Popov, G.
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SOME ESTIMATES OF GREEN'S FUNCTIONS IN THE SHADOW

GEORGI POPOV

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0. Introduction

The purpose of this work is to investigate the asymptotic behaviour of Green's functions in the so-called shadow for Laplace operator in an exterior domain. As a consequence a field scattered by a non-trapping obstacle will be examined at high frequencies.

These asymptotics have been studied by many authors since Keller's article [6] appeared. It was shown that for some convex obstacles the scattered field in the shadow should be as small as the exponent $\exp(-A|k|^{1/3})$, $A>0$, is when the frequency k tends to infinity. Such an estimate is believed to take place for a large class of domains but it has not been proved yet even for strictly convex obstacles except for some special cases. In [12], Ludwig constructed an asymptotic solution u_N for Helmholtz equation in the deep shadow which behaved like $\exp(-A|k|^{1/3})$, $A>0$, as $k\to\infty$, but he did not show that the difference between u_N and the exact solution could be estimated by the same exponent.

The asymptotics of Green's functions in the shadow were investigated in [1], [2], [3], [14]. Recently, an asymptotic solution of Green's functions in the deep shadow was obtained by Zayaev and Philippov [4] for planar strictly convex obstacles. Probably, the technique developed in [8], [9], [11] may be used to obtain the asymptotic expansions of Green's functions at high frequences for any strictly convex obstacle in $\boldsymbol{R}^n$, $n\geq 2$.

Let K be a compact in $\boldsymbol{R}^n$, $n\geq 2$, with a real analytic boundary Γ and let $\Omega=\boldsymbol{R}^n\backslash K$. The obstacle K is called non-trapping if for any $R>0$ with $K\subset B_R=\{x\in\boldsymbol{R}^n;\ |x|\leq R\}$ there exists $T_R>0$ such that there are no generalized geodesics, (for definition see [13]), with length T_R within $\bar{\Omega}\cap B_R$. Denote by Δ_0, respectively by Δ_D, the self-adjoint extension of the Laplace operator in $\boldsymbol{R}^n$, respectively in Ω with Dirichlet boundary conditions. Let

$$R_j^{\pm}(k)=(-\Delta_j-k^2)^{-1}$$

be the resolvent of the operator $-\Delta_j$, $j=0$, D in $\pm\mathrm{Im}k>0$. Consider the cut-off resolvents

(0.1) $\{k\in \boldsymbol{C};\ \pm\mathrm{Im}k>0\}\ni k\to R^{\pm}_{D,\chi}(k)=\chi R^{\pm}_{D}(k)\chi\in\mathcal{L}(L^2(\Omega),\ L^2(\Omega))$

where $\chi\in C^{\infty}_{(0)}(\bar{\Omega})=\{\varphi\in C^{\infty}(\bar{\Omega});\ \mathrm{supp}\varphi$ is compact$\}$ and $\chi(x)=1$ in a neighbourhood of Γ. Hereafter $\mathcal{L}(H_1, H_2)$ stands for the Banach space of bounded linear operators mapping from the Banach space H_1 into the Banach space H_2 and equipped with the usual norm. Obviously the functions (0.1) are analytic with respect to k in $\pm\mathrm{Im}k>0$.

Our first result is

Theorem 1. *Suppose K non-trapping. Then the function* (0.1) *admits an analytic continuation in the region*

$$U^{\pm}_{\alpha,\beta}=\{k\in \boldsymbol{C};\ \mp\mathrm{Im}k\leq\alpha|k|^{1/3}-\beta\}$$

for some positive constants α and β.

This theorem was proved for strictly convex obstacles with C^{∞} boundaries and for $n=3$ by Babich and Grigorieva [2]. Recently, in [8], [9], Bardos, Lebeau and Rauch showed that the region $U^{-}_{\alpha,\beta}$ is free of poles of the scattering matrix for any non-trapping obstacle with an analytic boundary, provided $n\geq 3$ odd. They investigated the generator B of the semi-group $Z(t)$ introduced by Lax and Phillips in [7]. Using the propagation of the Gevrey singularities of the solutions of the mixed problem for the wave equation they proved the estimate $\|B^jZ(t_0)\|\leq AC^j(3j)!$ for some t_0 and for any $j\in\boldsymbol{Z}^+$. Then the region $U^{-}_{\alpha,\beta}$ does not contain poles of the scattering matrix according to the results in [7], §3. This result can be obtained also from Theorem 1 since the poles of the scattering matrix coincide with the poles of the meromorphic continuation of $R^{-}_{D,\chi}(k)$.

A result close to Theorem 1 was proved by Vainberg [18] and Rauch [16] when K is non-trapping and Γ is smooth. In this case the functions (0.1) have analytic continuations in $\{k\in C;\ \mp\mathrm{Im}\,k\leq\alpha\,\mathrm{Log}|k|-\beta\}$. It is an open problem if Theorem 1 can be extended to hold for any smooth, non-trapping obstacle.

Let us now consider the distribution kernel $G^+(k, x, y)$ ($G^-(k, x, y)$) of the resolvent $R^{\pm}_{D}(k)$ in $\pm\mathrm{Im}\,k\geq 0$ which is usually called outgoing (incoming) Green's function. For any $k>0$ the distribution $G^{\pm}(k, x, y)$ solves the problem

$$(0.2)\qquad\begin{cases}(\Delta+k^2)G^{\pm}(k, x, y)=-\delta(x-y)\,, & (x, y)\in\Omega\times\Omega\\ B\,G^{\pm}=0 & \\ G^{\pm}(k, x, y)=0(r^{(1-n)/2})\,, & \dfrac{dG^{\pm}}{dr}\mp ik\,G^{\pm}=o(r^{(1-n)/2})\\ \text{as } r=|x-y|\to\infty & \text{and}\quad k\in\boldsymbol{R}^1_+=(0,\ \infty)\end{cases}$$

where $B\,u = u_{/\Gamma}$.

