}w_r\right|\\ &\leq C\lambda^{-1/4}\,\Phi_0(q, i_1)\,\Phi_{1/2}(q, j_2)\,.\end{aligned}$$

Then, we employ (ξ, ξ', ξ'')-technique for the triple $(\Phi_0(q, i_1)\,\Phi_{1/2}(q, j_3), G_{i_{\nu-1}q}\, G_{qi_{\nu+1}}, G_{j_{p'-1}q}\, G_{qj_{p'+1}})$ and we get

$$\mathrm{E}((5.42))\leq\lambda^{-(1/2)+\varepsilon'}\, E(\Phi_0(i_1, j_2)\, G_I[1, \nu\,|\, s]\, G_J[2, p'\,|\, t])\,.$$

Thus, we get $E(\mathrm{SUM}\,(\mathrm{D}(5)))\leq m^{2\varepsilon}(m^\beta/C'\lambda)^{s+t}\,\mathcal{F}^2\,\lambda^3 m^{-\beta}$.

Finally we study the case ((D(6)). We use (ξ, ξ')-technique for the pair $(K_{rq}\,\Phi_{1+\theta}(r, i_1), G_{j_{\mu-1}r}\, G_{rj_{\mu+1}})$ on the measure $\tilde{d}w_r$ and we see that E((5.42)) does not exceed

$$\lambda^{-(1/2)+\varepsilon'} E(\Phi_0(q, i_1)\, G_I\, \Phi_{1-\theta'}(q, j_1)\, G_J[1, \mu \,|\, t]) .$$

Then, we use (ξ, ξ', ξ'')-technique for the triple $(\Phi_0(q, i_1)\, \Phi_{1-\theta'}(q, j_1), G_{i_{\nu-1}q}\, G_{qi_{\nu+1}},$ q-factor in $G_J[1, \mu \,|\, t])$ and we get

$$E(\mathrm{SUM}\,(\mathrm{D}(6))) \le m^{2\varepsilon}(m^{\beta}/C'\lambda)^{s+t}\, \mathcal{F}^2\, \lambda^3 m^{-\beta} .$$

Summing up the above facts we get

$$E(\mathrm{SUM}\, \mathcal{S}^{(4)}) \le m^{2\varepsilon}(m^{\beta}/C'\lambda)^{s+t}\, \mathcal{F}^2(\lambda^3 m^{-\beta} + \lambda^{7/2} m^{-\beta-1}) .$$

Part 3=Estimation of $\mathcal{S}^{(2)}$. Terms in $\mathcal{S}^{(2)}$ up to factor $(\log \mathrm{m})^4 m^{\theta-\theta'}\mathcal{F}^2$ are represented as

$$K_{rq}\,\Phi_{1+\theta}(r, i_1)\, G_I\, \Phi_{1-\theta'}(q, j_1)\, G_J \tag{5.46}$$

with $i_p = q, j_{p'} = q$ $(p' \ge 2)$. Note that $r \notin J$. We have two cases.
$(\Gamma(1))$: $i_1 \neq q$
$(\Gamma(2))$: $i_1 = q$.

First we study the case $(\Gamma(1))$. We take $\theta = 1, \theta' > 0$. Since $r \notin J$, we have

$$\mathrm{E}((5.46)) \le \lambda^{-1/2} E\,\Phi_0(q, i_1)\, \Phi_{1-\theta'}(q, j_1)\, G_{IJ}) .$$

By (ξ, ξ', ξ'')-technique for the triple

$$(\Phi_0(q, i_1)\, \Phi_{1-\theta'}(q, j_1), G_{i_{p-1}q}\, G_{qi_{p+1}}, G_{j_{p'-1}q}\, G_{qj_{p'+1}})$$

we get

$$\begin{aligned} &\mathrm{E}((5.46)) \\ &\quad \le \lambda^{-1/2} E(\Phi_0(i_1, j_1)\, G_I[1, p \,|\, s]\, G_J[1, p' \,|\, t]) . \end{aligned}$$

Thus,

$$E(\mathrm{SUM}\,(\Gamma(1))) \le m^{2\varepsilon}(m^{\theta}/C'\lambda)^{s+t}\, \mathcal{F}^2\, \lambda^2 m^{-1}$$

for any $\varepsilon > 0$ letting $\theta' \to 0$.

We study the case $(\Gamma(2))$. We have terms of the form

$$K_{rq}\,\Phi_{1+\theta}(r, q)\, G_{qi_2} \cdots \Phi_{1-\theta'}(q, j_1)\, G_J$$

with $j_{p'} = q$ $(p' \ge 2)$. We put $\theta = 1, \theta' > 0$. We see that

$$\left| \int K_{rq}\,\Phi_{1+\theta}(r, q)\, \tilde{d}w_r \right| \le C\lambda^{-1/2} .$$

Thus, E((5.46)) does not exceed

$$\begin{aligned} &\lambda^{-1/2} E((\Phi_{1-\theta'}(q, j_1)\, G_J\, G_{qi_2} \cdots) \\ &\quad \le C\lambda^{-1/2} E(\Phi_0(i_2, j_1)\, G_J[1, p' \,|\, t]\, G_{i_2 i_3} \cdots) . \end{aligned}$$

Here we used (ξ, ξ')-technique. Therefore, we get the same bound for $E(\mathrm{SUM}(\Gamma(2)))$ as $E(\mathrm{SUM}(\Gamma(1)))$. Summing up these facts we see

$$E(\mathrm{SUM}\,\mathcal{S}^{(2)})\leq m^{2\varepsilon}(m^{\beta}/C'\lambda)^{s+t}\,\mathcal{F}^2\,\lambda^2 m^{-1}\,.$$

Part 4=Estimation of $\mathcal{S}^{(3)}$. Terms in $\mathcal{S}^{(3)}$ up to factor $(\log m)^4 m^{\theta-\theta'}\,\mathcal{F}^2$ are represented as

$$K_{rq}\,\Phi_{1+\theta}(r, i_1)\,\Phi_{1-\theta'}(q, j_1)\,G_I\,G_J \tag{5.47}$$

with $j_\nu=r$, $j_{p'}=q$ $(p'\geq 2)$. We have two cases.

$(\Lambda(1))$: $j_1=r$

$(\Lambda(2))$: $j_\nu=r$ $(\nu\geq 2)$.

We study the case $(\Lambda(1))$. For this case we examine the term $K_{rq}\,\Phi_{1+\theta}(r, i_1)\,\Phi_{1-\theta'}(q, r)\,G_{rj_2}\cdots$. We have two subcases $(\Lambda(3))$, $(\Lambda(4))$ of $(\Lambda(1))$.

$(\Lambda(3))$: $j_2=q$

$(\Lambda(4))$: $j_\nu=q$ $(\nu\geq 3)$.

We consider the case $(\Lambda(3))$. Then,

$$\begin{aligned}&\left|\int K_{rq}\,\Phi_{1+\theta}(r, i_1)\,\Phi_{1-\theta'}(q, r)\,G_{rq}\,G_{qj_3}\,\tilde{d}w_r\,\tilde{d}w_q\right|\\ &\quad\leq C\lambda^{-(2-\theta)/2}\,\Phi_0(i_1, j_3)\,.\end{aligned}$$

Thus, $E(\mathrm{SUM}(\Lambda(3)))\leq m^{2\varepsilon}(m^{\beta}/C'\lambda)^{s+t}\,\mathcal{F}^2\,\lambda^2 m^{-1}$ putting $\theta=1$, $\theta'=0$. We consider the case $(\Lambda(4))$. We put $\theta<1$, $\theta'>0$. By (ξ, ξ')-technique for the pair

$$(K_{rq}\,\Phi_{1+\theta}(r, i_1),\ \Phi_{1-\theta'}(q, r)\,G_{rj_2})$$

on the measure $\tilde{d}w_r$ we obtain

$$E((5.47))\leq CE(\Phi_0(q, i_1)\,G_I\,\Phi_{1-\rho}(q, j_2)\,G_{j_2 j_3}\cdots)\,.$$

Here $j_\nu=q$. Thus, we get the same bound for $E(\mathrm{SUM}(\Lambda(4)))$ as $E(\mathrm{SUM}(\Lambda(3)))$. Therefore,

$$E(\mathrm{SUM}(\Lambda(1)))\leq m^{2\varepsilon}(m^{\beta}/C'\lambda)^{s+t}\,\mathcal{F}^2\,\lambda^2 m^{-1}\,.$$

The same estimate for $E(\mathrm{SUM}(\Lambda(2)))$ is given which is left to the readers as an exercise.

