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SIMPLE BIRATIONAL EXTENSIONS OF A POLYNOMIAL RING $k[x, y]$

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Introduction. Let k be an algebraically closed field of characteristic zero and let $k[x, y]$ be a polynomial ring over k in two variables x and y. Let f and g be two elements of $k[x, y]$ without common nonconstant factors, and let $A=k[x, y, f/g]$. In the present article we consider the structures of the affine k-domain A under an assumption that $V:=\mathrm{Spec}(A)$ has only isolated singularities.

In the first section we describe how V is obtained from $\boldsymbol{A}^2:=\mathrm{Spec}(k[x, y])$ and we see that if V has only isolated singularities V is a normal surface whose singular points (if any) are rational double points. The divisor class group $Cl(V)$ can be explicitly determined (cf. Theorem 1.9); we obtain, therefore, necessary and sufficient conditions for A to be a unique factorization domain. If g is irreducible and if the curves $f=0$ and $g=0$ on $\boldsymbol{A}^2$ meet each other then A is a unique factorization domain if and only if the curves $f=0$ and $g=0$ meet in only one point where both curves intersect transversally. We consider, in the same section, a problem: When is every invertible element of A constant?

In the second section we prove the following:

Theorem. *Assume that V has only isolated singularities. Then A has a nonzero locally nilpotent k-derivation if and only if we have $g\in k[y]$ after a suitable change of coordinates x, y of $k[x, y]$.*

An affine k-domain of type A as above was studied by Russell [8] and Sathaye [9] in connection with the following result:

Assume that A is isomorphic to a polynomial ring over k in two variables. In a polynomial ring $k[x, y, z]$ over k in three variables x, y and z, let $u=gz-f$. Then there exist two elements v, w of $k[x, y, z]$ such that $k[x, y, z]=k[u, v, w]$.

Our terminology and notation are as follows:

k: an algebraically closed field of characteristic zero which we fix throughout the paper.

A^*: the group of all invertible elements of a ring A.

$Cl(V)$: the divisor class group of a normal surface V.

$\varphi'(C)$: the proper transform of a curve C on a normal surface Y by a birational morphism φ: $X \to Y$ from a normal surface X to Y.
$(t)_X$: the divisor of a function t on a normal surface X.
$p_a(D)$: the arithmetic genus of a divisor D on a nonsingular projective surface.
(C^2), $(C \cdot C')$: the intersection multiplicity.
$\boldsymbol{A}^n$: the n-dimensional affine space.
$\boldsymbol{P}^n$: the n-dimensional projective space.

1. The structures of the affine domain $k[x, y, f/g]$

1.1. Let $k[x, y, z]$ be a polynomial ring over k in three variables x, y and z, and let $\boldsymbol{A}^3 := \mathrm{Spec}(k[x, y, z])$. Let V be an affine hypersurface on $\boldsymbol{A}^3$ defined by $gz-f=0$, and let π: $V \to \boldsymbol{A}^2 := \mathrm{Spec}(k[x, y])$ be the projection π: $(x, y, z) = (x, y)$. Let F and G be respectively the curves $f=0$ and $g=0$ on $\boldsymbol{A}^2$. Then we have:

Lemma. (1) *For each point $P \in F \cap G$, $\pi^{-1}(P)$ is isomorphic to the affine line $\boldsymbol{A}^1$.*

(2) *If Q is a point on G but not on F, then $\pi^{-1}(P) = \phi$.*

Proof. Straightforward.

1.2. The Jacobian criterion of singularity applied to the hypersurface V shows us the following:

Lemma. *Let P be a point on F and G. Then the following assertions hold:*

(1) *If P is a singular point for both F and G then every point of $\pi^{-1}(P)$ is a singular point of V.*

(2) *If P is a singular point of F but not a singular point of G then the point $(P, z=0)$ is the unique singular point of V lying on $\pi^{-1}(P)$.*

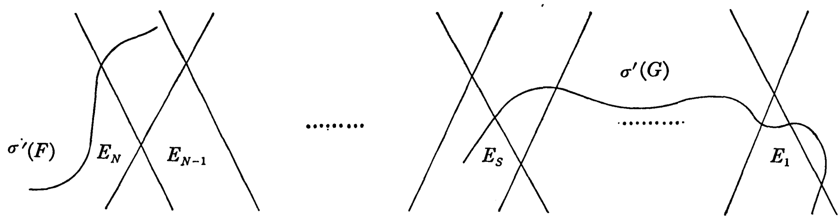
(3) *If P is a singular point of G but not a singular point of F then V is nonsingular at every point of $\pi^{-1}(P)$.*

(4) *If P is a nonsingular point of both F and G and if $i(F, G; P) \geqslant 2$ then the point $(P, z=\alpha)$ is the unique singular point of V lying on $\pi^{-1}(P)$, where $\alpha (\in k)$ satisfies: $\frac{\partial f}{\partial x}(P) = \frac{\partial g}{\partial x}(P)\alpha$ and $\frac{\partial f}{\partial y}(P) = \frac{\partial g}{\partial y}(P)\alpha$. If $i(F, G; P) = 1$ then V is nonsingular at every point of $\pi^{-1}(P)$.*

We assume, from now on, that V has only isolated singularities. Hence, if $P \in F \cap G$, either F or G is nonsingular at P. Furthermore, we assume that $F \cap G \neq \phi$. When $F \cap G = \phi$ then $A = k[x, y, 1/g]$ and A is a unique factorization domain.