The point $x_0 \in \bar{\Omega}$ belongs to the shadow $Sh(y_0)$ of K with respect to a given point $y_0 \in \bar{\Omega}$ if there are no generalized geodesics starting at y_0 and passing through x_0. Denote by $d(x, y)$ the distance function in $\bar{\Omega}$, i.e.

$$d(x, y) = \inf \{\text{length of } \gamma;\ \gamma \text{ is a path in } \bar{\Omega} \text{ connecting } x \text{ and } y\}\ .$$

Denote $D_x^p = D_1^{p_1} \cdots D_n^{p_n}$, where $D_j = i^{-1}\partial/\partial x_j$ and $p = (p_1, \cdots, p_n) \in \boldsymbol{Z}_+^n$, $\boldsymbol{Z}_+ = \{0, 1, \cdots\}$.

Theorem 2. *Suppose K non-trapping and $x_0 \in Sh(y_0)$. Then there exists a neighbourhood $\mathcal{O}$ of (x_0, y_0) in $\bar{\Omega} \times \bar{\Omega}$ such that*

$$|D_k^m D_x^p D_y^q G^{\pm}(k, x, y)| \leq C \exp(-A|k|^{1/3} \mp d(x, y) \operatorname{Im} k) \tag{0.3}$$

in $U_{\alpha,\beta}^{\pm} \times \mathcal{O}$ for any $(m, p, q) \in \boldsymbol{Z}_+^{2n+1}$ and for some positive constants α, β, A, and $C = C(m, p, q)$.

Now consider the scattering of plane waves by the obstacle K. Let $\omega \in S^{n-1} = \{\theta \in \boldsymbol{R}^n;\ |\theta| = 1\}$ and denote $L_s = \{x \in \boldsymbol{R}^n;\ \langle x, \omega\rangle = s\}$ where $\langle x, \omega\rangle = \sum_{j=1}^n x_j \omega_j$. Consider the solution $u_S(k, x)$ of the problem

$$\begin{cases} (\Delta + k^2)\, u_S(k, x) = 0 \\ u_{S/x\in\Gamma} = -e^{ik\langle x, \omega\rangle}/_{x\in\Gamma} \\ u_S = O(r^{(1-n)/2}),\ \dfrac{d}{dr} u_S - iku_S = o(r^{(1-n)/2}) \quad \text{as} \quad r = |x| \to \infty\ . \end{cases}$$

The point x_0 belongs to the shadow $Sh(K, \omega)$ of K with respect to a given direction ω if non of the generalized geodesics $\gamma(t)$, $t > 0$, starting at L_s for some $s < \min_{y\in\Gamma} \langle y, \omega\rangle$ and having ω as an initial direction passes through the point x_0 (t is the natural parameter on γ).

Theorem 3. *Suppose K non-trapping and $x_0 \in Sh(K, \omega)$. Then there exists a neighbourhood $\mathcal{O}$ of x_0 in $\bar{\Omega}$ such that*

$$|D_k^m D_x^p (u_S(k, x) + e^{ik\langle x, \omega\rangle})| \leq C \exp(-A|k|^{1/3}) \tag{0.4}$$

in $[k_0, \infty) \times \mathcal{O}$ for some $A > 0$ and any $k_0 > 0$, $m \geq 0$, $p \in \boldsymbol{Z}_+^n$.

An immediate consequence of (0.4) is the Kirchoff approximation of $\dfrac{\partial}{\partial \nu} u_{S/\Gamma}$ in the shadow, where ν is the outward normal to Γ.

An estimate close to (0.3) was obtained for strictly convex obstacles in [2]. Moreover, some asymptotic expansions in the shadow for x_0 and y_0 sufficiently close to Γ and $n = 2$ were recently obtained by Zayaev and Philippov in [4]. Provided $x_0 \in Sh(y_0)$ and Γ smooth the Green's functions $G^{\pm}$ were estimated in [14] as follows

$$|G^{\pm}(k, x, y)| \leq C_N k^{-N}$$

for any $N>0$ in $k \geq k_0 > 0$ and (x, y) in a neighbourhood of (x_0, y_0).

The estimate (0.4) was predicted by Keller's geometrical theory of diffraction [5], [6], see also [12].

The method we use is close to that developed by Vainberg [18] (see also [16]) in order to prove uniform decay of the local energy for hyperbolic equations. The propagation of Gevrey singularities for the mixed problem studied in [10], [11] and the non-trapping condition allow us to compare the solutions of the mixed problem with suitably chosen solutions of the Cauchy problem for the wave equation. This is used in Proposition 1 to prove that the kernels of the cut-off resolvents $R^{\pm}_{B,\chi}(k)$ coincide with the Fourier transforms of some compactly supported distributions modulo exponentially decreasing functions, holomorphic in $U^{\pm}_{\alpha,\beta}$. The theorems follow from Proposition 1 by using once more the results on the propagation of Gevrey singularities for the mixed problem.

