Komiya, K.
Osaka J. Math.
24 (1987), 299-305

THE LEFSCHETZ NUMBER FOR EQUIVARIANT MAPS

KATSUHIRO KOMIYA

(Received March 27, 1986)

1. Introduction and results

Let G be a compact Lie group. A G-ENR (Euclidean Neighborhood Retract) is a G-space which is a G-retract of some G-invariant open subspace in a Euclidean G-space. In this paper we will consider the Lefschetz number

$$\lambda(f) = \sum_i (-1)^i \operatorname{trace} f_{*,i}: H_i(X; Z)/\mathrm{Tor} \to H_i(X; Z)/\mathrm{Tor}$$

of a self G-map $f: X \to X$ of a compact G-ENR X. f restricts to the self map $f^G: X^G \to X^G$ of the G-fixed point set X^G of X. Then we will show

Theorem 1. *Let $f: X \to X$ be a self G-map of a compact G-ENR X.*

(i) *If X has only one isotropy type (H), then $\lambda(f) \equiv 0 \bmod \chi(G/H)$ where $\chi(\ )$ denotes the Euler characteristic.*

(ii) *If the G-action on X is semifree, then $\lambda(f) \equiv \lambda(f^G) \bmod \chi(G)$.*

(iii) *If G is finite and of prime power order p^k, then $\lambda(f) \equiv \lambda(f^G) \bmod p$.*

(iv) *If G is connected and abelian (i.e., torus), then $\lambda(f) = \lambda(f^G)$.*

In section 4 we will prove this theorem by using the fixed point index defined by Dold [2]. (i) of the theorem is a special case of Dold [3; (8.18)]. If G is finite and the G-action is free, related results are in Nakaoka [9] and Gottlieb [5]. As a corollary of the theorem we obtain

Corollary 2. (i) *If the G-action on X is semifree and $\lambda(f) \not\equiv 0 \bmod \chi(G)$, then f has a fixed point in X^G.*

(ii) *If G is of prime power order p^k and $\lambda(f) \not\equiv 0 \bmod p$, then f has a fixed point in X^G.*

Proof. In either case it follows $\lambda(f^G) \neq 0$ and by the Lefschetz fixed point theorem there exists a fixed point of $f^G: X^G \to X^G$. q.e.d.

If G is a compact monogenic Lie group (i.e., finite cyclic group, torus and product of these) and $f \in G$ is its generator, then we may regard f as a self G-map of a G-ENR X. In this case we can show, as in the proof of Theorem 1, that $\lambda(f) = \chi(X^G)$, although this has already appeared in the literature, tom

Dieck [1; (5.3.11)] and Huang [6; Corollary 1] for G a finite cyclic group, Kobayashi [7; p. 63] for X a Riemannian manifold. As applications of this we will show the following two results.

Proposition 3. *If X is a compact G-ENR and G is monogenic, then*

$$|\chi(X^G)| \leq \sum_i \operatorname{rank} H_i(X; Z).$$

In connection with this we note that if G is finite and of prime power order p^k, Floyd [4] shows

$$|\chi(X^G)| \leq \sum_i \dim H_i(X^G; Z_p) \leq \sum_i \dim H_i(X; Z_p).$$

Proposition 4. *Let G be of order 2 and f be its generator. Let M be a $2n$-dimensional closed smooth G-manifold and orientable over Z. If f is orientation preserving, then*

$$\chi(M^G) \equiv \operatorname{trace} f_{*,n} \mod 2.$$

If f is orientation reversing, then

$$\chi(M^G) = \operatorname{trace} f_{*,n} = 0.$$

Here $f_{*,n}$ is the automorphism of $H_n(M;Z)$ induced from f.

These two propositions will be proved in section 5.