1.3. Let P be a point on F and G. We first consider the case where F is nonsingular at P but G is not. Let $P_1 := P$ and let ν_1 be the multiplicity of G at P_1. Let σ_1: $V_1 \to V_0 := \boldsymbol{A}^2$ be the quadratic transformation with center at P_1, let

$P_2:=\sigma_1'(F)\cap\sigma_1^{-1}(P_1)$ and let ν_2 be the multiplicity of $\sigma_1'(G)$ at P_2. For $i\geqslant 1$ we define a surface V_i, a point P_{i+1} on V_i and an integer ν_{i+1} inductively as follows: When V_{i-1}, P_i and ν_i are defined, let σ_i: $V_i\to V_{i-1}$ be the quadratic transformation of V_{i-1} with center at P_i, let $P_{i+1}:=(\sigma_1\cdots\sigma_i)'(F)\cap\sigma_i^{-1}(P_i)$ and let ν_{i+1} be the multiplicity of $(\sigma_1\cdots\sigma_i)'(G)$ at P_{i+1}. Let s be the smallest integer such that $\nu_{s+1}=0$, and let $N:=\nu_1+\cdots+\nu_s$. We may simply say that $P_1,\cdots,P_s$ *are all points of* G *on the curve* F *over* P_1 *and* $\nu_1,\cdots,\nu_s$ *are the multiplicities of* G *at* $P_1,\cdots$, P_s, *respectively.* Let σ: $V_N\to V_0$ be the composition of quadratic transformations $\sigma:=\sigma_1\cdots\sigma_N$ and let $E_i:=(\sigma_{i+1}\cdots\sigma_N)'\sigma_i^{-1}(P_i)$ for $1\leqslant i\leqslant N$. In a neighborhood of $\sigma^{-1}(P_1)$, $\sigma^{-1}(F\cup G)$ has the following configuration:



(Fig. 1)

If $g=cg_1^{\beta_1}\cdots g_n^{\beta_n}$ $(c\in k^*)$ is a decomposition of g into n distinct irreducible factors, let G_j be the curve $g_j=0$ on $V_0=\boldsymbol{A}^2$ for $1\leqslant j\leqslant n$. Let $\nu_i(j)$ be the multiplicity of G_j at the point P_i for $1\leqslant i\leqslant s$ and $1\leqslant j\leqslant n$. Then it is clear that $\nu_i=\beta_1\nu_i(1)+\cdots+\beta_n\nu_i(n)$ for $1\leqslant i\leqslant s$.

1.4. We prove the following:

Lemma. *With the same assumption and notations as in* 1.3, V *is isomorphic, in a neighborhood of* $\pi^{-1}(P_1)$, *to* V_N *with the curves* $E_1,\cdots,E_{N-1}$ *and* $\sigma'(G)$ *deleted off.*

Proof. Let $\mathcal{O}:=\mathcal{O}_{V_0,P_1}$, $\tilde{V}_0:=\mathrm{Spec}(\mathcal{O})$ and $\tilde{V}=V\times_{V_0}\tilde{V}_0$. Since the curve F is nonsingular at P_1 there exist local parameters u, v of V_0 at P_1 such that $v=f$. Let $g(u,v)=0$ be a local equation of G at P_1. Then $\tilde{V}=\mathrm{Spec}(\mathcal{O}[v/g(u,v)])$. Note that V is nonsingular in a neighborhood of $\pi^{-1}(P_1)$ (cf. 1.2). Hence there exist a nonsingular projective surface $\bar{V}$ and a birational mapping φ: $V\to\bar{V}$ such that φ is an open immersion in a neighborhood of $\pi^{-1}(P_1)$ and a birational mapping $\bar{\pi}=\pi\cdot\varphi^{-1}$: $\bar{V}\to\boldsymbol{P}^2$ is a morphism, where V_0 is embedded canonically into the projective plane $\boldsymbol{P}^2$ as an open set. Since $\pi(\pi^{-1}(P_1))=P_1$ we know that $\bar{\pi}$ is factored by the quadratic transformation of $\boldsymbol{P}^2$ at P_1. Hence we know that π: $V\to V_0$ is factored by σ_1: $V_1\to V_0$, *i.e.*, π: $V\xrightarrow{\pi_1}V_1\xrightarrow{\sigma_1}V_0$.

Set $v=uv_1, u=vu_1, g(u,uv_1)=u^{\nu_1}g_1(u,v_1)$ and $g(vu_1,v)=v^{\nu_1}g_1'(u_1,v)$. Then $V_1\times_{V_0}\tilde{V}_0=\mathrm{Spec}(\mathcal{O}[v_1])\cup\mathrm{Spec}(\mathcal{O}[u_1])$; $\sigma_1^{-1}(P_1)$ and $\sigma_1'(G)$ are respectively defined by $u=0$ and $g_1(u, v_1)=0$ on $\mathrm{Spec}(\mathcal{O}[v_1])$, and by $v=0$ and $g_1'(u_1, v)=0$ on $\mathrm{Spec}(\mathcal{O}[u_1])$. Since $\tilde{V}:=V\times_{V_0}\tilde{V}_0=V\times_{V_1}(V_1\times_{V_0}\tilde{V}_0)=V\times_{V_1}\mathrm{Spec}(\mathcal{O}[v_1])\cup V\times_{V_1}\mathrm{Spec}(\mathcal{O}[u_1])$ $=\mathrm{Spec}(\mathcal{O}[v_1, v_1/u^{\nu_1-1}g_1(u, v_1)])\cup\mathrm{Spec}(\mathcal{O}[u_1, 1/v^{\nu_1-1}g_1'(u_1, v)])$ and since v is an invertible function on $\mathrm{Spec}(\mathcal{O}[u_1, 1/v^{\nu_1-1}g_1'(u_1, v)])$, we know that:

(i) $\tilde{V}=\mathrm{Spec}(\mathcal{O}[v_1, v_1/u^{\nu_1-1}g_1(u, v_1)])$,

(ii) $\tilde{\pi}:=\pi\times_{V_0}\tilde{V}_0$: $\tilde{V}\to\tilde{V}_0$ is a composition of $\tilde{\pi}_1:=\pi_1\times_{V_0}\tilde{V}_0$: $\tilde{V}\to\tilde{V}_1:=\mathrm{Spec}(\mathcal{O}[v_1])$ and $\tilde{\sigma}_1:=\sigma_1|_{\tilde{V}_1}$: $\tilde{V}_1\to\tilde{V}_0$,

(iii) if $Q\in(\sigma_1^{-1}(P_1)\cup\sigma_1'(G))-\sigma_1'(F)$ then $\tilde{\pi}_1^{-1}(Q)=\phi$.