1. Estimates of Green's functions

In this section we prove theorems 1 and 2. Let us denote by $U_0(t)$ and $U(t)$ the propagators of the Cauchy problem and the mixed problem respectively, i.e.

$$(1.1) \qquad \begin{cases} (\partial_t^2 - \Delta)\, U_0(t) f(x) = 0 & \text{in} \quad (t, x) \in \boldsymbol{R}^1 \times \boldsymbol{R}^n \\ U_0(0) f(x) = 0\,, \quad \partial_t U_0(0) f(x) = f(x)\,, & f \in C_0(\boldsymbol{R}^n)\,, \end{cases}$$

$$(1.2) \qquad \begin{cases} (\partial_t^2 - \Delta)\, U(t) f(x) = 0 & \text{in} \quad (t, x) \in \boldsymbol{R}^1 \times \Omega \\ B\; U(t) f(x) = 0 & \\ U(0) f(x) = 0\,, \quad \partial_t U(0) f(x) = f(x)\,, & f \in C_0^\infty(\Omega)\,. \end{cases}$$

Using standard energy estimates one can extend the operators $U_0(t)$ and $U(t)$ by continuity in $L^2(\boldsymbol{R}^n)$ and in $L^2(\Omega)$ respectively. Recall that a function $f(z)$ defined in a domain $M \subset \boldsymbol{R}^p$ belongs to the Gevrey class $G^s(M)$, $s \geq 1$, if for any compact $M_1 \subset M$ there exist some constants $A = A(M_1, f)$, $B = B(M_1, f)$ such that

$$\sup_{z \in M_1} |D^\alpha f(z)| \leq A\, B^{|\alpha|} (\alpha!)^s$$

for any α, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = (\alpha_1!) \cdots (\alpha_n!)$.

Let $\chi \in G^3(\boldsymbol{R}^n)$, $\chi(x) = 1$ in a neighbourhood of $B_R = \{x;\ x \leq R\}$ and $\chi(x) = 0$ for $x \notin B_{R_1}$ for some $R_1 > R$. In view of the non-trapping condition there exists $T > R_1$ such that any generalized geodesic starting at B_{R_1} leaves it by the time T. Then from the theorem about the propagation of Gevrey G^3 singularities proved by G. Lebeau [10] follows that the distribution kernel $U(t, x, y)$ of $U(t)$ is a G^3 function in

$$Q_0 = [\boldsymbol{R}^1 \backslash (-T, T)] \times (B_{R_1} \cap \bar{\Omega}) \times (B_{R_1} \cap \bar{\Omega}) .$$

Therefore the estimate

$$|D_t^j D_x^\alpha D_y^\beta U(t, x, y)| \leq A_Q C_Q^{j+|\alpha|+|\beta|}((j+|\alpha|+|\beta|)!)^3 \tag{1.3}$$

holds in $(t, x, y) \in Q$ for any compact $Q \subset Q_0$ and any j, α, β. Moreover the constants A_Q and C_Q do not depend on $(t, x, y) \in Q$ and on j, α, β.

Let $\zeta \in G^3(\boldsymbol{R}^{n+1})$, $\zeta = 1$ in a neighbourhood of the set $\{(t, x) \in \boldsymbol{R}^{n+1};\ ||x| -t| < T\}$ and $\zeta(t, x) = 0$ if $||x| - t| > T+1$. Consider the operators

$$U_\chi(t) = \chi\, U(t)\, \chi\,, \quad U_{0,\chi}(t) = \chi\, U_0(t)\, \chi\,, \quad E(t) = \zeta\, U(t)\, \chi\,.$$

Next we write the modified resolvent $R_{D,\chi}^+(k)$ in the form

$$R_{D,\chi}^+(k) = \chi\, \hat{E}(k) + Z_\chi(k) \tag{1.4}$$

where

$$\chi\, \hat{E}(k) = \int_0^\infty e^{ikt}\, \chi E(t) dt\,, \quad \operatorname{Im} k > 0\,,$$

denotes the Fourier-Laplace transform of $\chi E \in L^1(\boldsymbol{R}^1, \mathcal{L}(L^2(\Omega), L^2(\Omega)))$. Note that the operator-valued function $\chi E(t)$ has a compact support with respect to t since $\chi(x)\, \zeta(t, x)$ has. Therefore $\chi \hat{E}(k)$ is an analytic function with values in the space $\mathcal{L}(L^2(\Omega), L^2(\Omega))$, while $Z_\chi(k)$ is analytic in $\{k \in \boldsymbol{C};\ \operatorname{Im} k > 0\}$. Let $H^s(\Omega)$, $s \geqq 0$, $s \in \boldsymbol{Z}$, be the closure of $C_{(0)}^\infty(\Omega)$ with respect to the Sobolev norm $\|u\|_s^2 = \sum_{|\alpha| \leq s} \|D^\alpha u\|_{L^2(\Omega)}^2$ and let $H^{-s}(\Omega)$ be the dual space of $H^s(\Omega)$. We shall use also the domain D^s of the operator $(-\Delta_D)^{s/2}$, $s \geq 0$, $s \in \boldsymbol{Z}$, equipped with the graph topology, where the operator $(-\Delta_D)^{s/2}$ is given by the functional calculas. Denote by D^{-s} the dual space of D^s. Theorems 1 and 2 will follow from

Proposition 1. *The function $Z_\chi(k)$ can be extended as an analytic function*

$$\{k \in \boldsymbol{C};\ \operatorname{Im} k > 0\} \ni k \mapsto Z_\chi(k) \in \mathcal{L}(H^{-s}(\Omega), H^s(\Omega))$$

for any $s \geq 0$, $s \in \boldsymbol{Z}$. Moreover, there exist some positive constants α and β such that $Z_\chi(k)$ has an analytic continuation in $U_{\alpha,\beta}^+$ and

$$\|D_k^m Z_\chi(k)\|_{\mathcal{L}(H^{-s}, H^s)} \leq C \exp(-A|k|^{1/3} - T \operatorname{Im} K)\,, \quad m \geq 0\,, \tag{1.5}$$

in $k \in U_{\alpha,\beta}^+$ for some positive constants A and $C = C(m, s)$.