CONCLUSION. We get

$$\begin{aligned}&E(\mathrm{SUM}\,\mathcal{S}^{(1)}+\mathcal{S}^{(2)}+\mathcal{S}^{(3)}+\mathcal{S}^{(4)})\\ &\quad\leq m^{2\varepsilon}(m^{\beta}/C'\lambda)^{s+t}\,\mathcal{F}^2(m^{\beta-1}\lambda^{1/2}+\lambda^3 m^{-\beta}+\lambda^{7/2}m^{-\beta-1}+\lambda^2 m^{-1})\,.\end{aligned} \tag{5.48}$$

Therefore, we get Proposition 4.9.

Proof of Proposition 4.10. We have

$$\max_{\partial B_r\cap\partial\omega}|(Z_r^{m*}(\chi_\omega f))(x)|\leq m\,\mathcal{F}\,\textstyle\sum'_{(m*)} G_{I(m*)}\,.$$

We have

$$E(\sum'_{(m^*)} G_{I(m^*)}) \leq (C\lambda^{-1})^{m^*-1}(m^\beta)^{m^*}.$$

We can take sufficiently large T to get the desired result.

6. Summary of the techniques developed in 5

Calculation of multiple product of Green's function in this note is closely related to calculus using Feynman diagram in physist's literature. See for example Zaiman [14].

The author here makes a comment on distinction between our calculation and physicist's calculation. The large distinction between them lies in the following point. We use θ, θ'-parameters as in the proof of Propositions 4.5, 4.7, 4.9. Parameters of such kind, which correspond to management and moderation of accumulated singularity of Green's function, can not be seen in physicist's literature as far as the author knows. The usage of these parameters is a very delicate mathematical technique. Without it we can not make any rigorous mathematical argument when we want to show the important result (Theorem 7). The author thinks that this tehcnique developed in §5 will be a fundamental technique for a variety of problems concerning random media with many randomly distributed obstacles. The author also thinks that (ξ, ξ'), (ξ, ξ', ξ'')-techniques will be useful for other problems.

7. Facts which are used to prove Theorem 4

We assume that $w(m)$ satisfies $\mathcal{O}_1(m)$. We consider $\mathfrak{G}(s)$ given by

$$\mathfrak{G}(s) = \int_{\omega^C} (\sum_I G(x, w_{i_1})\, G_{I(s)})^2\, dx. \tag{7.1}$$

Then, $\|\tilde{\boldsymbol{H}}_{w(m)} f\|_{2,\omega^C}$ does not exceed

$$(\log m)\,(\|Gf_\omega\|^2_{2,\omega^C} + \sum_{s\geq 1} \tau^{2s}\, \mathfrak{G}(s)\, \mathcal{F}^2)^{1/2}.$$

Here we used the inequality with

$$(\sum_i^N a_i)^2 \leq N(\sum_i^N a_i^2)$$

with $N=(\log m)^2$. Note that $G_{I(s)} \equiv 1$ when $s=1$.

We have the following Propositions 7.1, 7.2 which are proved in the section 8. And we easily see that Theorem 4 follow from Propositions 7.1, 7.2.

Proposition 7.1. *Fix $\varepsilon>0$. Then, there exists $T \gg 1$ and $C'>0$ such that* $P(\mathfrak{G}(s) \leq C'(m^\beta/C\lambda)^{2s} m^{3\varepsilon} z_{m,\lambda}$ *hold for any* $1 \leq s \leq (\log m)^2) \geq 1-m^{-\varepsilon}$.

Here $z_{m,\lambda} = m^{-1-\beta}\lambda^2 + m^{-2}\lambda + m^{-(5/3)\beta-2}\lambda^{7/2} + m^{(\beta-5)/2}(\lambda^{1/2} + m^{-(5/3)\beta}\lambda^3 + m^{-\beta}\lambda^2)$.

Proposition 7.2. *Fix any $\varepsilon>0$. Then,*

$$P(\|Gf_\omega\|_{2,\omega^c}\leq C\, m^{2\varepsilon+(\beta-3)/2}\,\mathcal{F} \quad \textit{for any} \quad f\in L^\infty)\geq 1-m^{-\varepsilon}.$$

8. Proof of Propositions 7.1, 7.2

We have the following

Lemma 8.1. *Assume that $w(m)$ satisfies $\mathcal{O}_1(m)$. Then,*

$$\max_i \int_{\omega^C} G(x, w_i)^2\, dx \leq C\, m^{-1}(\log m)^4 \tag{8.1}$$

holds for a constant C independent of m.

Proof. See Lemma 12 in Ozawa [9] observing (8.1)$=U_2^2$ in [9].

Lemma 8.2. *Assume that $w(m)$ satisfies $\mathcal{O}_1(m)$. We put $H_i=\{x;\ |w_i-x|\geq 4m^{-\beta/3}\}$. Then,*

$$\max_i \int_{\omega^C\cap H_i} G(x, w_i)^2\, dx \leq C\, m^{(\beta-5)/2}(\log m)^8$$

holds for a constant C independent of m.

Proof. We modify the proof of Ozawa [9; Lemma 12]. Note that we have only to estimate

$$\sum_r^{(**)} G(w_r, x)^2\, (4\alpha m^{-1}(\log m)^2)^3$$

in [9; 351 p].

Lemma 8.3. *Assume that $w(m)$ satisfies $\mathcal{O}_1(m)$. Then,*

$$\int_{\omega^C\cap(H_i\cap H_j)} G(x, w_i)\, G(x, w_j)\, dx \leq C\, m^{(\beta-5)/2}(\log m)^8\, \Phi_0(\lambda/4, w_i, w_j)\,. \tag{8.2}$$

Proof. Note that $G(x, w_i)G(x, w_j)\leq C\Phi_0(\lambda/4, w_i, w_j)\times G^*(x, w_i; \lambda/4)G^*(x, w_j; \lambda/4)$. Here $G^*(x, y; \lambda/4)$ is the Green function of $-\Delta+(\lambda/4)$ in R^3. Therefore (8.2) does not exceed

$$\Phi_0(\lambda/4, w_i, w_j) \max_i \int_{\omega^C\cap H^i} G^*(x, w_i; \lambda/4)^2\, dx$$

by L^2 inequality. Then, the proof of Lemma 8.3 is reduced to the proof of similar inequality in Lemma 8.2. We have the desired result.

Lemma 8.4. *Assume that $w(m)$ satisfies $\mathcal{O}_1(m)$. Then,*

$$\int_{B_i''\cap\omega^C} G(x, w_i)\, dx \leq C\, m^{-2}(\log m)^6\,. \tag{8.3}$$

Here B_i'' is the ball of radius $4m^{-\beta/3}$ with the center w_i.