Set $v_1=uv_2, \cdots, v_{s-1}=uv_s$ and $g_1(u, v_1)=u^{\nu_2}g_2(u, v_2), \cdots, g_{s-1}(u, v_{s-1})=u^{\nu_s}g_s(u, v_s)$. Set $\tilde{V}_2=\mathrm{Spec}(\mathcal{O}[v_2]), \cdots, \tilde{V}_s=\mathrm{Spec}(\mathcal{O}[v_s])$. Then, by the same argument as above, we know that the following assertions hold for $2\leqslant i\leqslant s$:

(i) $\tilde{V}=\mathrm{Spec}(\mathcal{O}[v_i, v_i/u^{\nu_1+\cdots+\nu_i-i}g_i(u, v_i)])$;

(ii) $\tilde{\pi}$: $\tilde{V}\to\tilde{V}_0$ is a composition of a morphism $\tilde{\pi}_i$: $\tilde{V}\to\tilde{V}_i$ and $\tilde{\sigma}_1\tilde{\sigma}_2\cdots\tilde{\sigma}_i$: $\tilde{V}_i\to\tilde{V}_0$, where $\tilde{\sigma}_i:=\sigma_i|_{\tilde{V}_i}$: $\tilde{V}_i\to\tilde{V}_{i-1}$; moreover, $\tilde{\pi}_{i-1}=\tilde{\sigma}_i\cdot\tilde{\pi}_i$;

(iii) if $Q\in(\sigma_i^{-1}(P_i)\cup(\sigma_1\cdots\sigma_i)'(G))-(\sigma_1\cdots\sigma_i)'(F)$ then $\tilde{\pi}_i^{-1}(Q)=\phi$.

When $i=s$, the proper transform $(\sigma_1\cdots\sigma_s)'(G)$ of G on V_s does not meet the proper transform $(\sigma_1\cdots\sigma_s)'(F)$ of F on $\tilde{V}_s$ (cf. the definition of s in (1.3)). Therefore, in virtue of (iii) above, we know that $g_s(u, v_s)$ is an invertible function on $\tilde{V}$, where $g_s(u, v_s)=0$ is the equation of the proper transform $(\sigma_1\cdots\sigma_s)'(G)$ of G on $\tilde{V}_s$. Thus, $\tilde{V}=\mathrm{Spec}(\mathcal{O}[v_s, v_s/u^{N-s}])$.

Furthermore, set $v_s=uv_{s+1}, \cdots, v_{N-1}=uv_N$ and $\tilde{V}_{s+1}=\mathrm{Spec}(\mathcal{O}[v_{s+1}]), \cdots, \tilde{V}_N=\mathrm{Spec}(\mathcal{O}[v_N])$. Then it is easy to see that the following assertions hold for $s+1\leqslant i\leqslant N$:

(i) $\tilde{V}=\mathrm{Spec}(\mathcal{O}[v_i, v_i/u^{N-i}])$,

(ii) $\tilde{\pi}_s$: $\tilde{V}\to\tilde{V}_s$ is a composition of a morphism $\tilde{\pi}_i$: $\tilde{V}\to\tilde{V}_i$ and $\tilde{\sigma}_{s+1}\cdots\tilde{\sigma}_i$: $\tilde{V}_i\to\tilde{V}_s$, where $\tilde{\sigma}_i=\sigma_i|_{V_i}$: $\tilde{V}_i\to\tilde{V}_{i-1}$ and $\tilde{\pi}_{i-1}=\tilde{\sigma}_i\cdot\tilde{\pi}_i$.

Then $\tilde{V}=\tilde{V}_N=\mathrm{Spec}(\mathcal{O}[v_N])$. Hence, V is isomorphic, in a neighborhood of $\pi^{-1}(P_1)$, to V_N with the curves $E_1, \cdots, E_{N-1}$ and $\sigma'(G)$ deleted off. In particular, $\pi^{-1}(P_1)=\mathcal{E}:=E_N-E_N\cap E_{N-1}$. Q.E.D.

1.5. Assume that we are given two curves (not necessarily irreducible) F, G on a nonsingular surface V_0 and a point $P_1\in F\cap G$ at which one of F and G, say F, is nonsingular. Let $P_1, P_2, \cdots, P_s$ be all points of G on F over P_1, and let $\nu_1, \cdots, \nu_s$ be the multiplicities of G at $P_1, \cdots, P_s$, respectively. Let $N=\nu_1+\cdots+\nu_s$. As explained in 1.3, define σ: $V_N\to V_0$ as a composition of quadratic transformations with centers at N points $P_1, \cdots, P_N$ on F, each P_i $(2\leqslant i\leqslant N)$ being infinitely near to P_{i-1}. We call σ: $V_N\to V_0$ *the standard transformation of* V_0 *with respect to a triplet* (P_1, F, G). The configuration of $\sigma^{-1}(F\cup G)$ in a neighborhood of $\sigma^{-1}(P_1)$ is given by the Figure 1. With the notations in the Figure 1, we have a new surface V by deleting $E_1, \cdots, E_{N-1}$ from V_N. We then say that V is obtained from V_0 by *the standard process of the first kind with respect to* (P_1, F, G). On the other hand, note that $(E_i^2)=-2$ for $1\leqslant i\leqslant N-1$. Hence we obtain a new normal surface V' from V_N by contracting $E_1, \cdots, E_{N-1}$ to a point Q_1 on V' which is a rational double point (cf. Artin [2; Theorem 2.7]). We then say that V' is obtained from V_0 by *the standard process of the second kind with respect to* (P_1, F, G).