Proof. Let us denote $F(t) = [\partial_t^2 - \Delta, \zeta]\, U(t)\, \chi$, where $[F_1, F_2] = F_1 F_2 - F_2 F_1$

is the commutator of the operators F_1 and F_2 and ζ stands for the operator of multiplication by the function $\zeta(t, x)$. Then $E(t)$ is the propagator of the problem

$$(1.6)\qquad \begin{cases} (\partial_t^2-\Delta)\, E(t) f(x) = F(t) f(x) \\ BE(t) f = 0 \\ E(0) f(x) = 0\,, \quad \partial_t E(0) f(x) = \chi(x) f(x)\,, \quad f\in L^2(\Omega)\,. \end{cases}$$

The distribution kernel $F(t, x, y)$ of the operator $F(t)$ belongs to the Gevrey class $G^3(\boldsymbol{R}^1\times\bar{\Omega}\times\bar{\Omega})$ in view of the propagation of Gevrey singularities of $U(t, x, y)$ and the definition of the functions $\zeta(t, x)$ and $\chi(x)$. Moreover

$$(1.7)\qquad \operatorname{supp} F\subset \{(t, x, y)\in \boldsymbol{R}^1\times\bar{\Omega}\times\bar{\Omega};\ |t|>T,\ T\leq \big||x|-t\big|\leq T+1,\ |y|\leq R_1\}$$

in view of the finite propagation speed for the wave equation.

Let $\tilde{F}(t, x, y)$ be a G^3 continuation of the function $F(t, x, y)$ such that (1.7) continues to hold. Denote by $\tilde{F}(t)$ the operator with a distribution kernel $\tilde{F}(t, x, y)$ and consider the problem

$$(1.8)\qquad \begin{cases} (\partial_t^2-\Delta)\, W(t) f(x) = \tilde{F}(t) f(x) \\ W(0) f(x) = \partial_t W(0) f(x) = 0\,, \quad f\in C_0^\infty(\boldsymbol{R}^n) \end{cases}$$

The distribution kernel $W(t, x, y)$ of $W(t)$ is a G^3 function since the function $\tilde{F}(t, x, y)$ is such, $\tilde{F}(t)=0$ in $|t|<T$ and since

$$W(t) = \int_0^t U_0(s)\, \tilde{F}(t-s) ds$$

Let $\psi\in C^\infty(\boldsymbol{R}^n)$, $\psi(x)=0$ in a neighbourhood of B_R and $\chi(x)=1$ on supp $(1-\psi)$. Denote

$$\begin{aligned} Q(t) f(x) &= (\partial_t^2-\Delta)\,(E(t) f(x) - \psi W(t) f(x)) \\ &= (1-\psi)\, F(t) f(x) + [\Delta, \psi]\, W(t) f(x) \end{aligned}$$

in $x\in\bar{\Omega}$ for $f\in C_{(0)}^\infty(\bar{\Omega})$. In view of (1.6), (1.7) and Duhamel's formula we obtain

$$E(t) f - \psi W(t) f = U(t)\, \chi f + \int_0^t U(t-s)\, \chi Q(s) f ds\,, \quad f\in L^2(\Omega)\,.$$

Multiplying the last equality by χ and performing Fourier-Laplace transform with respect to t we obtain

$$(1.9)\qquad \chi\hat{E}(k) f = R_{D,\chi}^+(k) f + R_{D,\chi}^+(k)\, \hat{Q}(k) f + \psi\chi\hat{W}(k) f$$

for $\operatorname{Im} k>0$. We are going to prove that the functions $\psi\chi\hat{W}(k)$ and $\hat{Q}(k)$ can be continued analytically for $\operatorname{Im} k \leq 0$.

Let $\mathcal{H} \in C^\infty(\boldsymbol{R}^n)$, $\mathcal{H}(x)=0$ for $x \in B_T$, $\mathcal{H}(x)=1$ outside B_{T+1} and set

$$\begin{aligned} G(t)f(x) &= (\partial_t^2 - \Delta)(W(t)f(x) - \mathcal{H}(x)E(t)f(x)) \\ &= (1-\mathcal{H})\widetilde{F}(t)f(x) - [\Delta, \mathcal{H}]E(t)f(x)\,. \end{aligned}$$

The function $\boldsymbol{R}^1 \ni t \mapsto E(t) \in \mathcal{L}(D^{-s}, D^{-s+1})$ is bounded for any $s \in \boldsymbol{Z}$, $[\Delta, \mathcal{H}] \in \mathcal{L}(D^{-s+1}, H^{-s}(\boldsymbol{R}^n))$, and $H^{-s}(\Omega) \subset D^{-s}$ for any $s \geq 0$, $s \in \boldsymbol{Z}$. Then $\boldsymbol{R}^1 \ni t \mapsto [\Delta, \mathcal{H}]E(t)$ is a bounded function with values in $\mathcal{L}(H^{-s}(\Omega), H^{-s}(\boldsymbol{R}^n))$, $s \geq 0$, $s \in \boldsymbol{Z}$, and

$$\|G(t)\|_{\mathcal{L}(H^{-s}(\Omega),\, H^{-s}(\boldsymbol{R}^n))} \leq C \tag{1.10}$$

for any $t \in \boldsymbol{R}^1$.

In view of (1.6), (1.7), (1.10) and Duhamel's formula we write

$$W(t)f(x) = \mathcal{H}(x)E(t)f(x) + \int_0^t U_0(t-s)\,G(s)f(x)ds,\ f \in H^{-s}(\Omega)\,.$$

Note that the support of the distribution kernel of $G(t)$ is contained in $\{(t, x, y);\ |t| \leq 2T+2,\ |x| \leq T+1,\ |y| \leq T+1\}$. Therefore

$$\chi_2 W(t) f = \chi_2 \int_0^{T_1} U_0(t-s)\,\chi_1 G(s) f ds\,, \quad f \in H^{-s}(\Omega)\,, \tag{1.11}$$

for any $T_1 > 2T+2$, where $\chi_1 \in C_0^\infty(\boldsymbol{R}^n)$, $\chi_1(x)=1$ in B_{T+1} and $\chi_2 \in C_0^\infty(B_T)$.