Proof. Note that

$$\bigcup_{r=1}^{(\log m)^2} \{x;\ |x-w_r|\leq C'\, m^{-1}(\log m)^2\}$$

covers ω^C. Therefore we get (8.3) observing

$$\int_{|x-w^r|\leq C\,m^{-1}(\log m)^2} G(w_i, x)\,dx \leq C''\,m^{-2}(\log m)^4 .$$

Now we are in a line to prove Proposition 7.1. We devide $\mathfrak{G}(s)$ into some terms. We put

$$P_{ij} = \int_{\omega^C} G(x, w_i)\,G(x, w_j)\,dx .$$

Then, $\mathfrak{G}(s)=\sum_{I,J} P_{i_1 j_1} G_{IJ}=F_1+F_2+F_3$, where

$$F_k = \sum_{I,J}^{(k)} P_{i_1 j_1} G_{IJ}\,, \quad k = 1, 2, 3 .$$

Here the indices I, J in $\sum^{(k)}$ satisfies the following subset of I, J: When $k=1$, $i_1=j_1$. When $k=2$, $i_1\neq j_1$, $|w_{i_1}-w_{j_1}|\leq 9\,\mathrm{m}^{-\beta/3}$. When $k=3$, $i_1\neq j_1$, $|w_{i_1}-w_{j_1}| >9\,m^{-\beta/3}$.

We have the following Lemmas 8.5, 8.6.

Lemma 8.5. *Fix* $\varepsilon>0$. *Then,*

$$P(F_1\leq C'\,m^{2\varepsilon-1-\beta}\,\lambda^2(m^\beta/C\lambda)^{2s} \quad \textit{holds for} \quad 2\leq s\leq(\log m)^2) \geq 1-m^{-\varepsilon} .$$

Proof. We assume that $w(m)$ satisfies $\mathcal{O}_1(m)$. Then,

$$P_{i_1 j_1}\leq \max_i P_{ii}\leq C\,m^{-1}(\log m)^4$$

by Lemma 8.1. Therefore we get

$$F_1\leq C\,m^{-1}(\log m)^4 \sum{}^{(1)} E(G_{IJ}) .$$

By Lemma 3.7 we get the desired result.

Lemma 8.6. *Fix* $\varepsilon>0$. *Then, a similar estimate as in Lemma* 8.5 *holds for* F_2.

Proof. We have to estimate

$$\sum_{\substack{I,J\\ i_1\neq j_1}} G_{IJ}\,\chi_{D_{ij}}\,, \tag{8.4}$$

where $D_{ij}=\{(w_{i_1}, w_{j_1});\ |w_{i_1}-w_{j_1}|\leq 9\,m^{-\beta/3}\}$ and $\chi_{D_{ij}}$ is the characteristic function of the set D_{ij}.

First we treat the sum of all terms L_i in (8.4) where I and J in (8.4) have at least 1-intersection. Then, $E(L_1)\leq C'(m^\beta/C\lambda)^{2s}\,m^{-\beta}\,\lambda^2$. Next we treat the sum of all terms L_2 in (8.4) whose indices I and J have no intersection. Then, $E(L_2) \leq(m^\beta/C\lambda)^{2s}\,\lambda^2|D_{ij}|$. We have $|D_{ij}|\leq C\,m^{-\beta}$. Summing up these facts we get the desired result.

We consider F_3. We devide $P_{i_1 j_1}$ into two parts. We put $K_{i_1 j_1}=B''_{i_1}\cup B''_{j_1}$

and

$$P^{(k)}_{i_1j_1} = \int_{U_{(k)}} G(x, w_{i_1})\, G(x, w_{j_1})\, dx\,,$$

$k=0, 1$, where $U_{(0)}=\omega^C \cap K_{i_1j_1}$, $U_{(1)}=\omega^C \backslash K_{i_1j_1}$. We put

$$F_4 = \sum{}^{(3)} P^{(0)}_{i_1j_1} G_{IJ}$$
$$F_5 = \sum{}^{(3)} P^{(1)}_{i_1j_1} G_{JI}\,.$$

We have the following

Lemma 8.7. *Fix $\varepsilon>0$. Then,*

$$P(F_4 \leq C' m^{3\varepsilon}(m^{-2}\lambda + m^{-(5/3)\beta-2}\lambda^{7/2} \quad \textit{holds}$$
$$\textit{for} \quad 2\leq s\leq (\log m)^2) \geq 1-m^{-\varepsilon}\,.$$

Proof. Let $G^*(x, y; \lambda/4)$ be the Green function of $-\Delta+(\lambda/4)$ in R^3. We consider $P^{(0)}_{i_1j_1}$ when $|w_{i_1}-w_{j_1}|>9\, m^{-\beta/3}$. By Lemma 8.4 we get

$$\begin{aligned}(8.5)\qquad P^{(0)}_{i_1j_1} &\leq C\, G^*(w_{i_1}, w_{j_1}; \lambda 4) \max_{h=i_1,j_1} \int_{B_h''\cap\omega^C} G(x, w_h)\, dx \\ &\leq C' m^{-2}(\log m)^6\, G^*(w_{i_1}, w_{j_1}; \lambda/4)\,.\end{aligned}$$

Here we have used the fact that

$$\max_{x\in B_{i_1}''} G(x, w_{j_1}) \leq C_* G^*(w_{i_1}, w_{j_1}; \lambda/4)\,.$$

Now $F_4 \leq F_6+F_7+F_8$, where

$$F_k = \sum{}^{(h)} P^{(0)}_{i_1j_1} G_{IJ}\,,$$

$k=6, 7, 8$. Here the indices I, J in $\sum^{(6)}$ run over all I, J satisfying $|w_{i_1}-w_{j_1}|>9\, m^{-\beta/3}$, $i_1 \neq j_1$ and i_1 and j_1 are both single. The indices I, J in $\sum^{(7)}$ ($\sum^{(8)}$, respectively) run over all I, J satisfying $|w_{i_1}-w_{j_1}|>9\, \mathrm{m}^{-\beta/3}$ and exactly one of i_1 and j_1 has a pair (both i_1 and j_1 have a partner, respectively).

By (8.5) and Lemma 3.8 (case (1)) we get

$$\begin{aligned}E(F_6) &\leq C' m^{-2}(\log m)^6 \sum_{q\geq 0} (C\lambda^{-1})^{2s-1-(3/2)q}(m^\beta)^{2s-q} \\ &\leq C'(m^\beta/C\lambda)^{2s}\, m^{\varepsilon-2}\,\lambda\,.\end{aligned}$$

It should be noticed that $G^*(w_{i_1}, w_{j_1}, \lambda/4)$ is not equal to $G_{i_1j_1}$. However the estimate is quite similar to the case $G_{i_1j_1}$. By (8.5) and Lemma 3.8 (case (2)) we get the same estimates of $E(F_7)$ as in $E(F_6)$. We have

$$F_8 \leq C' m^{-2+(\beta/3)}\, (\log m)^6 \sum{}^{(8)} G_{IJ}\,,$$

since $|w_{i_1}-w_{j_1}|>9\, \mathrm{m}^{-\beta/3}$ in this case. In this case I and J have at least 2-intersections. Thus, we get

$$P(\sum{}^{(8)} G_{IJ} \leq C\, m^{2\varepsilon-2\beta}\, \lambda^{7/2}(m^\beta/C\lambda)^{2s}) \geq 1-m^{-\varepsilon}$$

using Lemma 3.7. Summing up these facts we get the estimate of F_8. We proved Lemma 8.7.

