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ON EMBEDDED PRIMARY COMPONENTS

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Introduction

Throughout this paper all rings will be commutative with identities and R will always denote a Noetherian local domain with maximal ideal M.

In section one, we assume that depth $R=1$, (Krull) dim $R>1$ and the integral closure of R is a finite R-module. It is well known that a non-zero principal ideal aR $(\neq R)$ has an embedded prime divisor M. Also, see [2, §5]. More generally, we consider the reason of the occurrence of an embedded primary component.

In section two, we assume that depth $R=d<\dim R$ and R is a Nagata local domain satisfying the demension formula. In treating this case, we can reduce to the case that depth $R=1$, using the theory of Rees rings. Hence we will study an embedded primary component in this manner.

Our general reference for undefined terminology is [4].

1. The case of Rings of depth one

Throughout this section, (R, M) denotes a Noetherian local domian such that depth $R=1$, dim $R<1$ and the integral closure $\bar{R}$ is a finite R-module. For an element α of the quotient field of R, we put $I_\alpha=\{x\in R/\alpha x\in R\}$. Moreover, we put

$$A = \{\alpha\in\bar{R}/I_\alpha\supset M^l \text{ for some positive integer } l\} .$$

From [1, 3. 24], it follows immediately that depth $R=1$ if and only if $I_\alpha=M$ for some element α of the quotient field of R. From [3, Exercise 3, p. 12] and $\dim R>1$, we have $\alpha\in\bar{R}$. Hence $\alpha\in A$ and $\alpha\notin R$. Thus $A\neq R$. Also it follows that A is an intermediate ring between R and $\bar{R}$. In fact, for any α, $\beta\in A$, there exist positive integers l and k such that $I_\alpha\supset M^l$ and $I_\beta\supset M^k$. Since $I_{\alpha+\beta}\supset I_\alpha\cdot I_\beta$ and $I_{\alpha\beta}\supset I_\alpha\cdot I_\beta$, we have $I_{\alpha+\beta}\supset M^{l+k}$ and $I_{\alpha\beta}\supset M^{l+k}$. Hence $\alpha+\beta\in A$ and $\alpha\beta\in A$. Moreover, the conductor ideal $\mathfrak{c}(A/R)=R\colon A$ is an M-primary ideal and A is the largest ring among the set $\{B/B$ is an intermediate ring between R and $\bar{R}$ such that $\mathfrak{c}(B/R)$ is M-primary$\}$. For, since $A=R\alpha_1+\cdots+R\alpha_n$ for some elements $\alpha_1, \cdots, \alpha_n$, there exist natural numbers l_i $(1\leq i\leq n)$

such that $I_{\alpha_i}\supset M^{l_i}$. Put $l=l_1+\cdots+l_n$. We have $M^lA\subset R$, that is, $M^l\subset \mathfrak{c}(A/R)$. Hence $\mathfrak{c}(A/R)$ is M-primary. Let B be an intermediate ring between R and $\bar{R}$ and $\mathfrak{c}(B/R)$ be M-primary. Since $M^lB\subset R$ for some integer l, we have $I_b\supset M^l$ for any element b of B. From the definition of A, it follows that $b\in A$, that is, $B\subset A$.

First we recall the following definitions.

DEFINITION. (1) Let I be an ideal of R. I is called *contractible* if $J\cap R=I$ for some intermediate ring $B\,(\neq R)$ between R and A and some ideal J of B.

(2) Let I be an ideal of R. Put $R(I)=\{\alpha\in A/\alpha I\subset I\}$.
This ring $R(I)$ is called the *coefficient ring of I*.

(3) Put $I_R^{-1}=\{\alpha\in A/\alpha I\subset R\}$.

REMARK. Let $I\,(\neq R)$ be an ideal of R. Then $I_R^{-1}\supsetneqq R$. In fact, since $A\neq R$, there exists an element $\alpha\in A$ such that $I_\alpha=M$. Hence $\alpha I\subset R$.