Lemma 1. *Let $\chi_2 \in C_0^\infty(B_T \backslash B_R)$. Then $\chi_2 U_0(t)\chi_1 \in \mathcal{L}(H^{-s}(\boldsymbol{R}^n), H^s(\Omega))$ for any $s \in \boldsymbol{R}^1$ and any $t \in [2T+3, \infty)$. Moreover the function*

$$[2T+3, \infty) \ni t \mapsto \chi_2 U_0(t)\chi_1 \in \mathcal{L}(H^{-s}(\boldsymbol{R}^n), H^s(\Omega))$$

can be continued analytically in $\{t \in \boldsymbol{C};\ |t| > 2T+3\}$ and

$$\|D_t^j \chi_2 U_0(t)\chi_1\|_{\mathcal{L}(H^{-s}(\boldsymbol{R}^n), H^s(\Omega))} \leq A(j!)|t|^{-2} \tag{1.12}$$

for any $t \in \boldsymbol{C}$, $|t| > 2T+3$, for $j \geq \max(0, 3-n)$, and for some A which does not depend on j.

Proof. The conclusion is obvious when n is odd because of Huyghens principle. Suppose $n \geq 2$ is even, $j \geq 1$, and set $\mathcal{O}_T = \{(t, x, y) \in \boldsymbol{C}^{2n+1};\ |t| > 2T+3,\ |x| \leq T,\ |y| \leq T+1\}$. Then $U_0(t, x, y) = C_n(t^2 - |x-y|^2)^{-(n-1)/2}$ for any $(t, x, y) \in \mathcal{O}_T$ and for some constant C_n. Using Cauchy integral formula we obtain for any $j \geq 1$, α, β the estimate

$$\begin{aligned} |D_t^j D_x^\alpha D_y^\beta U_0(t, x, y)| &\leq (2\pi)^{-2n-1}(j-1)!\,(\alpha+\beta)!\,2^{|\alpha+\beta|} \max\{|D_t U_0(z, \tilde{x}, \tilde{y})|; \\ &\quad |z-t| = 1,\ |x-\tilde{x}| + |y-\tilde{y}| = 1/2\} \leq A_{\alpha,\beta}(j!)|t|^{-2} \end{aligned}$$

in $\mathcal{O}_T$ which yields (1.12).

According to (1.10), (1.11) and lemma 1 the function

$$[T_2, \infty) \ni t \to \chi_2 W(t) \in \mathcal{L}(H^{-s}(\Omega), H^s(\Omega)), \quad T_2 = 2T_1+2,$$

can be continued as an analytical one in $\{t\in \boldsymbol{C};\ |t|>T_2\}$ for any $t\geq 0$ and any $\chi_2 \in C_0^\infty(B_T\backslash B_R)$. Moreover the estimate

$$\|D_t^j \chi_2 W(t)\|_{\mathcal{L}(H^{-s},H^s)} \leq A(3j)!\,|t|^{-2} \tag{1.13}$$

is valid in $|t|>T_2$ for any $j\geq \max(0, 3-n)$ and any $s\geq 0$, $s\in \boldsymbol{Z}$ where the constant A does not depend on j.

Now we can estimate the norm of the Fourier-Laplace transform of $\chi_2 W(t)$ in $\mathcal{L}(H^{-s}, H^s)$. Let $\operatorname{Re} k\geq k_0>0$ for some $k_0>0$. Since $W(t)=0$ in $|t|<T$ we can write

$$\chi_2 \hat{W}(k) = k^{-1}\int_0^{T_2} e^{ikt} D_t \chi_2 W(t)\,dt + k^{-1}\int_{T_2}^\infty e^{ikt} D_t \chi_2 W(t)\,dt\,.$$

Using (1.13) we can change the contour of integration in the second integral to obtain

$$\exp(C|k|^{1/3})\,\chi_2 \hat{W}(k) = \sum_{j=0}^\infty C^j |k|^{j/3-1}(j!)^{-1}\Big[\int_0^{T_2} e^{ikt} D_t \chi_2 W(t)\,dt + e^{ikT_2}\int_0^\infty e^{-kt}\chi_2(D_t W)(T_2+it)\,dt\Big]\,.$$

Integrating $[j/3]$ times by parts in any member of the last sum we have

$$\exp(C|k|^{1/3})\,\chi_2 \hat{W}(k) = \sum_{j=0}^\infty C^j k^{j/3-[j/3]-1}(j!)^{-1}\Big[\int_0^{T_2} e^{ikt}\chi_2 D_t^{[j/3]+1} W(t)\,dt + e^{ikT_2}\int_0^\infty e^{-kt}\chi_2(D_t^{[j/3]+1} W)(T_2+it)\,dt\Big]$$

where $[m]$ denotes the integer part of $m\in \boldsymbol{R}_+^1$. Since $W\in G^3$ and in view of (1.13) any member of the last sum can be estimated by

$$A_1 C^j B_1^{j/3} e^{-B\operatorname{Im} k}, \quad B = \begin{cases} T & \text{when } \operatorname{Im} k\geq 0 \\ T_2 & \text{when } \operatorname{Im} k<0 \end{cases}$$

in $\{k\in \boldsymbol{C};\ \operatorname{Re} k\geq k_0>0\}$, where the constants A_1 and B_1 do not depend on $j\in \boldsymbol{Z}$. Provided that $C<B_1^{-1/3}$ we obtain