2. A lemma

If M is a G-space and $x \in M$, then $G(x)$ denotes the orbit of x and G_x the isotropy subgroup at x. The conjugacy class (G_x) of an isotropy subgroup G_x is called an isotropy type. For a subgroup H of G let $M_{(H)} = \{x \in M \mid (G_x) = (H)\}$. If N is a G-invariant subspace of M and $h: N \to M$ a G-map, then the fixed point set $\mathrm{Fix}(h)$ of h is a union of orbits. If N and M are smooth G-manifolds, then for any fixed orbit $G(x) \subset \mathrm{Fix}(h)$ we may take G-invariant tubular neighborhoods T and T' of $G(x)$ such that $T \subset T'$ and $h(T) \subset T'$. We decompose T into $T = T_t \oplus T_n$, where $T_t = T \cap N_{(H)}$, $H = G_x$, is the component tangent to $N_{(H)}$, and T_n the component normal to $N_{(H)}$. Similarly we decompose T' into $T' = T'_t \oplus T'_n$. Then we see $h(T_t) \subset T'_t$. We may regard T and T' as G-vector bundles over $G(x) \approx G/H$.

Lemma 5. *Let M be a smooth G-manifold and N a G-invariant codimension 0 submanifold of M with finite isotropy types. If $f: N \to M$ is a G-map with* $\mathrm{Fix}(f)$ *compact, then there exists a G-map $h: N \to M$ such that*

(i) *h is G-homotopic to f relative to the outside of some G-invariant compact neighborhood of* $\mathrm{Fix}(f)$,

(ii) $\mathrm{Fix}(h)$ *consists of a finite number of orbits,*

(iii) *if* $f(N_{(H)})\cap M_{(H)}=\phi$ *then* $h(N_{(H)})\cap M_{(H)}=\phi$ *and hence* $\mathrm{Fix}(h)\cap N_{(H)}=\phi$,

(iv) *for any fixed orbit* $G(x)\subset\mathrm{Fix}(h)$ *if* $T=T_t\oplus T_n$ *and* $T'=T'_t\oplus T'_n$ *are G-invariant tubular neighborhoods of* $G(x)$ *as above, then* $h|T: T\to T'$ *is fibre preserving and decomposes into* $h|T=(h|T_t)\oplus 0$ *where* $0: T_n\to T'_n$ *maps any vector to* 0.

Proof. (I) *The case in which the G-action on N is free.* $N\times M$ is a G-manifold with diagonal G-action, and its action is also free. Thus the orbit spaces N/G and $N\times_G M$ are smooth manifolds. Define a G-map $\tilde{f}: N\to N\times M$ by $\tilde{f}(x)=(x, f(x))$ for $x\in N$. Passing to the orbit spaces, $\tilde{f}$ induces a map $\tilde{f}/G: N/G\to N\times_G M$. By the transversality theorem we obtain a smooth map $h_1: N/G\to N\times_G M$ such that

(i) h_1 is transverse to Δ/G, where Δ is the diagonal set in $N\times M$, and

(ii) h_1 is close enough and homotopic to $\tilde{f}/G$ relative to $N-V/G$, where V is some G-invariant compact neighborhood of $\mathrm{Fix}(h)$.

By the dimension reason $h_1^{-1}(\Delta/G)$ is a finite set, in particular it is empty if $\dim G>0$. If $f(N)\cap M_{(1)}=\phi$ where $M_{(1)}$ is the points of M with the identity isotropy subgroup, then $\mathrm{Fix}(f)=\phi$, $\tilde{f}/G(N/G)\cap\Delta/G=\phi$ and hence we may take $h_1=\tilde{f}/G$. By the equivariant covering homotopy property we may lift the homotopy of (ii) and obtain a G-map $h_2: N\to N\times M$ G-homotopic to $\tilde{f}$ relative to the outside of some G-invariant compact neighborhood of $\mathrm{Fix}(f)$. $h_2^{-1}(\Delta)$ consists of a finite number of orbits. Let $p_1: N\times M\to N$ and $p_2: N\times M\to M$ be the projections. $p_1h_2: N\to N$ is a diffeomorphism since it is close enough to $p_1\tilde{f}=$ identity. Let $h_3=h_2(p_1h_2)^{-1}: N\to N\times M$ and $h=p_2h_3: N\to M$. then $h_3(x)=(x, h(x))$ and $\mathrm{Fix}(h)=h_3^{-1}(\Delta)\approx h_2^{-1}(\Delta)$. h is a desired G-map.