Lemma 1. *Let I be an ideal of R. Then $I=J\cap R$ for some ideal J of A if and only if $IA\cap R=I$. Moreover, if these conditions are satisfied, $I_R^{-1}=R(I)$. (Consequently, I_R^{-1} is an intermediate ring between R and A.)*

Proof. The first statement is easy and so the second remains to be proved. We assume that $IA\cap R=I$. Take any element α of I_R^{-1}. Then $\alpha I\subset IA\cap R=I$. Hence $\alpha\in R(I)$. Thus $I_R^{-1}\subset R(I)$. Clearly $R(I)\subset I_R^{-1}$, which implies $I_R^{-1}=R(I)$.

Proposition 2. *Let $I\,(\neq R)$ be an ideal of R. Then I is not contractible if and only if $R(I)=R$.*

Proof. First, we prove the "only if" part. Put $B=R(I)$. Suppose that $B\supsetneqq R$. Since I is also an ideal of B, we have $IB\cap R=I$. Thus I is contractible. This contradicts the assumption.

Conversely, suppose that I is contractible. So there exists an intermediate ring $B\,(\neq R)$ between R and A such that $J\cap R=I$ for some ideal J in B. Clearly $IB\cap R=I$. Put $C=\{\beta\in B/\beta I\subset R\}$. Then $R\subsetneqq C\subset R(I)$. In fact, there exists an element $\alpha\in B-R$. Since I_α is an M-primary ideal, there exists some element a of R such that $M=I_\alpha : aR=I_{a\alpha}$ and so we can take $a\alpha$ instead of α. Since $I\subset M$, $\alpha\in C$ and $\alpha\notin R$. Since $CI\subset IB\cap R=I$, $C\subset R(I)$. Thus $R\subsetneqq C\subset R(I)$. The proof is complete.

Proposition 3. *Let I be an ideal of R and let $I=Q_1\cap\cdots\cap Q_t$ be an irredundant primary decomposition of I where Q_i is a P_i-primary ideal for $i=1,\cdots,t$. If $P_i\subsetneqq M$ for every $i\,(1\leq i\leq t)$, then $IA\cap R=I$.*

Proof. It is clear that $I\subset IA\cap R$. We shall prove that $IA\cap R\subset I$. Since $P_i\subsetneqq M$, we see that $P_i\not\supset \mathfrak{c}(A/R)$. Hence $R_{P_i}=A_{P_i}$ for $1\leq i\leq t$. Thus

$(IA\cap R)_{P_i}=IR_{P_i}\subset Q_iR_{P_i}$ and so $IA\cap R\subset Q_i$ for $1\leq i\leq t$. Consequently $IA\cap R\subset I$.

Theorem 4. *Let I be an ideal of R with* height $I<\dim R$. *If $R(I)=R$, then I has an embedded M-primary component.*

Proof. Suppose that I has no embedded M-primary components. From Proposition 3, we have $IA\cap R=I$. By Lemma 1, we have $I_R^{-1}=R(I)$. Since $I_R^{-1}\supsetneqq R$ by Remark, it contradicts the assumption. The proof is complete.

More precisely, Theorem 4 can be stated as follows:

Theorem 5. *Let I be an ideal of R with* height $I<\dim R$. *Also, let $I=Q_1\cap Q_2\cap\cdots\cap Q_t\cap Q$ be an irredundant primary decomposition, where Q_i is P_i-primary $(i=1,\cdots,t)$ and $P_i\neq M\,(i=1,\cdots,t)$. If $R(I)=R$, then Q is an M-primary ideal such that $R(Q)=R$.*

Proof. By Theorem 4, an M-primary component Q must occur in the primary decomposition. Put $J=Q_1\cap\cdots\cap Q_t$. By Proposition 3, we have $JA\cap R=J$. So $J_R^{-1}=R(J)$ by Lemma 1. Suppose that $R(Q)\supsetneqq R$. Then we claim that there exists an element $\alpha\in R(Q)-R$ such that $I_\alpha=M$. Since I_α is M-primary, there exists some element a of R such that $M=I_\alpha: aR$. On the other hand, $I_\alpha: aR=I_{a\alpha}$ and so we can take $a\alpha$ instead of α. By this claim, we see that $I_\alpha\supset J$ and so $\alpha J\subset R$. Thus $\alpha\in J_R^{-1}=R(J)$. Since $\alpha\in R(J)\cap R(Q)\subset R(I)$, it follows that $R(I)\supsetneqq R$. This contradicts the assumption. Hence $R(Q)=R$.