Lemma 8.8. *Fix* $\varepsilon>0$. *Then,*

$$P(F_5\leq m^{3\varepsilon+(\beta-5)/2}(\lambda^{1/2}+m^{-(5/3)\beta}\lambda^3)(m^\beta/C\lambda)^{2s}$$
$$\textit{holds for any}\quad 2\leq s\leq(\log m)^2)\geq 1-m^{-\varepsilon}.$$

Proof. We assume that $w(m)$ satisfies $\mathcal{O}_1(m)$. By Lemma 8.3 we have $P^{(1)}_{i_1j_1}\leq C\,m^{(\beta-5)/2}(\log m)^8\,\Phi_0(\lambda/8, w_{i_1}, w_{j_1})$. We know that $\Phi_0(\lambda/4, w_{i_1}, w_{j_1})\leq C'\lambda^{-1/2}\Phi_1(\lambda/16, w_{i_1}, w_{j_1})$. Thus,

$$F_5\leq C'(\log m)^8 m^{(\beta-5)/2}\lambda^{-1/2}\sum\nolimits^{(3)}\Phi_1 G_{IJ}\,. \tag{8.6}$$

Note that Φ_1 has a similar form as $G_{i_1j_1}$. Recall that $|w_{i_1}-w_{j_1}|>9\,\mathrm{m}^{-\beta/3}$. We know that $\sum\Phi_1 G_{IJ}$ in (8.6) is divided into three parts. $\sum^{(6)}, \sum^{(7)}, \sum^{(8)}$ in the proof of Lemma 8.7. We can apply the similar argument as in the proof of lemma 8.7 to the sum in (8.6) and we get the desired result.

Summing up Lemmas 8.5, 8.6, 8.7, 8.8 we get the following

Lemma 8.9. *Fix* $\varepsilon>0$. *Then,*

$$P(\mathfrak{G}(s)\leq m^{3\varepsilon}\,\nu_{m,\lambda}(m^\beta/C\lambda)^{2s}$$
$$\textit{holds for any}\quad 2\leq s\leq(\log m)^2)\geq 1-m^{-\varepsilon},$$

where $\nu_{m,\lambda}$ *is*

$$m^{-1-\beta}\lambda^2+m^{-2}\lambda+m^{-(5/3)\beta-2}\lambda^{(7/2)}+m^{(\beta-5)/2}(\lambda^{1/2}+m^{-5\beta/3}\lambda^3)\,.$$

We have the following

Lemma 8.10. *Fix* $\varepsilon>0$. *Then,*

$$P(\mathfrak{G}(1)\leq m^{3\varepsilon}\,\rho_{m,\lambda}(m^\beta/C\lambda)^2\geq 1-m^{-\varepsilon},$$

where $\rho_{m,\lambda}=m^{(\beta-5)/2}(\lambda^{1/2}+m^{-\beta}\lambda^2)$.

Proof. In this case $G_{IJ}\equiv 1$. Thus, we expand $\mathfrak{G}(1)$ as

$$\sum_i P_{ii}+\sum_{\substack{I,J\\ i\neq j}} P_{ij}\,.$$

We have

$$\sum_i P_{ii}\leq C\,m^{\beta-1}(\log m)^4\,.$$

We can devide $\sum_{\substack{i,j\\ i\neq j}} P_{ij}$ as $F_1^*+F_2^*+F_4^*+F_5^*$. Here F_j^* is the term which is obtained by replacing G_{IJ} in F_j by 1. For the case $s=1$ we have the same estimates of F_1^* and F_2^* as in Lemma 8.5, 8.6. For F_4^* we have $F_4^*\leq F_6^*$ and we get

$$E(F_6^*)\leq C'(m^\beta/C\lambda)^2 m^{-2}(\log m)^6\,\lambda\,.$$

We have

$$F_5^* \leq C' m^{(\beta-5)/2} (\log m)^8 \sum_{i,j}^{(3)} \Phi_0(\lambda/16, w_i, w_j)$$

which is estimated by Lemma 3.9. Thus,

$$P(F_5^* \leq m^{\varepsilon}(\lambda^{1/2} + m^{-\beta}\lambda^2)\, m^{(\beta-5)/2}(m^{\beta}/C\lambda)^2) \geq 1 - m^{-\varepsilon}.$$

Summing up these facts we get the desired result.

Summing up Lemmas 8.9, 8.10 we get Proposition 7.1.

Proof of Proposition 7.2. We assume that $\mathcal{O}_1(m)$ holds. Then, ω^C is contained in

$$\bigcup_{r=1}^{(\log m)^2} \{x;\ |x - w_r| \leq C\, m^{-1} (\log m)^2\}.$$

Thus, $|\omega^C| \leq C\, m^{\beta-3} (\log m)^6$. We then get the desired result.

9. The Hilbert-Schmidt norm of $\tilde{H}_{w(m)} - A'$

We treat a modification of calculations presented in [9]. It is suggested by Figari-Olrlandi-Teta [4].

We put $G_{(1)}(x, y) = G(x, y)$,

$$G_{(s+1)}(x, y) = \int_\Omega G_{(s)}(x, w)\, V(w)\, G(w, y)\, dw,$$

$s = 1, 2, \cdots$. We put $p(x, y; w(m))$ as the integral kernel of the operator $\tilde{H}_{w(m)} - A'$. We want to calculate

$$\int_\Omega \int_\Omega p(x, y; w(m))^2\, V(x)\, V(y)\, dx\, dy \tag{9.1}$$

instead of estimating

$$\int_\Omega \int_\Omega p(x, y; w(m))^2\, dx\, dy. \tag{9.2}$$

It is sufficient to calculate a bound for (9.1) to get an estimate of (9.2), since $V \in C^0(\bar{\Omega})$, $V > 0$ on $\bar{\Omega}$,

We see that (9.1) is equal to

$$\sum_{s,t=1}^{m^*} (-4\pi\alpha\rho\, n/m)^{s+t}\, Q(s, t),$$

where $Q(s, t)$ is defined by

$$\begin{aligned} & J_s J_t \int_\Omega G_{(s+t+2)}(x, x)\, V(x)\, dx \\ & - n^{-s} J_t \textstyle\sum_{(s)} \int_\Omega G_{(t+2)}(y, w_{i_1})\, G_{I(s)}\, G(w_{i_s}, y)\, V(y)\, dy \\ & - n^{-t} J_s \textstyle\sum_{(t)} \int_\Omega G_{(s+2)}(y, w_{i_1})\, G_{J(t)}\, G(w_{i_t}, y)\, V(y)\, dy \\ & + n^{-(s+t)} \sum_{I,J} G_{(2)}(w_{i_1}, w_{j_1})\, G_{(2)}(w_{i_s}, w_{j_t})\, G_{I(s)}\, G_{J(t)} \end{aligned} \tag{9.3}$$

for $s, t \geq 2$. We have the same formula for $s, t \geq 1$ using the notational conventions $G_{I(1)} \equiv 1$, Here we used the fact that the indices $i_1, \cdots, i_s, j_1, \cdots, j_t$ in $\sum_{(s)}$, $\sum_{(t)}$ run over self-avoiding sum.

We have

$$(9.4) \qquad E(Q(s, t)) = (J_{s+t} - J_s J_t) \int_\Omega G_{(s+t+2)}(x, x) V(x)\, dx \\ + n^{-(s+t)} E(\textstyle\sum^* G_{(2)}(w_{i_1}, w_{j_1}) G_{(2)}(w_{i_s}, w_{j_t}) G_{IJ}) .$$

Here $\sum^*$ is the sum of terms whose indices I, J have at least one intersection. We know that $J_{s+t} - J_s J_t = -\mathrm{st}\ m^{-\beta} + \cdots = O((\log m)^4 m^{-\beta})$. Thus, the first term in the right hand side of (9.4) does not exceed $O((\log m)^4 m^{-\beta}(C\lambda^{-1})^{s+t+(1/2)})$. We have

$$(9.5) \qquad E(G_{(2)}(w_{i_1}, w_{j_1}) G_{(2)}(w_{i_s}, w_{j_t}) G_{IJ}) \\ = E(G(w_{i_0}, w_{i_1}) G(w_{i_s}, w_{i_{s+1}}) G(w_{j_0}, w_{j_1}) G(w_{j_t}, w_{j_{t+1}}) G_{IJ}) .$$

Here the indices $i_0, i_1, \cdots, i_{s+1}, j_0, \cdots, j_{t+1}$ in (9.5) satisfies $i_0 = j_0$ and $i_{s+1} = j_{t+1}$ and it is of $q+2$ intersections when $(i_1, \cdots, i_s)$ and $(j_1, \cdots, j_t)$ have q intersections. Therefore, we see that

$$E(\textstyle\sum^* \cdots) \leq \sum (m^\beta)^{s+t-q} (C\lambda^{-1})^{s+t+4-(3/2)(q+2)-(1/2)} \\ \leq C'(m^\beta / C\lambda)^{s+t} m^{-\beta} \lambda^1 .$$

by Lemma 3.7.