$$\|\chi_2 \hat{W}(k)\|_{\mathcal{L}(H^{-s}(\Omega),H^s(\Omega))} \leq C_s \exp(-C|k|^{1/3} - B\operatorname{Im} k) \tag{1.14}$$

for $\operatorname{Re} k\geq k_0>0$, where $C_s = A_1(1-CB_1^{1/3})^{-1}$. Proceeding in the same way when $\operatorname{Re} k\leq -k_0<0$ we can continue $\chi_2 \hat{W}(k)$ analytically in $\boldsymbol{C}\backslash\{k;\ \operatorname{Im} k\leq 0,\ |\operatorname{Re} k|\leq k_0\}$ so that (1.14) holds in this region for any $k_0>0$. Then the Fourier-Laplace transform $\hat{Q}(k)$ of $Q(t)=(1-\psi)F(t)+[\Delta, \psi]\,W(t)$ can be continued analytically in $\boldsymbol{C}\backslash[i0, -i\infty)$ and

$$\|\hat{Q}(k)\|_{\mathcal{L}(H^{-s}(\Omega),H^{s}(\Omega))}\leq C_s \exp(-C|k|^{1/3}-B\,\mathrm{Im}\,k) \tag{1.15}$$

is fulfilled in $\boldsymbol{C}\backslash\{k;\ \mathrm{Im}\,k\leq 0,\ |\mathrm{Re}\,k|\leq k_0\}$ for any $k_0>0$.

Lemma 2. *The function* $\boldsymbol{C}\ni k\mapsto \chi\hat{E}(k)\in\mathcal{L}(H^s(\Omega), H^s(\Omega))$ *is analytic and*

$$\|\chi\hat{E}(k)\|_{\mathcal{L}(H^s(\Omega),H^s(\Omega))}\leq C(1+|k|)^{2s}\, e^{(2T+1)\max(0,-\mathrm{Im}\,k)} \tag{1.16}$$

for any $s\geq 0$, $s\in\boldsymbol{Z}$.

Proof. The assertion is obvious for $t=0$ since $U(t)$ is a bounded function in $\boldsymbol{R}^1$ with values in $\mathcal{L}(L^2(\Omega), L^2(\Omega))$ and $\chi\,\zeta(t,x)=0$ for any $t>2T+1$. Suppose $s\geq 1$ and consider

$$\begin{cases} (\Delta-1)\,\chi\hat{E}(k)f = L(k)f-\chi f+([\Delta,\chi]-(k^2+1)\,\chi)\,\hat{E}(k)f \\ \chi\hat{E}(k)f_{/\Gamma}=0 \end{cases} \tag{1.17}$$

for $f\in H^s(\Omega)$. Here

$$L(k)f=-\int_0^\infty e^{ikt}\,\chi F(t)\,f dt\in H^s(\Omega)$$

and $L(k)$ satisfies the estimate (1.16) for any $s\geq 0$ since the distribution kernel of the operator $\chi F(t)$ is smooth and $\mathrm{supp}(\chi F)\subset\{|x|\leq R_1,\ |y|\leq R_1,\ |t|<2T+1\}$ in view of (1.7). Then

$$\|\chi\hat{E}(k)f\|_s\leq C((1+|k|^2)\|\chi_1\hat{E}(k)f\|_{s-1}+e^{(2T+1)\max(0,-\mathrm{Im}\,k)}\|f\|_s)$$

$f\in H^s(\Omega)$, for some $\chi_1\in C^\infty_{(0)}(\bar{\Omega})$, $\chi_1=1$ in a neighbourhood of $\mathrm{supp}(\chi)$ which proves (1.16) by induction. Differentiating (1.17) with respect to k and using (1.16) it is easy to prove that $\frac{d}{dk}\chi\hat{E}(k)\in\mathcal{L}(H^s(\Omega), H^s(\Omega))$ for any $s\geq 0$, $s\in\boldsymbol{Z}$. Thus $\chi\hat{E}(k)$ is an analytic function.

According to (1.15) the operator $I+\hat{Q}(k)$: $H^s(\Omega)\mapsto H^s(\Omega)$ is invertable for any $k\in U^+_{\alpha,\beta}$ and for some α, β. Then $R^+_{D,\chi}(k)$ is an analytic function in $U^+_{\alpha,\beta}$ with values in $\mathcal{L}(H^s(\Omega), H^s(\Omega))$ and satisfies (1.16) in view of (1.9) and Lemma 2. Now, (1.5) follows for $m=0$ from (1.9), (1.14) and (1.15), choosing α and β small enough. Using Cauchy integral formula we obtain (1.5) for any $m\in\boldsymbol{Z}_+$.

To prove theorem 2 we choose some neighbourhoods $\mathcal{O}_1$ and $\mathcal{O}_2$ of x_0, respectively y_0, $\mathcal{O}_j\subset\bar{\Omega}$, so that none of the generalized geodesics starting at $\mathcal{O}_2$ passes through $\mathcal{O}_1$. Set $\mathcal{O}=\mathcal{O}_1\times\mathcal{O}_2$ and suppose that $\mathcal{O}_j\subset B_R$ and $T>\sup\{d(x,y);\ (x,y)\in\mathcal{O}\}$. According to proposition 1 we have

$$G^+(k,x,y)=\int_0^\infty e^{ikt}\,\zeta(t,x)\,U(t,x,y)\,dt+Z_\chi(k,x,y)$$

where

$$|D_k^m D_x^p D_y^q Z_\chi(k, x, y)| = |\langle D_x^p \delta_x, D_k^m Z_\chi(k) D_y^q \delta_y \rangle| \leq \|D_k^m Z_\chi(k)\|_{\mathcal{L}(H^{-s}, H^s)} \|\delta_x\|_{p+q-s}^2$$
$$\leq C \exp(-A|k|^{1/3} - T \operatorname{Im} k) \leq C \exp(-A_0|k|^{1/3} - d(x, y) \operatorname{Im} k)$$