REMARK. We can give another proof of the following well-known result:
Let $a\neq 0$ be a non-unit element of R. Then aR has M as an embedded prime divisor. In fact, since $R(aR)=R$ and height $(aR)\leqq 1$, it follows from Theorem 4.

3. The Rees Rings and embedded primary components

Throughout this section, (R, M) denotes a Nagata local domain satisfying the dimension formula and depth $R=d<\dim R=n$.

We recall the following two definitions:

DEFINITION. A Noetherian domain R satisfies the *dimension formula* if for any finitely generated extension domain T of R, and for any $Q\in\operatorname{Spec} T$ with $P=Q\cap R$, we have height $P+\text{tr.deg}_R T=$ height $Q+\text{tr.deg}_{R/P}(T/Q)$. Here $\text{tr.deg}_A B$ is the transcendence degree of the quotient field of a domain B over that of a subdomain A of B.

DEFINITION (cf. [4, (31.A)]. A ring B is a *Nagata ring* if it is Noetherian and if, for any finite extension L of the quotient field of B/P, the integral closure of B/P in L is a finite B/P-module for every $P\in\operatorname{Spec} B$.

Let $a_1, \cdots, a_d$ be a maximal regular sequence of elements in R and write $I=(a_1, \cdots, a_d)$. Then $\operatorname{Ass}_R(R/I)=\operatorname{Ass}_T(R/I^l)$ for all $l>0$ (cf. [3, p. 103, Exercise 13]). Since $M\in\operatorname{Ass}_R(R/I)$, we put $\operatorname{Ass}_R(R/I)=\{p_1, \cdots, p_u, M\}$. Let $I^l=q_{1,l}\cap q_{2,l}\cap\cdots\cap q_{u,l}\cap Q_l$ be an irredundant primary decomposition where the $q_{1,l}$ is p_i-primary and Q_l is M-primary. Put $J_l=q_{1,l}\cap q_{2,l}\cap\cdots\cap q_{u,l}$. J_l is independent of the irredundant primary decomposition of I^l. In fact, $J_l=I^l\cap R[1/a]\cup R$ for some $a\in M-\bigcup_{i=1}^{u} p_i$. Thus $J_lJ_m\subset J_{l+m}$. Let A be the Rees ring of R with respect to I, that is, the ring $A=R[t^{-1}, It]$ with an indeterminate t. Put $\tilde{A}=R[t^{-1}]\oplus(\oplus_{l>0}J_lt^l)$. $\tilde{A}$ is a $\mathbf{Z}$-graded ring containing A. Since R is a Nagata ring, A and $\tilde{A}$ are also Nagata rings by [4, (31. H)]. In the following let A, $\tilde{A}$, I, I^l and J_l be as above.