Summing up these facts we get $\mathrm{E}(9.2)) \leq C\, m^{-\beta} \lambda^1$. We complete our proof of Theorem 5.

10. Proof of Theorem 6

Fix $f \in L^2(\Omega)$. Put $(\tilde{H}_{w(m)} - A')f = g$. Then,

$$E(\langle g, Vg \rangle) = \sum_{s,t=1}^{m^*} (-4\pi\rho\, n/m)^{s+t} E(R_f(s, t)) ,$$

where $R_f(s, t)$ is given by

$$n^{-(s+t)} \sum_{I,J} G_{(2)}(w_{i_1}, w_{j_1}) G_{IJ}(\boldsymbol{G}f)(w_{i_s})(\boldsymbol{G}f)(w_{j_t}) \\ - n^{-s} \sum_I (\boldsymbol{G}(V\boldsymbol{G})^{t+1} f)(w_{i_1}) G_{I(s)}(\boldsymbol{G}f)(w_{i_s}) \\ - n^{-t} \sum_J (\boldsymbol{G}(V\boldsymbol{G})^{s+1} f)(w_{j_1}) G_{J(t)}(\boldsymbol{G}f)(w_{j_t}) \\ + J_s J_t \langle \boldsymbol{G}(V\boldsymbol{G})^{s+t+1} f, f \rangle .$$

Therefore, $E(R_f(s, t))$ is equal to

$$(10.1) \qquad (J_s J_t - J_{s+t}) \langle \boldsymbol{G}(V\boldsymbol{G})^{s+t+1} f, f \rangle \\ + n^{-(s+t)} E(\sum_{I,J}^* G_{(2)}(w_{i_1}, w_{j_1}) G_{IJ}(\boldsymbol{G}f)(w_{i_s})(\boldsymbol{G}f)(w_{i_t})) .$$

Here $\sum^*$ is the sum of terms whose indices I, J have at least one intersection. The first term in (10.1) is $O((\log m)^4 m^{-\beta}(C\lambda)^{-s-t-2}\|f\|_2^2)$ observing $J_s J_t - J_{s+t} = O(m^{-\beta}(\log m)^4)$. We know

$$\|Gf\|_\infty \leq C \min(\lambda^{-1/4}\|f\|_2, \lambda^{-1}\|f\|_\infty).$$

We have

$$\begin{aligned} &E(\sum_{I,J}^* G_{(2)}(w_{i_1}, w_{j_1})\, G_{IJ}) \\ &\leq \sum_q (m^\beta)^{s+t-q}(C\lambda^{-1})^{s+t+2-(3/2)(q+1)-(1/2)} \\ &\leq (m^\beta/C\lambda)^{s+t} m^{-\beta}\lambda^{3/2}. \end{aligned}$$

Therefore, the second term in (10.1) does not exceed $(m^\beta/Cn\lambda)^{s+t} m^{-\beta}\min(\lambda \|f\|_2^2, \lambda^{-1/2}\|f\|_\infty^2)$.

Summing up these facts we get

$$E(\langle g, Vg\rangle) \leq \min(m^{-\beta}\lambda\|f\|_2^2, m^{-\beta}\lambda^{-1/2}\|f\|_\infty^2).$$

We get the desired result.

11. Spectral properties I

We want to deduce spectral properties of $G_{w(m)}$. Let $\lambda_j^{(i)}(w(m))$ $(i=1, 2, 3)$ and $\lambda_j^{(4)}$ be the j-th eigenvalue of the operators $G_{w(m)}$ $(i=1)$, $\tilde{H}_{w(m)}(i=2)$, $\tilde{H}_{w(m)}$ $(i=3)$, $A(i=4)$.

By the spectral theory of operators applied to Theorems 3.5, (2.4), (2.5) we see that the measure of the set $w(m)$ satisfying (11.1), (11.2) and (11.3) tends to 1 as $m\to\infty$. Here

$$|\lambda_j^{(1)}(w(m)) - \lambda_j^{(2)}(w(m))| \leq m^{5\varepsilon} m^{(\beta-5)/4} \tag{11.1}$$

$$|\lambda_j^{(2)}(w(m)) - \lambda_j^{(3)}(w(m))| \leq m^{2\varepsilon}\lambda^{3/4} U_m \tag{11.2}$$

$$|\lambda_j^{(3)}(w(m)) - \lambda_j^{(4)}| \leq m^{2\varepsilon-(\beta/2)}\lambda^{1/2}. \tag{11.3}$$

Here we used $\tilde{H}_{w(m)} - \chi_\omega \tilde{H}_{w(m)}\chi_\omega = (1-\chi_\omega)\tilde{H}_{w(m)}\chi_\omega + \tilde{H}_{w(m)}(1-\chi_\omega)$ and the fact that the adjoint operator of $\tilde{H}_{w(m)}(1-\chi_\omega)$ is equal to $(1-\chi_\omega)\tilde{H}_{w(m)}$ and

$$\|S^*\|_{\mathcal{L}(L^2(\Omega), L^2(\Omega))} = \|S\|_{\mathcal{L}(L^2(\Omega), L^2(\Omega))}$$

for $S = \tilde{H}_{w(m)}(1-\chi_\omega)$.

We put $\mathcal{H}(m) = m^{(\beta-5)/4} + \lambda^{3/4} U_m + m^{-\beta/2}\lambda^{1/2}$. If $w(m)$ satisfies $\mathcal{O}_1(m)$ we say that $w(m)\in\mathcal{O}_1(m)$. We say that $w(m)\in\mathcal{H}$ if $w(m)\in\mathcal{O}_1(m)$ and (11.1), (11.2) and (11.3) holds for $w(m)$.

Hereafter we assume that $V\equiv|\Omega|^{-1}$. We consider the case $\lambda = Tm^{\beta-1}$. Then, $\lambda_j^{(1)}(w(m)) = (\mu_j(w(m)) + Tm^{\beta-1})^{-1}$ and $\lambda_j^{(4)} = (\mu_j(V; m) + Tm^{\beta-1})^{-1}$ with $\mu_j(V; m) = \mu_j + 4\pi\alpha\rho\, m^{\beta-1}|\Omega|^{-1}$.

Assume that $w(m)\in\mathcal{H}$. Then,

(11.4) $$|\mu_j(w(m))-\mu_j(V;m)|\leq C_j\, m^{5\varepsilon+2(\beta-1)}\,\mathcal{H}(m)\,.$$

Here C_j may depend on j.

It is easy to check the following Lemma 11.1 observing

$$m^{2(\beta-1)}\,\mathcal{H}(m)=m^{2(\beta-1)}\,(m^{3(\beta-3)/4}+m^{(7\beta-17)/8}+m^{(\beta-5)/4})+m^{2\beta-(5/2)}\,.$$

Lemma 11.1. *There exists $\varepsilon'>0$ such that $m^{2(\beta-1)}\,\mathcal{H}(m)=O(m^{\beta-1-5\varepsilon'})$ for $\beta\in[1, 3/2)$ and $m^{2(\beta-1)}\,\mathcal{H}(m)=O(m^{-5\varepsilon'})$ for $\beta\in[1, 5/4)$.*

As a corollary we get the following.