1.6. We next consider the case where, at a point $P_1\in F\cap G$, the curve G is nonsingular. Indeed, we prove the following:

Lemma. *With the assumption as above, let V' be the surface obtained from $V_0:=A^2$ by the standard process of the second kind with respect to (P_1, G, F). Then, in a neighborhood of $\pi^{-1}(P_1)$, V is isomorphic to V' with the proper transform of G deleted off. If either F is singular at P_1 or $i(F, G; P_1)\geqslant 2$, V has a unique rational double point on $\pi^{-1}(P_1)$.*

Proof. Let $P_1, P_2, \cdots, P_r$ be all points of F on G over P_1, and let $\mu_1, \cdots, \mu_r$ be the multiplicities of F at $P_1, \cdots, P_r$, respectively. Let $M:=\mu_1+\cdots+\mu_r$. We prove the assertions by induction on M. Note that $M=1$ if and only if $i(F, G; P_1)=1$. It is then easy to see that V is isomorphic, in a neighborhood of $\pi^{-1}(P_1)$, to a surface V_1' obtained as follows: Let $\sigma_1\colon V_1\to V_0$ be the quadratic transformation of $V_0:=A^2$ with center at P_1, and let $V_1':=V_1-\sigma_1'(G)$. Now, assume that $M>1$. Since G is nonsingular at P_1 there exist local parameters u, v of V_0 at P_1 such that $v=g$. Let $f(u, v)=0$ be a local equation of F at P_1. Then, V is isomorphic, in a neighborhood of $\pi^{-1}(P_1)$, to an affine hypersurface $vz=f(u, v)$ in the affine 3-space A^3. There exists only one singular point $Q_1'\colon (u, v, z)=(0, 0, 0)$ of V lying on $\pi^{-1}(P_1)$. Let $\rho_1\colon W_1\to A^3$ be the blowing-up of A^3 with center the curve $\pi^{-1}(P_1)\colon u=v=0$, let V_1' be the proper transform of V on W_1, and let $\tau_1:=\rho_1|_{V_1'}\colon V_1'\to V$ be the restriction of ρ_1 onto V_1'.

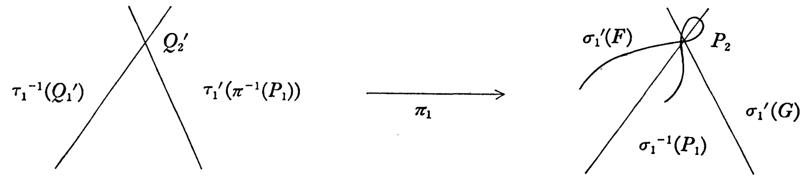
Set $v=uv_1$, $u=vu_1$ and $f(u, uv_1)=u^{\mu_1}f_1(u, v_1)$, $f(vu_1, v)=v^{\mu_1}\tilde{f}_1(u_1, v)$. Then V_1' is given by $v_1z=u^{\mu_1-1}f_1(u, v_1)$ with respect to the coordinate system (u, v_1, z) and by $z=v^{\mu_1-1}\tilde{f}_1(u_1, v)$ with respect to the coordinate system (u_1, v, z). By construction of V_1', V_1' dominates the surface V_1 obtained from V_0 by the quadratic transformation σ_1 with center at P_1;

$$\begin{array}{ccc} V_1' & \xrightarrow{\tau_1} & V \\ \Big\downarrow{\scriptstyle \pi_1} & & \Big\downarrow{\scriptstyle \pi} \\ V_1 & \xrightarrow{\sigma_1} & V_0\,. \end{array}$$

The proper transform $\tau_1'(\pi^{-1}(P_1))$ of $\pi^{-1}(P_1)$ on V_1' is given by $u=v_1=0$; the curve $\tau_1^{-1}(Q_1')$ is given by $u=z=0$; $\tau_1\colon V_1'-\tau_1^{-1}(Q_1')\cong V-\{Q_1'\}$; the singular point of V_1' is possibly $Q_2'\colon (u, v_1, z)=(0, 0, 0)$.

The morphism $\pi_1\colon V_1'\to V_1$ is isomorphic at every point of $\tau_1^{-1}(Q_1')-\{Q_2'\}$.

Indeed, if $v_1 \neq 0$ or ∞, π_1 is given by $(u, v_1, z)=(u, v_1, u^{\mu_1-1}f_1(u, v_1)/v_1) \mapsto (u, v_1)$ which is clearly isomorphic; if $v_1=\infty$, π_1 is given by $(u_1, v, v^{\mu_1-1}\tilde{f}_1(u_1, v)) \mapsto (u_1, v)$ which is isomorphic as well. Under this isomorphism, $\tau_1^{-1}(Q_1')$ corresponds to $\sigma_1^{-1}(P_1)$:



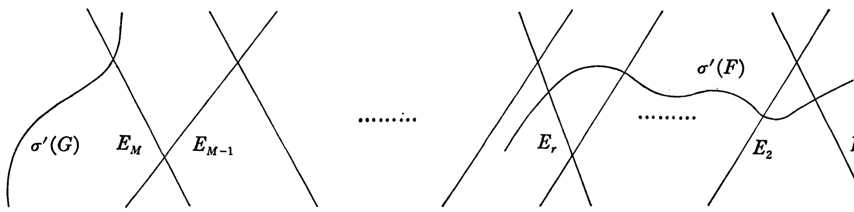
Note that the following assertions hold:

(i) V_1' is isomorphic, in a neighborhood of $\pi_1^{-1}(P_2)$, to an affine hypersurface $v_1 z=u^{\mu_1-1}f_1(u, v_1)$ on A^3;

(ii) in a neighborhood of P_2, $\sigma_1'(G)$ is defined by $v_1=0$ and $\sigma_1'(F)$ is defined by $f_1(u, v_1)=0$;

(iii) $P_2, \cdots, P_r$ are all points of the curve F_1: $u^{\mu_1-1}f_1(u, v_1)=0$ on $\sigma_1'(G)$ over P_2, and the sum of multiplicities of the curve F_1 at $P_2, \cdots, P_r$ is $M-1$.

