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ON THE SMALL HULLS OF A COMMUTATIVE RING

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This short note is a continuous one of [1]. We always assume that every ring R is a commutative ring with identity 1 and every ring homomorphism f is unitary, i.e. $f(1)$ is the identity. Let R and R' be rings and $f\colon R \to R'$ a ring homomorphism. Then every R'-module may be regarded as an R-module through f. If $f(R)$ is a small submodule of R' as an R-module, we say *f being small* or *R being small in R'* [1]. In this note, we shall find the smallest small homomorphism of R.

We shall make use of the same notations as in [1]. We define the smallest small homomorphism as follows:

DEFINITION. Let R and $\tilde{R}$ be rings and $f\colon R \to \tilde{R}$ a ring homomorphism. We say *$\tilde{R}$ being a small hull of R* if the following conditions are satisfied.

1) *f is small.*

2) *For any small* (*ring*) *homomorphism $g\colon R \to R'$, there exists a unique* (*ring*) *homomorphism $h\colon \tilde{R} \to R'$ such that $g=hf$.*

We say *$\tilde{R}$ being a local small hull of R* if the following conditions are satisfied.

$\tilde{R}$ is a local ring and the properties 1), 2) *above are satisfied whenever R' is a local ring* (*R is not necessarily local*).

It is clear that (local) small hulls of R are unique up to isomorphism if they exist.

Proposition 1. *We assume R has a* (*local*) *small hull $f\colon R \to \tilde{R}$. Then for any* (*local*) *ring R' and a ring homomorphism $g\colon R \to R'$ g is small if and only if there exists a homomorphism $h\colon \tilde{R} \to R'$ such that $g=hf$.*

Proof. It is clear from the definition and [1], Remark 3.

Proposition 2. *Let R be a ring. Then the trivial ring* 0 *is a small hull of R if and only if* K. dim $R=0$.

Proof. We assume K. dim $R=0$ and $f\colon R \to R'$ a ring homomorphism. Let M' be a maximal ideal in R'. If $f \neq 0$, $M' \cap R$ is maximal. Hence f is not small by [1], Theorem 1. The converse is clear from Proposition 1 and [1], Proposition 9.

From the definition of tbe small hull, we have

Proposition 3. *If R has a small hull, every small ring extension of R contains the unique minimal small extension of R.*

From now on, we assume that a (local) small hull means a non-trivial one unless otberwise stated.

Theorem 4. *Let R be a commutative ring. Then R has a local small hull if and only if there exists a unique maximal one $\mathfrak{P}$ among non-maximal prime ideals in R. In this case, $\mu_{\mathfrak{P}}\colon R\to R_{\mathfrak{P}}$ is a local small hull and a small hull, too.*

Proof. "If" part. $\mu_{\mathfrak{P}}\colon R\to R_{\mathfrak{P}}$ is small by [1], Theorem 1. Let R' be a local ring with maximal ideal M' and $g\colon R\to R'$ a small homomorphism. Since $p=R\cap M'$ is not maximal by [1], Theorem 1, $p\subseteq\mathfrak{P}$. Therefore, tbere exists a unique homomorphism $h\colon R_{\mathfrak{P}}\to R'$ such that $h\mu_{\mathfrak{P}}=g$, since $R'=R'_{M'}$. Hence, $R_{\mathfrak{P}}$ is a local small bull of R.

"Only if" part. We assume $f\colon R\to\tilde{R}$ is a local small hull. Let p be a non-maximal prime ideal in R. Then $\mu_p\colon R\to R_p$ is small. Hence, there exists a unique homomorphism $h\colon \tilde{R}\to R_p$ such that $\mu_p=hf$. Put $R_0=h(\tilde{R})$ and $M_0=h(\tilde{M})$, where $\tilde{M}$ is the maximal ideal of $\tilde{R}$. Since f is small, $\mathfrak{P}=f^{-1}(\tilde{M})$ is not maximal by [1], Theorem 1. Furthermore, $p=\mu_p^{-1}(pR_p)=(hf)^{-1}(pR_p)=(hf)^{-1}(pR_p\cap R_0)\subseteq (hf)^{-1}(M_0)=f^{-1}(\tilde{M})=\mathfrak{P}$.We shall show that $R_{\mathfrak{P}}$ is also a small hull of R. Let $g\colon R\to R'$ be a small homomorphism and $s\in R-\mathfrak{P}$. If $g(s)R'\neq R'$, we can take a maximal ideal M' in R' containing $g(s)R'$. Then $s\in R\cap M'$. Since g is small, $R\cap M'\subseteq\mathfrak{P}$, which is a contradiction. Hence, $g(s)R'=R'$ i.e. $g(s)$ is unit in R'. Therefore, we obtain a unique homomorphism $h\colon R_{\mathfrak{P}}\to R'$ such that $h\mu_{\mathfrak{P}}=g$.

The following theorem shows that the existence of a small hull does not quarantee the existence of a local small hull.