in $U_{\alpha,\beta}^+ \times \mathcal{O}$ for some $\alpha>0$ and $A_0>0$. Here $\langle \delta_x, \varphi \rangle = \varphi(x)$ for any $\varphi \in C_{(0)}^\infty(\bar{\Omega})$ and $s>n+p+q$. On the other hand $\zeta(t, x)\, U(t, x, y)$ is a G^3 function in $\boldsymbol{R}^1 \times \mathcal{O}$ with a compact support with respect to t. Moreover, $U(t, x, y)=0$ for $|t|< d(x, y)$ since the propagation speed for the solutions of the mixed problem for the wave equation equals one (see [17]). Now the arguments used in the proof of (1.14) yield (0.3).

Denote by $e(\lambda, x, y)$ the spectral function of the operator $-\Delta_D$ given as the distribution kernel of the spectral projector E_λ of $-\Delta_D$. Since $E_\lambda \to I$ in $L^2(\Omega)$ as $\lambda \to \infty$ and

$$\frac{de}{d\lambda}(\lambda^2, x, y) = (2\pi i)^{-1}\{G^+(\lambda, x, y) - G^-(\lambda, x, y)\} \qquad \text{for} \quad x \neq y,\ \lambda>0\,,$$

it is easy to obtain from theorem 2 the following

Corollary 1. *Suppote K non-trapping and* $x_0 \in Sh(y_0)$. *Then*

$$|D_\lambda^m D_x^p D_y^q e(\lambda, x, y)| \leq C \exp(-A\lambda^{1/6})\,, \quad A>0\,,$$

in $[\lambda_0, \infty) \times \mathcal{O}$ *for* $(m, p, q) \in \boldsymbol{Z}_+^{2n+1}$, $\lambda_0>0$.

2. Asymptotics of the scattered waves

In this section we prove theorem 3. Translating the origin to a given point $z_0 \in \boldsymbol{R}^n$ the function $u_S(k, x)$ is multiplied by $\exp(ik\langle z_0, \omega\rangle)$. Thus we can suppose that $K \subset B_R(x_0) = \{x \in \boldsymbol{R}^n;\ |x-x_0| \leq R\}$ and $\langle x, \omega\rangle > 0$ for any $x \in B_{R+1}(x_0)$. Consider the function

$$v(k, x) = u_S(k, x) + \varphi(x)\, e^{ik\langle x, \omega\rangle}$$

where $\varphi \in G^3(B_{R+1}(x_0))$ and $\varphi(x)=1$ on $B_R(x_0)$, supp $\varphi \subset B_{R+1}(x_0)$. Then

$$\begin{cases} (\Delta+k^2)\, v(k, x) = [\Delta, \varphi]\, e^{ik\langle x, \omega\rangle} \\ v(k, x)_{/\Gamma} = 0 \end{cases}$$

and $v(k, x)$ satisfies the outgoing Sommerfeld's condition at infinity. Therefore

$$\begin{aligned} v(k, x) &= R_{D,\chi}^+(k)\,([\Delta, \varphi]\, e^{ik\langle x, \omega\rangle}) \\ &= Z_\chi(k)([\Delta, \varphi]\, e^{ik\langle x, \omega\rangle}) + \chi\hat{E}(k)\,([\Delta, \varphi]\, e^{ik\langle x, \omega\rangle}) \end{aligned}$$

for $x \in B_R(x_0)$ where $\chi \in G^3(\boldsymbol{R}^n)$, $\chi=1$ on $B_{R+1}(x_0)$, $\operatorname{supp}(\chi) \subset B_{R+2}(x_0)$.

The first term of the last equality is estimated by proposition 1. The second one is equal to the Fourier-Laplace transform of the distribution

$$v_1(t, x) = \chi(x) \int_{-\infty}^{t} \zeta(t-s, x)\, U(t-s)\, [\Delta, \varphi]\, \delta(s-\langle x, \omega\rangle)\, ds$$

since $v_2(s, y)=[\Delta, \varphi]\, \delta(s-\langle y, \omega\rangle)$ vanishes for $s<0$. The distribution v_1 is well-defined since v_2 has a compact support, $v_2 \in D^{-m}$ for $m>3$ and $\zeta(t-s)$ $U(t-s)$ is a continuous function with valued in $\mathcal{L}(D^{-m}, D^{-m})$.

We are going to prove that there exists a neighbourhood $\mathcal{O}$ of x_0 such that v_1 is a G^3 function in $\boldsymbol{R}^1 \times \mathcal{O}$.

Let us write $v_1 = Q(v_2)$ where the operator Q has a distribution kernel $Q(t, s, x, y)=\chi(x)\, \zeta(t-s)\, H(t-s)\, U(t, x, y)\, \chi(y)$ and $H(s)=0$ for $s \leq 0$, $H(s)=1$ for $s>0$. We shall evaluate the Gevrey G^3 wave front $SS^3(v_1)$ of v_1 using the relation $SS^3(v_1) \subset SS^3(Q)^1 \circ SS^3(v_2)$. We have

$$SS^3(v_2) \subset \{(s, y; \tau, \eta);\ s=\langle y, \omega\rangle>0,\ y \notin B_R(x_0),\ \eta=-\tau\omega,\ \tau \neq 0\}\ .$$