Proopsition 11.2. *Fix $\beta\in[1, 3/2)$. Then, there exists $\delta(\beta)>0$ such that*

$$\lim_{m\to\infty} P(w(m);\ |m^{\delta(\beta)-(\beta-1)}(\mu_j(w(m))-(\mu_j+4\pi\alpha\rho\, m^{\beta-1}|\Omega|^{-1}))|<\varepsilon)=1$$

holds for any $\varepsilon>0$.

REMARK: We can take $\delta(\beta)$ as $(3/2)$-β.

The following result is crucial to study fluctuation of $\mu_j(w(m))$ around its mean.

Proposition 11.3. *Fix $\beta\in[1, 5/4)$. Assume that μ_j is simple. Then, the measure of the set $w(m)$ satisfying*

$$\min\,(|\mu_j(w(m))-\mu_{j+1}(w(m))|,\ |\mu_j(w(m))-\mu_{j-1}(w(m))|)>C_0>0$$

tends to 1 *as m tends to infinity.*

Proof. Note that $\mu_j(V;m)$ is simple. Write $\mu_k(w(m))$ as $\mu_k(V;m)+(\mu_k(w(m))-\mu_k(V;m))$. If $\mu_k(w(m))-\mu_k(V;m)=o(1)$ for any fixed k, then $\mu_j(w(m))$ and $\mu_k(w(m))$ are separated as $m\to\infty$. By (11.4) and Lemma 11.1 we see that the measure of the set $w(m)$ satisfying

$$|\mu_k(w(m))-\mu_k(V;m)|<m^{-5\varepsilon'}$$

tends to 1 as m tends to infinity.

Let $\psi_{j,w(m)}$ be the normalized eigenfunction of $\boldsymbol{H}_{w(m)}$ associated with $\lambda_j^{(3)}(w(m))$. Then, we have the following.

Proposition 11.4. *Under the same assumption as in Proposition* 11.3 *we see that the measure of the set $w(m)$ satisfying*

$$\|\psi_{j,w(m)}-\varphi_j\|_2=O(m^{2(\beta-1)+\varepsilon-(\beta/2)}\,\lambda^{-1/4})=O(m^{-5\varepsilon'})\,.$$

Proof. Fix j. By the eigenfunction expansion we have

(11.5) $$\begin{aligned}&\|(\tilde{\boldsymbol{H}}_{w(m)}-\lambda_j^{(4)})\,\varphi_j\|_2^2\\ &=\sum_{k=1}|\mu_k(w(m))-\lambda_j^{(4)}|^2\,|\langle\psi_{k,w(m)},\varphi_j\rangle|^2\\ &=\|(\tilde{\boldsymbol{H}}_{w(m)}-\boldsymbol{A}')\,\varphi_j\|_2^2+O((\log m)^8 m^{-2\beta}\,\lambda^{-2})\end{aligned}$$

observing (2.5). By Theorem 6 we have

$$(11.5) \leq C\, m^{4\varepsilon-\beta}\, \lambda^{-1/2}\,.$$

Since we assume $\beta \in [1, 5/4)$ we see that there exists a constant $c_0 > 0$ such that $|\lambda_k^{(3)}(w(m)) - \lambda_j^{(4)}| \geq c_0 m^{-2(\beta-1)}$ holds for any $k (k \neq j)$. which is a conclusion of (11.4) and Proposition 11.3. Therefore,

$$\sum_{\substack{k=1 \\ k \neq j}} |\langle \psi_{k,w(m)}, \varphi_j \rangle|^2 \leq C\, m^{4(\beta-1)}(11.5)\,.$$

Thus,

$$\|\varphi_j - \langle \psi_{j,w(m)}, \varphi_j \rangle\, \psi_{j,w(m)}\|_2 = O(m^{2(\beta-1)}\, m^{2\varepsilon-(\beta/2)}\, \lambda^{-1/4})\,.$$

We get the desired result.

12. Spectral properties II

In this section we prove the following result which is a generalization of Figari-Orlandi-Teta's result for $\beta > 1$.

Theorem 12.1. *Fix $\beta \in [1, 3/2)$. The random variable*

$$\Lambda(w(m)) = m^{1-(\beta/2)}\, (\lambda_j^{(4)})^{-2} (\varphi_j, (\boldsymbol{H}_{w(m)} - \boldsymbol{A}')\, \varphi_j)_{L^2}$$

tends in distribution to Gaussian random variable $\prod_j$ of mean $E(\prod_j) = 0$ and variance $E(\prod_j^2)$ in Theorem 1..

Proof of Theorem 12.1 is divided into two parts. The first part is the following.

Proposition 12.2. *We take sufficiently large T and we fix it. Then, the random variable given by $U_*(w(m)) = m^{\beta/2}(\varphi_j, (\boldsymbol{H}_{w(m)} - \boldsymbol{A}')\, \varphi_j)_{L^2}$ has mean $E(U_*(\cdot)) = 0$ and variance*

$$\begin{aligned} &E(U_*(\cdot)^2) \\ &\quad = -(4\pi\rho\alpha)^2 m^{2(\beta-1)}((\boldsymbol{A}\varphi_j, \boldsymbol{V}\boldsymbol{A}\varphi_j)_{L^2}^2 - (\boldsymbol{A}\varphi_j\, \boldsymbol{A}\varphi_j, \boldsymbol{V}\boldsymbol{A}\varphi_j\, \boldsymbol{A}\varphi_j)_{L^2}) \\ &\qquad + O(\mathcal{P}(m))\,, \end{aligned}$$

where $\mathcal{P}(m) = m^{2-3\beta}(\log m)^6 + m^{(1-3\beta)/2}(\log m)^8 + m^{-1}(\log m)^8$.

Proof. See Proposition 6 in [9] which is a generalization of the result in Figari-Orlandi-Teta [4].

As a corollary we have the following.

Proposition 12.3. *The random variable $\Lambda(w(m))$ have mean zero and variance*

$$\begin{aligned} E(\Lambda(\cdot)^2) &= (4\pi\alpha\rho)^2 |\Omega|^{-1} \Big(\int_\Omega \varphi_j(x)^4\, dx - |\Omega|^{-1}\Big) \\ &\quad + O((\lambda_j^{(4)})^{-4} m^{2-2\beta}\, \mathcal{P}(m))\,. \end{aligned}$$

The second part of our proof of Theorem 12.1 *is the following result. We put*

$$\Theta_j(\cdot) = (4\pi\alpha\rho)\, n^{-1/2} \sum_{i=1}^{n} (\varphi_j(w_i)^2 - |\Omega|^{-1})\,.$$

Proposition 12.4. *We have*

$$E(\Lambda(\cdot)\,\Theta_j(\cdot))$$
$$= (4\pi\alpha\rho)^2\, |\Omega|^{-1}(|\Omega|^{-1} - \int_\Omega \varphi_j(w)^4\, dw) + O(m^{-\beta}(\log m)^6)\,.$$

As a corollary we have

Proposition 12.5.

$$E((\Lambda(\cdot)+\Theta_j(\cdot))^2) = O(m^{-\beta}(\log m)^6 + (\lambda_j^{(4)})^{-4} m^{2-2\beta}\, \mathcal{P}(m))\,.$$

REMARK. The right hand side of the above equation is $O(m^{-\beta}(\log m)^6 + m^{(\beta-3)/2}(\log m)^8 + m^{2\beta-3}$—$(\log m)^8)$.