(II) *The general case.* Let $\{(H_1), (H_2), \cdots, (H_a)\}$ be the set of isotropy types on N ordered in such a way that if H_i is conjugate to a subgroup of H_j then $j\leq i$. Consider the following assertion A(i) for $0\leq i\leq a$:

A(i). *There exist a G-map* $h_i: N\to M$ *and a G-invariant neighborhood* U_i *of* $X_i=N_{(H_1)}\cup\cdots\cup N_{(H_i)}$ *such that*

(i) h_i *is G-homotopic to f relative to the outside of some G-invariant compact neighborhood of* $\mathrm{Fix}(f|X_i)$,

(ii) $\mathrm{Fix}(h_i)\cap(U_i-X_i)=\phi$,

(iii) $h_i|U_i: U_i\to M$ *satisfies the conditions* (ii), (iii) *and* (iv) *of the lemma.*

If $i=0$, then $X_i=\phi$ and hence we may take $U_i=\phi$, $h_i=f$. Thus A(0) is valid. A(a) is equivalent to the lemma since $X_a=N$. Thus, to prove the lemma it suffices to prove that A(i) implies A($i+1$).

Now suppose A(i). As in the author [8; Lemma 3.1] there exists a G-invariant codimension 0 submanifold P (with boundary) of N such that $X_i\subset \mathrm{Int}\,P\subset P\subset\mathrm{Int}\,U_i$. Let $Q=N-\mathrm{Int}\,P$ and $K=H_{i+1}$. Consider an $N(K)$-map

$h_i | Q^K$: $Q^K \to M^K$, where $N(K)$ is the normalizer of K in G. $h_i | Q^K$ may also be considered as an $N(K)/K$-map. Since K is the maximal isotropy subgroup on Q, then the action of $N(K)/K$ on Q^K is free. Thus we may apply the preceding argument (I) to the $N(K)/K$-map $h_i | Q^K$, and obtain a resulting $N(K)/K$-map $Q^K \to M^K$. By G-equivariancy it extends to a G-map f_1: $Q_{(K)} = G(Q^K) \to M$, which satisfies the conditions (i)$\sim$(iv) of the lemma. To be precise for the condition (i) it says that f_1 is G-homotopic to $h_i | Q_{(K)}$ relative to the outside of some G-invariant compact neighborhood (in $Q_{(K)}$) of $\mathrm{Fix}(h_i | Q_{(K)})$. Moreover its G-homotopy may be so taken as to be relative to a neighborhood of $\partial Q_{(K)}$, since h_i has no fixed point in a neighborhood of $\partial Q_{(K)}$. Let T be a G-invariant tubular neighborhood of $Q_{(K)}$ in Q and π: $T \to Q_{(K)}$ be the projection. Then we may extend f_1 to a G-map f_2: $T \to M$ such that

(i) for some two neighborhoods $U \subset U'$ (U' compact) of $\mathrm{Fix}(f_1)$ in $Q_{(K)}$, $f_2 = f_1 \circ \pi$ on $T | U$ and $f_2 = h_i$ on $T | Q_{(K)} - U'$,

(ii) $\mathrm{Fix}(f_2) \cap (T - Q_{(K)}) = \phi$.

From $h_i | Q$ and f_2, as in the author [8; Lemma 3.2], we obtain a G-map f_3: $Q \to M$ such that

(i) $f_3 = h_i$ on a neighborhood A of ∂Q, $f_3 = f_2$ on a neighborhood of $Q_{(K)}$, $f_3 = h_i = f$ on the outside of a G-invariant compact neighborhood B of $\mathrm{Fix}(f_1)$ ($= \mathrm{Fix}(f_2)$),

(ii) f_3 is G-homotopic to $h_i | Q$ relative to $A \cup (Q - B)$.

Define h_{i+1}: $N \to M$ as $h_{i+1} = h_i$ on P and $h_{i+1} = f_3$ on Q. Then h_{i+1} is a G-map required in A($i+1$). q.e.d.