Theorem 5. *Let R be a commutative noetherian ring. Then R has a small hull if and only if* K. dim $R=1$. *In this case, R_S is a small hull of R, where* $S=R-\bigcup_{i=1}^{r} p_i$ and $\{p_i\}_{i=1}^{r}$ *is the set of minimal but not maximal prime ideals.*

Proof. We assume that R had a small hull and K. dim $R\geqslant 2$. Let M be a maximal ideal and $M\supsetneqq P'\supsetneqq P$ a chain of prime ideals in R. Then $R_1=R/P$ is a noetherian domain of K. dim $\geqslant 2$. Let $R_1{}^*$ be the integral closure of R_1. Then we can take a maximal ideal $M_1{}^*$ in $R_1{}^*$ sucb that $M_1{}^*\cap R_1=M/P$ and height $M_1{}^*\geqslant 2$ ([2], p. 30). Since $R_1{}^*{}_{M_1{}^*}$ is a Krull ring by [2], Theorem 33.10,

${R_1^*}_{M_1^*}=\bigcap_{p^*} {R_1^*}_{p^*}$, where the p^* ranges over all prime ideals in M_1^* of height 1 and so the p^* is not maximal. Hence, R_1^* is small in ${R_1^*}_{p^*}$. Therefore, R is small in ${R_1^*}_{p^*}$ by [2], p. 30 and [1], Proposition 4. Accordingly, R is small in ${R_1^*}_{M_1^*}$ by Proposition 3. On the other hand, $M_1^* {R_1^*}_{M_1^*} \cap R = M_1^* \cap R_1 \cap R = M/P \cap R = M$. Therefore, R is not small in ${R_1^*}_{M_1^*}$ by [1], Theorem 1. Thus, we have proved $\mathrm{K.dim}\, R \leqslant 1$. Therefore, $\mathrm{K.dim}\, R=1$ by Proposition 2. Conversely, we assume $\mathrm{K.dim}\, R=1$. Let $g: R \to R'$ be a small homomorphism and M' a maximal ideal in R'. Since $\mathrm{K.dim}\, R=1$ and $M' \cap R$ is not maximal by [1], Theorem 1, $R-(M' \cap R) \subset S$. Therefore, there exists a unique homomorphism $h: R_S \to R'$ such that $h\mu_S = g$ as in the proof of Theorem 4. Accordingly, R_S is a small hull of R, since R is small in R_S by [1], Theorem 1.

REMARK. If R is a valuation ring of $\mathrm{K.dim}\, R \geqslant 2$, then $R_{\mathfrak{P}}$ is a small hull of R for a prime ideal $\mathfrak{P}$ of depth 1. Hence, Theorem 5 is not true if R is not noetherian.

Proposition 6. *Let R be a domain. Then the quotient field Q of R is a (local) small hull of R if and only if* $\mathrm{K.dim}\, R=1$.

Proof. It is clear from Theorem 4 and [1], Proposition 11.

Proposition 7. *Let R be a local ring with principal maximal ideal (m). Then if m is nilpotent, the trivial ring is a small hull of R and if m is not nilpotent, $R_{\{m^i\}}$ is a small hull of R.*

Proof. The first part is clear from Proposition 2. If $g: R \to R'$ is small, $g(m)R'=R'$ by [1], Theorem 1. Hence, $R_{\{m^i\}}$ is a small hull of R.

Proposition 8. *Let R be a domain. We assume that every non-zero prime ideal contains a prime element* (e.g. U.F.D., see [2], p. 42). *Then R has a small hull if and only if* $\mathrm{K.dim}\, R=1$. *In this case, R is noetherian.*

Proof. If $\mathrm{K.dim}\, R=1$, Q is a small hull of R by Proposition 6. Conversely, we assume that R has a small hull. Let M be a maximal ideal and $R_1=R_M$. Then R_M satisfies the same condition as in the proposition. Let m be a prime element in $M_1=MR_1$. Let n be any non-zero element in M_1. We assume $nR_1 \not\subseteq mR_1$. Let x be in $R_{1\{m^t\}} \cap R_{1\{n^s\}}$. Then $x=a/m^{t'}=b/n^{s'}$; $a, b \in R_1$. Since mR_1 is prime, $a \in mR_1$. Therefore, $R_1=R_{1\{m^t\}} \cap R_{1\{n^s\}}$. Since R_1 is small in $R_{1\{m^t\}}$ and $R_{1\{n^s\}}$, R is small in R_1 by Proposition 3 and [1], Proposition 4. On the other hand, R is not small in R_1 by [1], Theorem 1. Therefore, $M_1=(m)$. Since every non-zero prime ideal contains a prime element, M_1 is a unique non-zero prime ideal. Therefore, $\mathrm{K.dim}\, R=1$ and R is noetherian by [2], Theorem 3.4.

A Krull ring R satisfies

(*) *any principal ideal* $aR(\neq 0)$ *of* R *is the intersection of a finite number of primary ideals of height* 1.

Proposition 9. *Let* R *be a domain satisfying* (*). *Then* R *has a small hull if and only if* $\mathrm{K.dim}\, R=1$.

Proof. If $K.\dim R=1$, Q is a small hull of R from Proposition 6. We assume R has a small hull and $\mathrm{K.}\dim R\geqslant 2$. If we use the same argument as in the proof of Theorem 5, we may assume R is local. Then $R=\bigcap_{p} R_p$ by [2], p. 115, where the p ranges over all prime ideals of height 1. Since $\mathrm{K.dim}\, R\geqslant 2$, R is small in R_p, which is a contradiction from Proposition 3.

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References

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