Proof of Proposition 12.5. We have

$$E(\Theta_j(\cdot)^2) = (4\pi\alpha\rho)^2 |\Omega|^{-1}(\int_\Omega \varphi_j(w)^4\, dx - |\Omega|^{-1})\,.$$

Thus, we get the desired result.

Proposition 12.6. *Fix $\beta \in [1, 3/2)$. Then, the random variable $\Lambda(\cdot)$ tends in distribution to Gaussian random variable $\prod_j$ of Theorem* 1.

Proof. We know that $E((\Lambda(\cdot)+\Theta_j(\cdot))^2) \to 0$ for $\beta \in [1, 3/2)$. Thus, we get the desired result.

Proof of Proposition 12.4. For the sake of simplicity we write φ_j as φ and μ_j as μ. We have $E(\langle\varphi, (\boldsymbol{H}_{w(m)} - \boldsymbol{A}')\,\varphi\rangle) = 0$. Thus, $E(\Lambda(\cdot)\,\Theta_j(\cdot)) = m^{1-(\beta/2)} (\lambda_j^{(4)})^{-2}((*)_1 + (*)_2)$, where

$$(*)_1 = (4\pi\alpha\rho)\, n^{-1/2} \sum_{\nu=1}^{n} E((\varphi, \tilde{\boldsymbol{H}}_{w(m)}\, \varphi)\, \varphi(w_\nu)^2)$$

and

$$(*)_2 = -(4\pi\alpha\rho)\, n^{-1/2} \sum_{\nu=1}^{n} E((\varphi, \boldsymbol{A}'\, \varphi)\, \varphi(w_\nu)^2)\,.$$

There are two cases (i), (ii) in the indices I in $\tilde{\boldsymbol{H}}_{w(m)}$ and ν.

(i) $i_h = \nu$ for some h and $i_k \neq \nu$ for other $k \neq h$.

(ii) $i_1, \cdots, i_s$ in I and ν are all distinct.

We fix ν. We write

$$(\varphi, \tilde{\boldsymbol{H}}_{w(m)}\, \varphi) = \textstyle\sum_{1,\nu} + \sum_{2,\nu} \tag{12.1}$$

where $\sum_{1,\nu}$ ($\sum_{2,\nu}$; respectively) represents sum of terms in the left hand side of (12.1) whose indices $i_1, \cdots, i_s$ run over the case (i), ((ii), respectively). Put $E(\varphi(\cdot)^4) = E_4$. Then,

$(12.2)_\nu$ $\quad E(\varphi(w_\nu)^2 \sum_{1,\nu})$

$$
\begin{aligned}
&= -(4\pi\alpha\rho/m)(\lambda+\mu)^{-2} E_4 \\
&\quad +(4\pi\alpha\rho/m)(\lambda+\mu)^{-3}\, 2|\Omega|^{-1} E_4(n-1) \\
&\quad +(4\pi\alpha\rho/m)^k(\lambda+\mu)^{-(k+1)}\, k|\Omega|^{-1} E_4(n-1)\cdots(n-(k-1))+\cdots .
\end{aligned}
$$

Each term in $(12.2)_\nu$ comes from

$$
(12.3) \qquad E((G\varphi)(w_{i_1})\, G_I(G\varphi)(w_{i_s})\, \varphi(w_\nu)^2)\, .
$$

Note that the number of pairs such that the case (i) holds in (12.3) is $k(n-1)\cdots(n-(k-1))$. Therefore, we have $(12.2)_\nu$.
As a conclusion we get

$$
\begin{aligned}
(12.4) \qquad & \sum_{\nu=1}^{n} (10.6)_\nu \\
&= \sum_{s=1}^{(\log m)^2} (-4\pi\alpha\rho/m)^s(\lambda+\mu)^{-(s+1)}\, s|\Omega|^{-(s-1)}\, m^{\beta s} E_4 \\
&\quad +0(m^{-\beta}\sum_{s=2}^{m^*}(4\pi\alpha\rho/m)^s(\lambda+\mu)^{-(s+1)}\, m^{\beta s} s^3) \\
&= -4\pi\alpha\rho\, m^{\beta-1}(\lambda+\mu+4\pi\alpha\rho\, m^{\beta-1}|\Omega|^{-1})^{-2} E_4+0(m^{-2\beta+1}(\log m)^6)
\end{aligned}
$$

using $\lambda=\mathrm{T}m^{\beta-1}$.

Next we consider $E(\varphi(w_\nu)^2(\Sigma_{2,\nu}-(\varphi, A'\varphi)))$. It is easy to see that

$$
\begin{aligned}
(12.5) \qquad & \sum_{\nu=1}^{n} E(\varphi(w_\nu)^2\, \Sigma_{2,\nu}) = \sum_{\nu=1}^{n} E((\varphi, G\varphi)\, \varphi(w_\nu)^2) \\
&\quad +\sum_{s=1}^{m^*}\sum_{\nu=1}^{n}(-4\pi\alpha\rho/m)^s(\lambda+\mu)^{-(s+1)}|\Omega|^{-(s-1)} \\
&\quad \times(n-1)\cdots(n-s)\, E(\varphi(\cdot)^2)^2 \\
&= m^\beta E(\varphi(\cdot)^2)(\varphi, G\varphi)+m^\beta|\Omega|^{-1}\sum_{s=1}^{m^*}(-4\pi\alpha\rho\, m^{\beta-1}/|\Omega|)^s J_{s+1}\, .
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
(12.6) \qquad & \sum_{\nu=1}^{n} E(\varphi(w_\nu)^2(\varphi, A'\varphi)) = \sum_{\nu=1}^{n} E(\varphi(w_\nu)^2(\varphi, G\varphi)) \\
&\quad +m^\beta\sum_{s=1}^{(\log m)^2}(-4\pi\alpha\rho\, m^{\beta-1})^s(\lambda+\mu)^{-(s+1)}|\Omega|^{-s} J_s\, E(\varphi(\cdot)^2)\, .
\end{aligned}
$$

Comparing the terms J_{s+1} and J_s in (12.5), (12.6), we have

$$
\begin{aligned}
(12.7) \qquad & \sum_{\nu=1}^{n} E(\varphi(w_\nu)^2(\Sigma_{2,\nu}-(\varphi, A'\varphi))) \\
&= 4\pi\alpha\rho\, m^{\beta-1}(\lambda+\mu+4\pi\alpha\rho\, m^{\beta-1}|\Omega|^{-1})^{-2} E(\varphi(\cdot)^2)|\Omega|^{-1} \\
&\quad +O(m^{-2\beta+1}(\log m)^4)
\end{aligned}
$$

noticing $J_{s+1}-J_s=-m^{-\beta}s+O((\log m)^4 m^{-2\beta})$. Notice that we used $\lambda=Tm^{\beta-1}$.

Summing up (12.4) and (12.6) we get $E(\Lambda(\cdot)\Theta_j(\cdot))=(4\pi\alpha\rho)^2(-E_4+|\Omega|^{-2})+O(m^{-\beta}(\log m)^6)$.

13. Fluctuation of spectra

We are now in a position to prove Theorem 1.

Fix $\beta\in[1, 13/9)$. By (11.1) and (11.2) we see that the measure of the set $w(m)$ satisfying

$$(13.1) \qquad |\lambda_j^{(1)}(w(m))-\lambda_j^{(2)}(w(m))|+|\lambda_j^{(2)}(w(m))-\lambda_j^{(3)}(w(m))| = o(m^{-(\beta/2)-(\beta-1)})$$

tends to 1 as m tends to infinity. The following result is a direct consequence of Theorem 12.1 and which tells that

$$(\lambda_j^{(3)}(w(m))-\lambda_j^{(4)}(w(m)))\, m^{-1+(3\beta/2)}$$

has non trivial distribution as m tends to infinity.