3. Fixed point index

We first recall the definition of the fixed point index from Dold [2]. Let $F \subset N \subset R^n \subset R^n \cup \{\infty\} = S^n$, where F is compact and N is open. The *fundamental class* $o_F \in H_n(N, N-F; Z)$ is the image of 1 under the composite homomorphism

$$Z = H_n(S^n; Z) \to H_n(S^n, S^n - F; Z) \cong H_n(N, N-F; Z).$$

Let h: $N \to R^n$ be a map with $\mathrm{Fix}(h)$ compact. Define the map $1-h$: $(N, N-F) \to (R^n, R^n - 0)$ by $(1-h)(x) = x - h(x)$ for $x \in N$. Then the *fixed point index* $\mathrm{ind}(h)$ of h is defined as $\mathrm{ind}(h) = (1-h)_* o_F \in H_n(R^n, R^n - 0; Z) = Z$. Dold uses the symbol I_h for the index, but we use the symbol $\mathrm{ind}(h)$ to facilitate the printing.

Let R^n be a Euclidean G-space, N be a G-invariant open subspace of R^n, and h: $N \to R^n$ be a G-map satisfying the conditions (ii) and (iv) of Lemma 5. Let $\mathrm{Fix}(h) = G(x_1) \cup G(x_2) \cup \cdots \cup G(x_a)$ with $G_{x_i} = H_i$ ($1 \le i \le a$). If T_i is a small G-invariant open tubular neighborhood of $G(x_i)$ in N, then by the additivity of the index [2; (1.5)] it follows that

$$\mathrm{ind}(h) = \sum_{i=1}^{a} \mathrm{ind}(h \mid T_i) .$$

For a while let $x=x_i$, $H=H_i$, $T=T_i$. We may consider that a fibre in T over $g(x)\in G(x)$ is a subspace in R^n which is a parallel translation to $g(x)$ of (a small open disc in) a linear subspace through the origin. Let π: $T\to G(x)\subset R^n$ be the projection, and T' be the other G-invariant open tubular neighborhood of $G(x)$ as in (iv) of Lemma 5. Define $1-h+\pi$: $T\to T'$ as $(1-h+\pi)(v)=v-h(v)+\pi(v)$. This map is fibre preserving, and the following diagram is commutative for any $g\in G$.

$$\begin{array}{ccc} H_n(T,\ T-G(x)) & \xrightarrow{j_*} & H_n(T,\ T-g(x)) = Z \\ {\scriptstyle (1-h+\pi)^*}\downarrow & & \downarrow{\scriptstyle (1-h+\pi)_*} \\ H_n(T',\ T'-G(x)) & \xrightarrow[j_*]{} & H_n(T',\ T'-g(x)) = Z , \end{array}$$

where j: $(T,\ T-G(x))\to(T,\ T-g(x))$ is the inclusion. Let $o=o_{G(x)}\in H_n(T,\ T-G(x))$ be the fundamental class. Let $\alpha_g=j_*(1-h+\pi)_*o\in Z$. By the commutativity of the diagram, $\alpha_g=(1-h+\pi)_*j_*o$ and $j_*o=1$ in $H_n(T,\ T-g(x))=Z$. Since $1-h+\pi$ is G-equivariant, α_g are all equal for every $g\in G$. So, if α is its the same value, then we see that $(1-h+\pi)_*o=\alpha\cdot o$ in $H_n(T',\ T'-G(x))$.

$$(1-h)_*:\ H_n(T,\ T-G(x)) \to H_n(R^n,\ R^n-0)$$

factors as

$$H_n(T,\ T-G(x)) \xrightarrow{(1-h+\pi)_*} H_n(T',\ T'-G(x)) \xrightarrow{(1-\pi)_*} H_n(R^n,\ R^n-0) .$$

Thus we see that in $H_n(R^n,\ R^n-0)$

$$(1-h)_*o = (1-\pi)_*(1-h+\pi)_*o = \alpha\cdot(1-\pi)_*o ,$$

and hence $\mathrm{ind}(h\mid T)=\alpha\cdot\mathrm{ind}(\pi)$. Since $\mathrm{ind}(\pi)=\chi(G/H)$ by [2; (4.1)], it follows that $\mathrm{ind}(h\mid T_i)$ is a multiple of $\chi(G/H_i)$ for $i=1, 2, \cdots, a$.