Then, by the assumption of induction applied to V_1', we obtain V_1' from the surface V_1'', which is obtained from V_1 by the standard process of the second kind with respect to a triplet $(P_2, \sigma_1'(G), F_1)$, by deleting the proper transform of $\sigma_1'(G)$ on V_1'':



(Fig. 2)

where the surface V_1'' is obtained by contracting $E_2, \cdots, E_{M-1}$. Then it is easy to see that V is isomorphic, in a neighborhood of $\pi^{-1}(P_1)$, to the surface V', which is obtained from V_1'' by contracting $E_1, \cdots, E_{M-1}$, with the proper transform of $\sigma'(G)$ deleted off. Hence, the unique singular point of V lying on $\pi^{-1}(P_1)$ is a rational double point. Q.E.D.

1.7. Let $P_1 \in F \cap G$, and assume that G is nonsingular at P_1. Let $P_1, P_2, \cdots, P_r$ be all points of F on G over P_1, and let $\mu_1, \cdots, \mu_r$ be the multiplicities of F at $P_1, \cdots, P_r$, respectively. If $f=cf_1^{\alpha_1}\cdots f_m^{\alpha_m}$ $(c \in k^*)$ is a decomposition of f into distinct irreducible factors, let F_j $(1 \leq j \leq m)$ be the curve on V_0 defined by $f_j=0$. Let $\mu_i(j)$ be the multiplicity of F_j at P_i for $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant m$. Then it is clear that $\mu_i=\alpha_1\mu_i(1)+\cdots+\alpha_m\mu_i(m)$ for $1 \leqslant i \leqslant r$.

1.8. As a consequence of Lemmas 1.4 and 1.6, we have the following:

Theorem. *Assume that V has only isolated singularities. Let W be the surface obtained from $V_0:=A^2$ by the standard processes of the first (or the second) kind at every point of $F\cap G$. Then V is isomorphic to the surface W with the proper transform of G on W deleted off. The surface V is, therefore, a normal surface whose singular points (if any) are rational double points.*

1.9. In the paragraphs 1.9$\sim$1.11 we shall study the divisor class group $Cl(V)$. Let $g=cg_1^{\beta_1}\cdots g_n^{\beta_n}$ $(c\in k^*)$ be a decomposition of g into distinct irreducible factors, and let G_j be the curve $g_j=0$ on V_0 for $1\leqslant j\leqslant n$. Assume that $F\cap G\neq\phi$. Let $F\cap G=\{P_1^{(1)},\cdots,P_1^{(e)}\}$. For $1\leqslant l\leqslant e$, either F is nonsingular at $P_1^{(l)}$ but G is not, or G is nonsingular at $P_1^{(l)}$. We may assume that F is nonsingular at $P_1^{(1)},\cdots,P_1^{(a)}$ but G is not, and G is nonsingular at $P_1^{(a+1)},\cdots,P_1^{(e)}$. (The number a may be 0.) For $l\leqslant a$, let $P_1^{(l)},\cdots,P_{s_l}^{(l)}$ be all points of G on F over $P_1^{(l)}$, and let $\nu_i^{(l)}(j)$ be the multiplicity of G_j at $P_i^{(l)}$ for $1\leqslant i\leqslant s_l$ and $1\leqslant j\leqslant n$; let $N^{(l)}(j)=\nu_1^{(l)}(j)+\cdots+\nu_{s_l}^{(l)}(j)$, let $\nu_i^{(l)}=\beta_1\nu_i^{(l)}(1)+\cdots+\beta_n\nu_i^{(l)}(n)$ and let $N^{(l)}=\beta_1N^{(l)}(1)+\cdots+\beta_nN^{(l)}(n)$. For $a+1\leqslant l\leqslant e$, let $P_1^{(l)},\cdots,P_{r_l}^{(l)}$ be all points of F on G over $P_1^{(l)}$, and let $\mu_i^{(l)}$ be the multiplicity of F at $P_i^{(l)}$ for $1\leqslant i\leqslant r_l$. Let $M^{(l)}=\mu_1^{(l)}+\cdots+\mu_{r_l}^{(l)}$. Since G is nonsingular at $P_1^{(l)}$, there exists a unique G_j $(1\leqslant j\leqslant n)$ such that $P_1^{(l)},\cdots,P_{r_l}^{(l)}$ lie on G_j. Then we set $M^{(l)}(j)=M^{(l)}$ and $M^{(l)}(j')=0$ for $j'\neq j$. Let $\varepsilon^{(l)}=\pi^{-1}(P_1^{(l)})$ for $1\leqslant l\leqslant e$.