Proposition 13.1. *Fix $\beta\in[1, 5/4)$. By $\varepsilon_{w(m)}$ we denote the random variable*

$$m^{1-(\beta/2)}(\lambda_j^{(4)})^{-2}\,(\lambda_j^{(3)}(w(m))-\lambda_j^{(4)})\,.$$

Then, $\varepsilon_{w(m)}$ tends in distribution to Gaussian random variabole Π_j of mean 0 *and variance $E(\Pi_j^2)$ in Theorem* 1.

Proof. Note that $\beta\in[1, 5/4)$. Then, Proposition 11.3 holds. For the sake of simplicity we write φ_j as φ and $\psi_{j,w(m)}$ as $\psi_{w(m)}$. We have

$$(13.2) \qquad \begin{aligned}&(\lambda_j^{(3)}(w(m))-\lambda_j^{(4)})-(\varphi, (\boldsymbol{H}_{w(m)}-\boldsymbol{A}')\,\varphi)\\ &= (\varphi, (\boldsymbol{H}_{w(m)}-\boldsymbol{A}')\,\varphi)\,((\varphi, \psi_{w(m)})_{L^2}^{-1}-1)\\ &\quad +(\psi_{w(m)}-\varphi, (\boldsymbol{H}_{w(m)}-\boldsymbol{A}')\,\varphi)_{L^2}\,(\varphi, \psi_{w(m)})_{L^2}^{-1}\,.\end{aligned}$$

By Proposition 11.4 and Theorem 6 we see that the measure of the set $w(m)$ such that the first term in the right hand side of (13.2) does not exceed

$$(13.3) \qquad m^{3\varepsilon-(\beta/2)}\,\lambda^{-1/4}\,m^{2\varepsilon+2(\beta-1)-(\beta/2)}\,\lambda^{-1/4}$$

tends to 1 as m tends to infinity for any $\varepsilon>0$. We see that (13.3) does not exceed $m^{1-(3\beta/2)-\delta}$ for some $\delta>0$ when $\beta\in[1, 5/4)$. And we know from Proposition 12.3 that $(\lambda_j^{(3)}(w(m))-\lambda_j^{(4)})\, m^{(2\beta/2)-1}$ has limit distribution as m tends to infinity. We get Proposition 13.1 observing Proposition 12.3.

Summing up Proposition 13.1, (1.31) we get Theorem 1.

14. Comment

We consider the case $V(x)\equiv|\Omega|^{-1}$. If L^4 norm of the j th normalized eigenfunction and the $j+1$ th or the $j-1$ th normalized eigenfunction are distinct, then we know that a phenomena of transition of fluctuation of spectre occurs, that is, we do not have Theorem 1 for $\beta>2$. See [11]. It is very intersting to give complete statistical properties of $\mu_j(w(m))$ for any $\beta\in[1, 3)$.

15. On the previous literature of the author

In [9] there is a lack of terms of $\tilde{I}_r^s g$ in rearrangement of $H_{w(m)}\, g$.

The formula (36) in [9] is not correct. Mistake lies in the same reason as in important remark below Theorem 4 in the present paper. We can use Hilbert-Schmidt norm to get the correct estimate which is given in section 8 of this note.

Appendix

Proof of Lemma 2.1. We will show that the measure of the set $w(m)$ such that $\mathcal{O}_1(m)$ does not hold is less than m^{-N} for any N. If $\mathcal{O}_1(m)$ does not hold, then there exists a ball of radius $m^{-\beta/3}$ with center $\eta_m \in \Omega$ such that the number of $i \in (1, \cdots, m^\beta)$ satisfying $|w_i - \eta_m| \leq 2m^{-\beta/3}$ exceeds $(\log m)^2$. Then, there exists a ball of radius $m^{-\beta/3}(\log m)^{-k}$ with center η_m^* such that the number of $i \in (1, \cdots, m^\beta)$ satisfying $|w_i - \eta_m^*| \leq m^{-\beta/3}(\log m)^{-k}$ exceed $C(\log m)^{2-3k}$ for $k>0$. We used Dirichlet's Schbfachverfahren.

We count the probability such that $w_{i_k}(k=1, \cdots, M)$ for some $i_1, \cdots, i_M$ are in $B(\delta; \eta)$ for some $\eta \in \Omega$ with $\delta \ll 1$. It does not exceed $(m^\beta)^M(4\pi\delta/3|\Omega|)^{3(M-1)}$. Thus, $P(\mathcal{O}_1(m)$ does not hold$) \leq (m^\beta)^M(C'm^{-\beta/3}(\log m)^{-1/4})^{3(M-1)}$ with $M = C(\log m)^{2-3k}$, if we take $k=1/4$.

Proof of Lemmas 3.4, 3.5, 3.6. Recall the notations in Lemma 3.2. In general B_r and B_i may have an intersection. We put $\mathrm{Cone}(B_r \backslash B_i) = \{y = \theta w_r + (1-\theta)x;\ x \in \partial B_r \backslash B_i,\ \theta \in [0, 1]\}$. By a simple geometrical observation we see that there exists a constant C independent of w_r, w_i, m such that $|w_r - w_i| \leq C\,\mathrm{dist}(\mathrm{Cone}(B_r \backslash B_i), w_i)$ holds. We see that the left hand side of (3.2) does not exceed

$$C_\xi(\alpha/m)^\xi \|G(\cdot, w_i)\|_{C^\xi(F)},$$

where $F = \mathrm{Cone}(B_r \backslash B_i)$ and $C^\xi(F)$ denote the usual Hölder space. Thus, we get Lemma 3.6 for (3.2).

We want to show Lemma 3.6 for (3.3). Take r such that $\partial B_r \cap \partial\omega \neq \phi$. Then, $\mathrm{dist}(w_r, \partial\omega \cap \partial\Omega) \geq (\alpha/m)$. By a simple geometrical observation using $\mathcal{O}_1(m)$, we prove the following: Fix r and i. Then, there exists a constant $C'>1$ independent of m, r, i such that we can take $w^*(r, i) \in \partial\omega \cap \partial\Omega$ satisfying

$$\mathrm{dist}(w_r, w^*(r, i)) \leq C'(\log m)^2\, \mathrm{dist}(w_r, \partial\Omega \backslash B_r)$$
$$\mathrm{dist}(w_i, S_{r,i}) \geq (C')^{-1}|w_r - w_i|,$$

where

$$S_{r,i} = \bigcup_{0 \leq \theta \leq 1} \{\theta w_r + (1-\theta)\, w^*(r, i)\}.$$

Take $\tilde{w}(r) \in \partial\Omega \backslash B_r$ such that $\mathrm{dist}(w_r, \tilde{w}(r)) = \mathrm{dist}(w_r, \partial\Omega \backslash B_r)$. Then,

$$\text{(A.1)} \quad |w_r - \tilde{w}(r)|^{-1} G(w_r, w_i) \leq C(\log m)^2 |w_r - w^*(r, i)|^{-1}\, |G(w_r, w_i) - G(w^*(r, i), w_i)|$$

$$\leq C(\log m)^2 (\alpha/m)^{\xi-1} |w_r - w^*(r, i)|^{-\xi} \times$$
$$|G(w_r, w_i) - G(w^*(r, i), w_i)| \,.$$

We see that

$$\text{(A.2)} \qquad ||G(\cdot, w_i)||_{C^\xi(S(r,i))} \leq C\Phi_{1+\xi}(\lambda/4, w_i, w_r)$$

holds. By a simple observation on the boundary behaviour of Green's function we have

$$\text{(A.3)} \qquad \max_{x \in \partial B_r \cap \partial \omega} |S(x, w_r)| \operatorname{dist}(w_r, \tilde{w}(r)) \leq C \,.$$

by (A.1), (A.2) and (A.3) we get Lemma 3.6 for (3.3).

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