Let $\mathrm{Fix}(h)\cap N^G=\{x_1, x_2, \cdots, x_b\}$ $(1\le b\le a)$. For $1\le i\le b$ the tubular neighborhood T_i is a disc with x_i as its center. As before T_i decomposes into the direct sum $T_i=T_{i,t}\oplus T_{i,n}$ where $T_{i,t}=T_i^G$ is the component tangent to N^G and $T_{i,n}$ is the component normal to N^G. Then, from the condition (iv) of Lemma 5 we see that h on T_i decomposes into $h(u, v)=(h(u), 0)$. Thus, by [2; (1.4), (1.6)], $\mathrm{ind}(h\mid T_i)=\mathrm{ind}(h\mid T_i^G)$ and hence

$$\sum_{i=1}^{b} \mathrm{ind}(h\mid T_i) = \mathrm{ind}(h^G) .$$

From the above argument it follows the following.

(i) If $\mathrm{Fix}(h)\subset N^G\cup N_{(H)}$, then $\mathrm{ind}(h)\equiv\mathrm{ind}(h^G) \bmod \chi(G/H)$.

(ii) If G is finite and of prime power order p^k, then $\mathrm{ind}(h)\equiv\mathrm{ind}(h^G) \bmod p$. For $\chi(G/H)\equiv 0 \bmod p$ for any proper subgroup H of G.

(iii) If G is connected and abelian, then $\operatorname{ind}(h)=\operatorname{ind}(h^G)$. For $\chi(G/H)=0$ for any proper subgroup H of G.

4. Proof of Theorem 1

Let $f: X\to X$ be as in the theorem. Let N be a G-invariant open subspace in a Euclidean G-space R^n, and $i: X\to N$, $r: N\to X$ be G-maps such that ri=identity. We easily see that X^H is also an ENR for any $H<G$. We apply Lemma 5 to the G-map $ifr: N\to R^n$ and obtain a G-map $h: N \dot{\to} R^n$ satisfying the conditions (i)~(iv). By [2; (1.7), (4.1)] it follows that $\lambda(f^H)=\operatorname{ind}((ifr)^H)=\operatorname{ind}(h^H)$. From this and (ii), (iii) in the preceding section, (iii) and (iv) of the theorem immediately follow. If X has only one isotropy type (H), then $(ifr)(N_{(K)})\subset R^n_{(H)}$ for any $K<G$. Thus, from (iii) of Lemma 5 it follows $\operatorname{Fix}(h)\subset N_{(H)}$ and from (i) in the preceding section it follows $\lambda(f)=\operatorname{ind}(h)\equiv 0 \bmod \chi(G/H)$. This proves (i) of the theorem. If the G-action on X is semifree, from (iii) of the lemma it follows $\operatorname{Fix}(h)\subset N^G\cup N_{(1)}$. Thus (ii) of the theorem follows from (i) in the preceding section.

5. Proof of Proposition 3 and 4

Let X be a compact G-ENR and N, i, r be as in section 4. If G is a compact monogenic Lie group, we regard a generator f of G as a G-map $f: X\to X$. Then $\operatorname{Fix}(f)=X^G$. Let $h: N\to R^n$ be a G-map obtained by Lemma 5 from the G-map $ifr: N\to R^n$. In this case we may construct h satisfying the additional condition $\operatorname{Fix}(h)\subset N^G$. This is ensured by the fact $\operatorname{Fix}(ifr)=i(X^G)\subset N^G$. Then we see that $\lambda(f)=\operatorname{ind}(h)=\operatorname{ind}(h^G)=\lambda(f^G)$. Since f^G is the identity map of X^G, it follows $\lambda(f)=\chi(X^G)$. As noticed in Introduction this has already appeared in the literature. Using this result, Proposition 3 and 4 are proved as follows.