1.10. The structure of the divisor class group $Cl(V)$ is given by the following:

Theorem. *With the notations as above, the divisor class group $Cl(V)$ is isomorphic to:*

$$\{\boldsymbol{Z}\varepsilon^{(1)}+\cdots+\boldsymbol{Z}\varepsilon^{(e)}\}/\{\sum_{l=1}^{a}N^{(l)}(j)\varepsilon^{(l)}+\sum_{l=a+1}^{e}M^{(l)}(j)\varepsilon^{(l)};\ 1\leqslant j\leqslant n\}\ .$$

Proof. Embed $V_0:=A^2$ into the projective plane $\boldsymbol{P}^2$ in a canonical way as an open set, and let $l_\infty:=\boldsymbol{P}^2-V_0$. For $1\leqslant l\leqslant e$, let $E_1^{(l)},\cdots,E_q^{(l)}$ be all exceptional curves which arise by the standard transformation of V_0 with respect to a triplet $(P_1^{(l)},F,G)$ (or $(P_1^{(l)},G,F)$) where $q=N^{(l)}$ (or $M^{(l)}$). Let $\tau\colon W\to\boldsymbol{P}^2$ be a composition of standard transformations of $\boldsymbol{P}^2$ with respect to triplets $(P_1^{(l)},F,G)$ for $1\leqslant l\leqslant a$ and triplets $(P_1^{(l)},G,F)$ for $a+1\leqslant l\leqslant e$. Then it is easy to see that the divisor

$$(g_j)_W-\{\sum_{l=1}^{a}N^{(l)}(j)E_{N^{(l)}}^{(l)}+\sum_{l=a+1}^{e}M^{(l)}(j)E_{M^{(l)}}^{(l)}\}\ (1\leqslant j\leqslant n)$$

has support on $\tau'(G_j)$, $\tau'(l_\infty)$, $E_1^{(l)},\cdots,E_{q-1}^{(l)}$ $(q=N^{(l)}$ or $M^{(l)})$ for $1\leqslant l\leqslant e$. Hence we have:

$$\sum_{l=1}^{a}N^{(l)}(j)\varepsilon^{(l)}+\sum_{l=a+1}^{a}M^{(l)}(j)\varepsilon^{(l)}\sim 0\qquad(1\leqslant j\leqslant n)\,.$$

Now, let C be an irreducible curve on V such that $\pi(C)$ is not a point, and let the closure of $\pi(C)$ be defined by $h=0$ with $h\in k[x, y]$. Then, by considering the divisor $(h)_W$ on W, we easily see that C is linearly equivalent to an integral combination of $\mathcal{E}^{(1)}, \cdots, \mathcal{E}^{(e)}$. Hence , by setting

$$\mathcal{G}:=\{Z\bar{\mathcal{E}}^{(1)}+\cdots+Z\bar{\mathcal{E}}^{(e)}\}/\{\sum_{l=1}^{a} N^{(l)}(j)\bar{\mathcal{E}}^{(l)}+\sum_{l=a+1}^{e} M^{(l)}(j)\bar{\mathcal{E}}^{(l)};\ 1\leqslant j\leqslant n\}\ ,$$

we have a surjective homomorphism:

$$\theta\colon \mathcal{G}\to Cl(V);\ \theta(\bar{\mathcal{E}}^{(l)})=\mathcal{E}^{(l)}\ (1\leqslant l\leqslant e)\,.$$

Assume that Ker $\theta\neq(0)$, and let $d_1\mathcal{E}^{(1)}+\cdots+d_e\mathcal{E}^{(e)}=(t)_V$ on V, where $d_l\in Z$ $(1\leqslant l\leqslant e)$ and $t\in k(V)$. Let $(t)_{V_0}=\sum m_iC_i$ with irreducible curves C_i and $m_i\in Z$. Let $t_i\in k[x, y]$ be such that C_i is given by $t_i=0$, and write:

$$(t_i)_V=\pi'(C_i)+\sum_{l=1}^{e} b_{il}\mathcal{E}^{(l)}\ \text{ with } b_{il}\in Z\,.$$

Then we have:

$$(t)_V=\sum_i\ \{m_i\pi'(C_i)+\sum_{i=1}^{e} m_ib_{il}\mathcal{E}^{(l)}\}\ =\sum_{l=1}^{e} d_l\mathcal{E}^{(l)}\,.$$

Therefore, either $m_i=0$ for every i, or $\pi'(C_i)=\phi$ for every i. In the first case, t is a constant $\in k$, whence $d_l=0$ for $1\leqslant l\leqslant e$. In the second case, C_i must coincide with one of G_j's $(1\leqslant j\leqslant n)$. Then $d_1\bar{\mathcal{E}}^{(1)}+\cdots+d_e\bar{\mathcal{E}}^{(e)}=0$ in $\mathcal{G}$. This is a contradiction. Therefore, θ is an isomorphism. Q.E.D.

1.11. The affine domain $A=k[x, y, f/g]$ is a unique factorization domain if and only if $Cl(V)=(0)$. We have the following two consequences of 1.10.

1.11.1. **Corollary.** *With the notation of* 1.9, *if* $e>n$ *then* A *is not a unique factorization domain.*

1.11.2. **Corollary.** *Assume that g is irreducible and that* $F\cap G\neq\phi$. *Then* A *is a unique factorization domain if and only if the curves* F *and* G *meet each other in only one point where they intersect each other transversally.*

1.12. Let A^* be the group of all invertible elements of $A=k[x, y, f/g]$. Then A^* contains $k^*=k-(0)$ as a subgroup. By virtue of ([4], Remark 2, p. 174) we know that A^*/k^* is a torsion-free Z-module of finite rank and A^* is isomorphic to a direct product of k^* and A^*/k^*. The purpose of this paragraph is to determine the group A^*/k^*. Let H be the subgroup of $Z\mathcal{E}^{(1)}+\cdots+Z\mathcal{E}^{(e)}$ generated by

$$\{\sum_{l=1}^{a} N^{(l)}(j)\mathcal{E}^{(l)}+\sum_{l=a+1}^{e} M^{(l)}(j)\mathcal{E}^{(l)};\ 1\leqslant j\leqslant n\}\ .$$

Let $T_1, \cdots, T_n$ be n-indeterminates, and let $\eta\colon Z^{(n)}:=ZT_1+\cdots+ZT_n\to H$ be a homomorphism such that, for $1\leqslant j\leqslant n$,

$$\eta(T_j) = \sum_{l=1}^{a} N^{(l)}(j)\mathcal{E}^{(l)} + \sum_{l=a+1}^{e} M^{(l)}(j)\mathcal{E}^{(l)} .$$