Proposition 6. *$\tilde{A}$ is integral over A.*

Proof. Let $\bar{A}$ be the integral closure of A. Since $\bar{A}$ is a Krull domain, we have $\bar{A}=\cap\bar{A}_{\bar{P}}$, the intersection being taken over all $\bar{P}\in\operatorname{Ht}_1(\bar{A})$ where $\operatorname{Ht}_1(\bar{A})$ denotes the set of all prime ideals of height one in $\bar{A}$. Put $P=\bar{P}\cap A$ for $\bar{P}\in\operatorname{Ht}_1(\bar{A})$. Since R satisfies the dimension formula and A is a finitely generated R-algebra, it follows that A satisfies the dimension formula. Hence $P\in\operatorname{Ht}_1(A)$. Put $P\cap R=p$. We shall prove that $\tilde{A}\subset\bar{A}_{\bar{P}}$ for any $\bar{P}\in\operatorname{Ht}_1(\bar{A})$. First, we consider the case that $t^{-1}\in P$. Using the dimension formula, we have height $p=\operatorname{tr.deg}_{R/p}(A/P)$. Since $t^{-1}\in P$, it follows that $P\supset I=(a_1, \cdots, a_d)$. Hence $p\supset I$. Thus height $p\geqq$ height $I=d$. Since $I=(a_1, \cdots, a_d)$ and $a_1, \cdots, a_d$ is a regular sequence, it follows that $\oplus_{i\geq 0} I^i/I^{i+1}\cong(R/I)[X_1, \cdots, X_d]$, where $X_1, \cdots, X_d$ are indeterminates over R/I. We see that the canonical homomorphism $A/t^{-1}A=\oplus_{i\geq 0}I^i/I^{i+1}\to A/P$ is surjective, and so height $p=\operatorname{tr.deg}_{R/p}A/P\leq\operatorname{tr.deg}_{R/p}(R/p)[X_1, \cdots, X_d]=d$. Hence height $p=d$. Since height $M=n>d$, we see that $M\supsetneqq p$. Therefore $(I^l)_p=(J_l)_p$. Since $A_p=R[t^{-1}]_p\oplus(\oplus_{l>0}(I^l)_pt^l)=R[t^{-1}]_p\oplus(\oplus_{l>0}(J_l)_pt^l)=\tilde{A}_p$, we have $\bar{A}_{\bar{P}}\supset\tilde{A}_p$. Next, we consider the case that $t^{-1}\notin P$. Since $\tilde{A}=R[t^{-1}]\oplus(\oplus_{l>0}J_lt^l)$ by definition, $R_p[t, t^{-1}]\supset\tilde{A}$. Since $t^{-1}\notin P$, we have $A_P\supset R_p[t, t^{-1}]$. Thus $\tilde{A}\subset A_P\subset\bar{A}_{\bar{P}}$. Hence $\tilde{A}\subset\bigcap_{\bar{P}\in\operatorname{Ht}_1(\bar{A})}\bar{A}_{\bar{P}}=\bar{A}$. Therefore $\tilde{A}$ is integral over A. The proof is complete.

Put $\bar{A}_R=\bar{A}\cap R[t, t^{-1}]$.

Lemma 7. *$\tilde{A}=\{\alpha\in\bar{A}_R/M^l\alpha\subset A$ for some $l>0\}$.*

Proof. Put $A'=\{\alpha\in\bar{A}_R/M^l\alpha\subset A$ for some $l>0\}$. First we shall prove that $\tilde{A}\subset A'$. Take a homogeneous elment at^n $(a\in J_n)$. Then there exists a positive integer l such that $J_nM^l\subset I^n$. Hence $M^l(at^n)\subset A$. Thus $\tilde{A}\cap A'$. Next, we shall prove that $A'\subset\tilde{A}$. Take an element α of A'. Since A is a graded ring over R, we can assume that α is a homogeneous element. Let $\alpha=at^n$ where $a\in R$. It is obvious that $\alpha\in\tilde{A}$ in the case that $n\leq 0$. We suppose that $n>0$. Since $M^t\alpha\subset A$, we have $M^t a\subset I^n$. Hence $a\in(I^n)_{p_i}\cap R\subset q_{i,n}$. Thus $a\in\bigcap_{i=1}^n q_{i,n}=J_n$. Therefore $\alpha\in\tilde{A}$. Thus we prove that $A'\subset\tilde{A}$. The proof is complete.

Lemma 8. $\mathrm{Ass}_R(\tilde{A}/A)=\{M\}$.

Proof. It is enough to prove that "if $P\in\mathrm{Ass}_A(\tilde{A}/A)$, then $P\cap R=M$" (cf. [4, p. 57,9. A]). Since $\tilde{A}$ and A are graded rings, there exists $\alpha=at^n(a\in J_n)$ such that $P=A:\alpha$. Hence $P\cap R=I^n:a$. Since $a\in J_n$, it follows that I^n: $a\supset Q_n$. Therefore I^n: a is an M-primary ideal. Thus $P\cap R=M$. The proof is complete.