(1) Proof of Proposition 3. Let $f_{*,i}: H_i(X; C)\to H_i(X; C)$ be the automorphism induced from f, where C is the complex numbers. Let $z_1, z_2, \cdots, z_r \in C$ be the eigenvalues of $f_{*,i}$ where $r=\dim H_i(X; C)=\operatorname{rank} H_i(X; Z)$. Since the $\chi(G)$ times composition of $f_{*,i}$ is the identity, then $z_j^{\chi(G)}=1$ and thus $|z_j|=1$ for $1\leq j\leq r$. We see that

$$|\operatorname{trace} f_{*,i}|=|\textstyle\sum_{j=1}^r z_j|\leq \sum_{j=1}^r |z_j| = \operatorname{rank} H_i(X; Z),$$

and

$$|\chi(X^G)| = |\lambda(f)|\leq \textstyle\sum_i \operatorname{rank} H_i(X; Z). \qquad \text{q.e.d.}$$

(2) Proof of Proposition 4. Let G, f and M be as in the proposition. Note that a smooth G-manifold with a finite number of isotropy types is a G-ENR. Let $z\in H_{2n}(M; Z)$ be the fundamental class defined from an orientation of M for which f is either orientation preserving or reversing. Consider the following commutative diagram.

$$
\begin{array}{ccc}
H_i(M; Z) & \xrightarrow{\ \cap z\ } & H^{2n-i}(M; Z) \\
{\scriptstyle f_{*,i}}\downarrow & & \uparrow{\scriptstyle f^{*,2n-1}} \\
H_i(M; Z) & \xrightarrow{\ \cap f_{*,2n}(z)\ } & H^{2n-1}(M; Z)\,,
\end{array}
$$

where $\cap$ denotes the cap product and the horizontal homomorphisms are the isomorphisms of the Poincaré duality. Note that the inverse of the isomorphism $f^{*,2n-i}$ is itself since f is an involution. It follows that if f is orientation preserving then $\operatorname{trace} f_{*,i} = \operatorname{trace} f^{*,2n-i}$, and if f is orientation reversing then $\operatorname{trace} f_{*,i} = -\operatorname{trace} f^{*,2n-i}$. From the universal coefficient theorem it follows that $\operatorname{trace} f^{*,i} = \operatorname{trace} f_{*,i}$. It thus follows that if f is orientation preserving then $\lambda(f) \equiv \operatorname{trace} f_{*,n} \bmod 2$, and if f is orientation reversing then $\lambda(f) = \operatorname{trace} f_{*,n} = 0$. Thus $\lambda(f) = \chi(M^G)$ implies the proposition. q.e.d.

In case M is odd dimensional in Proposition 4, we then easily see that $\lambda(f) = \chi(M^G) = 0$ if f is orientation preserving, and $\lambda(f) = \chi(M^G) \equiv 0 \bmod 2$ if f is orientation reversing.

References

[1] T. tom Dieck: Transformation groups and representation theory, Lecture notes in Math. 766, Springer-Verlag, Berlin, Heidelberg and New York, 1979.

[2] A. Dold: *Fixed point index and fixed point theorem for Euclidean neighborhood retracts,* Topology **4** (1965), 1–8.

[3] A. Dold: *The fixed point transfer of fibre-preserving maps,* Math. Z. **148** (1976), 215–244.

[4] E.E. Floyd: *On periodic maps and the Euler characteristics of associated spaces,* Trans. Amer. Math. Soc. **72** (1952), 138–147.

[5] D.H. Gottlieb: *The Lefschetz number and Borsuk-Ulam theorems,* Pacific J. Math. **103** (1982), 29–37.

[6] W.-H. Huang: *Equivariant method for periodic maps,* Trans. Amer. Math. Soc. **189** (1974), 175–183.

[7] S. Kobayashi: Transformation groups in differential geometry, Springer-Verlag, Berlin, Heidelberg and New York, 1972.

[8] K. Komiya: *Equivariant immersions and embeddings of smooth G-manifolds,* Osaka J. Math. **20** (1983), 553–572.

[9] M. Nakaoka: *Coincidence Lefschetz numbers for a pair of fibre preserving maps,* J. Math. Soc. Japan **32** (1980), 751–779.

Department of Mathematics
Yamaguchi University
Yamaguchi 753
Japan