Let L be the kernel of η. Since $N^{(l)}(j)$ and $M^{(l)}(j)$ are non-negative integers for $1 \leqslant l \leqslant e$ and $1 \leqslant j \leqslant n$, each nonzero element of L is written in the form: $\gamma_1 T_1 + \cdots + \gamma_n T_n$ ($\gamma_i \in \boldsymbol{Z}$), where some of γ_i's are negative. Define a homomorphism ξ: $L \rightarrow K^*$ (where $K = k(x, y)$) by $\xi(\gamma_1 T_1 + \cdots + \gamma_n T_n) = g_1^{\gamma_1} \cdots g_n^{\gamma_n}$. Then we have the following:

Lemma. *The homomorphism ξ induces an isomorphism ξ: $L \xrightarrow{\sim} A^*/k^*$.*

Proof. (1) Since $(g_j)_V = \sum_{l=1}^{a} N^{(l)}(j)\mathcal{E}^{(l)} + \sum_{l=a+1}^{e} M^{(l)}(j)\mathcal{E}^{(l)} = \eta(T_j)$ for $1 \leqslant j \leqslant n$, we have:

$$\eta(\gamma_1 T_1 + \cdots + \gamma_n T_n) = (g_1^{\gamma_1} \cdots g_n^{\gamma_n})_V .$$

Therefore, if $\gamma_1 T_1 + \cdots + \gamma_n T_n \in L$ then $g_1^{\gamma_1} \cdots g_n^{\gamma_n}$ is an invertible element of A, which is a constant if and only if $\gamma_1 = \cdots = \gamma_n = 0$. Thus, ξ is a monomorphism from L into A^*/k^*.

(2) Let t be a non-constant invertible element of A. Write $(t)_{V_0} = \sum_i m_i C_i$ with irreducible curves C_i and $m_i \in \boldsymbol{Z}$. Let C_i be defined by $t_i = 0$ with $t_i \in k[x, y]$. As in the proof of 1.10, write:

$$(t_i)_V = \pi'(C_i) + \sum_{l=1}^{e} b_{il}\mathcal{E}^{(l)} \quad \text{with} \quad b_{il} \in \boldsymbol{Z} .$$

Then we have:

$$(t)_V = \sum_i \{m_i \pi'(C_i) + \sum_{l=1}^{e} m_i b_{il} \mathcal{E}^{(l)}\} = 0 .$$

Therefore, either $m_i = 0$ for every i, or $\pi'(C_i) = \phi$ for every i. The first case does not occur because, if otherwise, t is a constant. In the second case, C_i must coincide with one of G_j's. Hence we could write:

$$(t)_{V_0} = \sum_{i=1}^{n} m_i G_i .$$

Then $t = c g_1^{m_1} \cdots g_n^{m_n}$ with $c \in k^*$. It is then clear that $m_1 T_1 + \cdots + m_n T_n \in L$ and $\xi(m_1 T_1 + \cdots + m_n T_n) = t$. Therefore, ξ: $L \rightarrow A^*/k^*$ is an isomorphism. Q.E.D.

1.13. By virtue of 1.10 and 1.12, we have the following:

Theorem. *Assume that V has only isolated singularities. Then we have the following exact sequence of $\boldsymbol{Z}$-modules:*

$$0 \rightarrow A^*/k^* \rightarrow \boldsymbol{Z}^{(n)} \rightarrow \boldsymbol{Z}^{(e)} \rightarrow Cl(V) \rightarrow 0 ,$$

where $\boldsymbol{Z}^{(r)}$ stands for a free $\boldsymbol{Z}$-module of rank r; n is the number of distinct irreducible *factors of* g; e *is the number of distinct points of* $F\cap G$.

1.14. REMARKS. (1) It is clear from 1.13 that if g is irreducible then $A^*=k^*$.

(2) rank $(Cl(V))-$rank $(A^*/k^*)=e-n$.

(3) Though we proved Theorem 1.13 under the assumption that $F\cap G\neq\phi$ it is clear that the theorem is valid also in the case where $F\cap G=\phi$.

2. Locally nilpotent derivation on $k[x, y, f/g]$

2.1. Let A be an affine k-domain. A k-derivation D on A is said to be *locally nilpotent* if, for every element a of A, $D^n(a)=0$ for sufficiently large n. If D is a locally nilpotent k-derivation on A, we define a k-algebra homomorphism

$$\Delta\colon A\to A[X] \quad \text{(with an indeterminate } X\text{)}$$

by

$$\Delta(a)=\sum_{n\geq 0}(1/n!)D^n(a)X^n\,.$$

Then it is known (cf. [6]) that Δ gives rise to an action of the additive group scheme G_a on Spec(A). Conversely, every action of G_a on Spec(A) is expressed in the above-mentioned way with some locally nilpotent k-derivation on A. We set $A_0:=\{a\in A\,|\,D(a)=0\}$. Then A_0 is an inert subring of A, and A_0 is, in fact, the ring of G_a-invariant elements of A with respect to the corresponding G_a-action on Spec(A). For other relevant results on these materials, the readers are referred to [4] and [6].

2.2. In this section, we set $A=k[x,y,f/g]$, and assume that A is normal. Assume that A has a nonzero locally nilpotent k-derivation D. Then we assert the following:

Lemma. *The subring* A_0 *of* D*-constants is a finitely generated, normal, rational* k*-domain of dimension* 1.