Now, we consider the problem when M is a prime divisor of an ideal N containing I. We recall the definition:

$$R_{\tilde{A}}(IA)=\{\alpha\in\tilde{A}/\alpha IA\subset IA\}\ .$$

Theorem 9. *Let (R, M) be a Nagara local domain satisfying the dimension formula and* $\mathrm{depth}\,R=d<\dim R=n$. *Let N be an ideal of R containing I. If* $\mathrm{height}\,N<n$ *and $R_{\tilde{A}}(NA)=A$, then M is an embedded prime divisor of N.*

Proof. First, we shall prove that "if M is not a prime divisor of N then $N\tilde{A}\cap A=NA$". For this, it is enough to prove that $N\tilde{A}\cap A\subset NA$, that is, $NJ_n\cap I^n\subset NI^n$ for any $n>0$. Take an element α of $NJ_n\cap I^n$,

$$\alpha=\sum x_{i_1,\cdots,i_d}a_1^{i_1}\cdots a_d^{i_d}\ ,$$

the sum being taken over the integers $i_1,\cdots,i_d$ such that $i_1+i_2+\cdots+i_d=n$. We claim that $x_{i_1,\cdots,i_d}\in N$. Let $N=q_1\cap\cdots\cap q_s$ be an irredundant primary decomposition of N. Let $p_i'=\mathrm{rad}\,(q_i)$ where $\mathrm{rad}\,(q_i)$ denotes the radical of q_i. It follows that $p_1'\subsetneqq M$ by the assumption. Put $p=p_i'$. Then $(J_n)_p=(I^n)_p$ (cf. The proof in Proposition 6). Since $\alpha\in(NJ_n)_p=(NI^n)_p$, it follows that

$$\alpha=\sum y_{i_1,\cdots,i_d}a_1^{i_1}\cdots a_d^{i_d}$$

where $y_{i_1,\cdots,i_d}\in N_p$. Since $\alpha\in(I^n)_p$, we have

$$\bar{\alpha}\in I_p^n/I_p^{n+1}\subset\bigoplus_{i\geqq 0}I_p^i/I_p^{i+1}\cong(R_p/I_p)\,[X_1,\cdots,X_d]\ .$$

Therefore

$$\bar{\alpha}=\sum\bar{y}_{i_1,\cdots,i_d}\bar{a}_1^{i_1}\cdots\bar{a}_d^{i_d}=\sum\bar{x}_{i_1,\cdots,i_d}\bar{a}_1^{i_1}\cdots\bar{a}_d^{i_d}\ .$$

Thus $y_{i_1,\cdots,i_d}\equiv x_{i_1,\cdots,i_d}\ (\mathrm{mod}\ I_p)$, that is,

$$x_{i_1,\dots,i_d}=y_{i_1,\dots,i_d}+z_{i_1,\dots,i_d} \quad \text{for some} \quad z_{i_1,\dots,i_d}\in I_p\,.$$

Since $y_{i_1,\dots,i_d}\in N_p$ and $z_{i_1,\dots,i_d}\in I_p\subset N_p$, we see that $x_{i_1,\dots,i_d}\in N_p\cap R\subset q_i$. Therefore $NA\cap A=NA$.

Next, we shall prove that $R_{\tilde{A}}(NA)=(NA)_{\tilde{A}}^{-1}\supsetneqq A$. We recall the definition:

$$(NA)_{\tilde{A}}^{-1}=\{\alpha\in\tilde{A}/\alpha NA\subset A\}\,.$$

It is clear that $R_{\tilde{A}}(NA)\subset(NA)_{\tilde{A}}^{-1}$ and so we prove that $(NA)_{\tilde{A}}^{-1}\subset R_{\tilde{A}}(NA)$. Take any element θ of $(NA)_{\tilde{A}}^{-1}$. Then $\theta\in\tilde{A}$ and $\theta NA\subset A$. Since $N\tilde{A}\cap A=NA$, we have $\theta(NA)\subset N\tilde{A}\cap A=NA$. Thus $\theta\in R_{\tilde{A}}(NA)$. Hence $R_{\tilde{A}}(NA)=(NA)_{\tilde{A}}^{-1}$. Now, we shall prove that $(NA)_{\tilde{A}}^{-1}\supsetneqq A$. From Lemma 8, there exists some $\alpha\in\tilde{A}-A$ such that $M=A:_R\alpha$. Since $N\subset M$, it follws that $\alpha N\subset A$, that is, $\alpha\in(NA)_{\tilde{A}}^{-1}$. Hence $R_{\tilde{A}}(NA)\supsetneqq A$. This is a contradiction.

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