Moreover, theorem 1.4 in [10] yields

$$SS^3(Q)^1 \subset \{(\phi^{t-s}(s, y, \tau, \eta);\ s, y, \tau, \eta);\ s \leq t,\ \tau \neq 0\} \cup \{(0, y, \tau, \xi;\ 0, y, \tau, \eta)\}$$

where $\phi^t(s, y, \tau, \eta)=(t+s,\ x^t(s, y, \tau, \eta),\ \tau,\ \xi^t(s, y, \tau, \eta))$ is the generalized bicharacteristic starting at (s, y, τ, η) and t is the natural parameter on it. Thus we have

$$SS^3(v_1) \subset \{(t, x^{t-s}(s, y, \tau, -\tau\omega), \tau, \xi);\ \tau \neq 0,\ 0<s=\langle y, \omega\rangle \leq t,\ y \notin B_R(x_0)\}\ .$$

Note that the initial codirection of the generalized geodesic $\gamma(t)=x^t(s, y, \tau, \eta)$ is $\frac{d\gamma}{dt}(0)=-\eta/\tau$ for any $y \in \Omega$. Then

$$SS^3(v_1) \subset \{(t, \gamma(t-s), \tau, \xi);\ \gamma \text{ is a generalized geodesic with } \gamma(0) \notin B_R(x_0),\ \frac{d\gamma}{dt}(0) = \omega,\ 0<s=\langle \gamma(0), \omega\rangle \leq t\}\ .$$

Moreover $\gamma(t) \notin B_R(x_0)$ for any $t \geq 0$ when $\gamma(0) \notin B_R(x_0)$ and $\langle \gamma(0), \omega\rangle \geq \langle x_0, \omega\rangle$ while $\gamma(t-s)=\gamma_1(t)$, $\gamma_1(t)$ is the generalized geodesic with initial data $\gamma_1(0)=\gamma(0)-s\,\omega \in L_0$, $\frac{d\gamma_1}{dt}(0)=\omega$, when $\gamma(0) \notin B_R(x_0)$ and $\langle \gamma(0), \omega\rangle \leq \langle x_0, \omega\rangle$. Therefore

$$(\text{sing supp}_{G^3}(v_1)) \cap B_R(x_0) \subset \{x=\gamma(t);\ t>0 \text{ and } \gamma \text{ is a generalized geodesic with } \gamma(0) \in L_0,\ \frac{d\gamma}{dt}(0) = \omega\}\ .$$

Since $x_0 \in Sh(K, \omega)$ we can choose a neighbourhood $\mathcal{O}$ of x_0 such that (sing $\text{supp}_{G^3}(v_1)) \cap \mathcal{O}=\phi$ which proves theorem 3 since $\text{supp}(v_1)$ is compact.

References

[1] V. Babich and V. Buldyrev: Asymptotic methods in short wave diffraction problems, Nauka, Moscow, 1972 (in Russian).

[2] V. Babich and N. Grigorieva: *Asymptotical properties of solutions of some three dimensional problems*, Notes Sci. Sem. Steklov Math. Inst. (Leningrad Branch) **127** (1975), 20–78 (in Russian).

[3] V. Philippov: *On the exact justification of the short wave approximation in a shadow region*, Notes Sci. Sem. Steklov Math. Inst. **34** (1973), 142–206 (in Russisian).

[4] A. Zayaev and V. Philippov: *On exact justification of Friedlander-Keller formula*, Notes Sci. Sem. Steklov Math. Inst. **140** (1984), 49–60 (in Russian).

[5] J. Keller: *Geometrical theory of diffraction*, J. Optical Soc. Amer. **58** (1962), 116–130.

[6] J. Keller: *Diffraction by a convex cylinder*, Trans. IRE, Ant. and Prop. 4, 3, (1956).

[7] P. Lax and R. Phillips: Scattering theory, Academic Press, 1967.

[8] C. Bardos, G. Lebeau, J. Rauch: Estimation sur les poles de la diffusion, Methodes semi-classiques en mecanique quantique, Publ. de l'Universite de Nantes, (1985), 71.

[9] C. Bardos, G. Lebeau, J. Rauch: *Estimation sur les poles de la diffusion*, preprint.

[10] G. Lebeau: *Propagation des singularités Gevrey pour le probleme de Dirichlet*, Advances in microlocal analysis, NATO ASI Series, Ser. C, Math. Phys. Sci. 168 (1985), 203–223.

[11] G. Lebeau: *Regularité Gevrey 3 pour la diffraction*, Comm. Partial Differential Equations **9** (1984), 1437–1497.

[12] D. Ludwig: *Uniform asymptotic expansion of the field scattered by a convex object at high frequencies*, Comm. Pure Appl. Math. **20** (1967), 103–138.

[13] R. Melrose and J. Sjöstrand: *Singularities of boundary value problems*, I, II, Comm. Pure Appl. Math. **31** (1978), 593–617 and **35** (1982), 129–158.

[14] G. Popov: *Spectral asymptotics for elliptic second order differential operators*, J. Math. Kyoto Univ. **25** (1985), 659–681.

[15] G. Popov and M. Shubin: *Complete asymptotic expansion of the spectral function for elliptic second order differential operators in* $\boldsymbol{R}^n$, Functional Anal. i Pril. **17** (1983), n°3, 37–45 (in Russian).

[16] J. Rauch: *Asymptotic behaviour of solutions to hyperbolic partial differential equations with zero speeds*, Comm. Pure Appl. Math. **31** (1978), 431–480.

[17] J. Rauch: *The leading wavefront for hyperbolic mixed problems*, Bull. Soc. Roy. Sci. Liege **46** (1977), 156–161.

[18] B. Vainberg: *On the short wave asymptotic behaviour of solutions of stationary problems and the asymptotic behaviour as* $t\to\infty$ *of solutions of nonstationary problems*, Russian Math. Surveys **30** (1975), 1–53.

Institute of Mathematics
Bulgarian Academy of Sciences
1090, Sofia, Bulgaria