Proof. Since A_0 is the ring of G_a-invariants in a normal domain A, A_0 is integrally closed in the quotient field $Q(A_0)$ of A_0 and $A_0=A\cap Q(A_0)$, where $Q(A_0)$ is the field of G_a-invariants in the quotient field $Q(A)$ of A. Then, by virtue of Zariski's Theorem (cf. Nagata [7; p. 52]), A_0 is a finitely generated normal k-domain of dimension 1. Besides, A_0 is rational over k by Lüroth's Theorem because A is rational over k. Q.E.D.

2.3. Let $V:=$Spec(A). Then V has a nontrivial G_a-action corresponding to the derivation D on A. Let $U:=$Spec(A_0); U is isomorphic to an open set of the affine line A^1 (cf. 2.2). Let $q\colon V\to U$ be the morphism defined by the canonical inclusion $A_0\hookrightarrow A$. By 2.2, we know that $A_0=k[t, 1/h(t)]$ with $h(t)\in k[t]$. For almost all elements α of k such that $h(\alpha)\neq 0$, the fibre $q^{-1}(\alpha)$ is a G_a-orbit and is, therefore, isomorphic to the affine line. Let $\rho\colon V'\to V$ be the minimal resolution of singularities of V. As we saw in 1.8, singular points of V are rational double points. Hence, ρ is a composition of quadratic transformations with centers at singular points. Let $q' := q \cdot \rho$: $V' \rightarrow U$. Almost all fibres of q' are therefore isomorphic to the affine line. Now we shall prove the following:

Lemma. *There exist a nonsingular projective surface W and a surjective morphism p: $W \rightarrow \boldsymbol{P}^1$ satisfying the following conditions*:

(1) *Almost all fibres of p are isomorphic to $\boldsymbol{P}^1$.*

(2) *There exists an open immersion ι: $V' \rightarrow W$ such that $p \cdot \iota = \overline{\iota} \cdot q'$, where $\overline{\iota}$: $U \hookrightarrow \boldsymbol{P}^1$ is the canonical open immersion via $U \hookrightarrow \boldsymbol{A}^1 := \mathrm{Spec}(k[t])$.*
Then the fibration p has a cross-section S such that $S \subset W - \iota(V')$.

Proof. Let $\bar{V}$ be a nonsingular projective surface containing V' as an open set. Then, a subfield $k(t)$ of $k(V') = k(\bar{V})$ defines a linear pencil $\bar{\Lambda}$ of effective divisors on $\bar{V}$ such that a general member of $\bar{\Lambda}$ cuts out a general fibre of q' on V'. The base points of $\bar{\Lambda}$ are situated on $\bar{V} - V'$. Let θ: $W \rightarrow \bar{V}$ be the shortest succession of quadratic transformations of $\bar{V}$ with centers at the base points of $\bar{\Lambda}$ such that the proper transform Λ of $\bar{\Lambda}$ by θ has no base points, and let p: $W \rightarrow \boldsymbol{P}^1$ be the morphism defined by Λ. Since V' is canonically embedded into W as an open set, let ι: $V' \rightarrow W$ be the canonical immersion. Then it is not hard to see that p: $W \rightarrow \boldsymbol{P}^1$ and ι: $V' \rightarrow W$ satisfy the conditions (1), (2) of Lemma.
Q.E.D.

2.4. We shall prove the following:

Lemma. (cf. [3]). *Let p: $W \rightarrow \boldsymbol{P}^1$ be a surjective morphism from a nonsingular projective surface W onto $\boldsymbol{P}^1$ such that almost all fibres are isomorphic to $\boldsymbol{P}^1$. Let $F = n_1C_1 + \cdots + n_rC_r$ be a reducible fibre of p, where C_i is an irreducible curve and $n_i > 0$. Then we have*:

(1) *For $1 \leqslant i \leqslant r$, C_i is isomorphic to $\boldsymbol{P}^1$ and $(C_i^2) < 0$.*

(2) *For $i \neq j$, C_i and C_j do not intersect or intersect transversally at a single point.*

(3) *For distinct indices i, j and l, $C_i \cap C_j \cap C_l = \phi$.*

(4) *One of C_i's, say C_1, is an exceptional curve of the first kind. If τ: $W \rightarrow W_1$ is the contraction of C_1, then p factors as p: $W \xrightarrow{\tau} W_1 \xrightarrow{p_1} \boldsymbol{P}^1$, where p_1: $W_1 \rightarrow \boldsymbol{P}^1$ is a fibration by $\boldsymbol{P}^1$.*

Proof. For each i, $n_i(C_i^2) + \sum_{i \neq j} n_j(C_i \cdot C_j) = 0$, where $(C_i \cdot C_j) > 0$ for some j because F is connected. Hence $(C_i^2) < 0$. To prove the remaining assertions we have only to show that one of C_i's is an exceptional curve of the first kind. Let K be the canonical divisor of W. Then $(F \cdot K) = -2$ because $p_a(F) = 0$. Hence, $-2 = (F \cdot K) = \sum_i n_i(C_i \cdot K) = \sum_i n_i(2p_a(C_i) - 2 - (C_i^2))$, where $2p_a(C_i) - 2 - (C_i^2) \geqslant -1$ and the equality holds if and only if C_i is an exceptional curve of the first kind. However, it is impossible that $2p_a(C_i)-2-(C_i^2)\geqslant 0$ for every i. Therefore, $2p_a(C_i)-2-(C_i^2)=-1$ for some i. Q.E.D.

2.5. With the notations of 2.3, Lemma 2.4 implies:

Lemma. *Write $W-\iota(V')=\bigcup_{i=1}^{r} C_i$ with irreducible curves C_i. Then we have:*

(1) *Every C_i is isomorphic to $\boldsymbol{P}^1$.*

(2) *For $i\neq j$, C_i and C_j meet each other (if at all) in a single point where they intersect transversally.*

(3) *For distinct indices i, j and l, $C_i\cap C_j\cap C_l=\phi$.*

(4) *$\bigcup_{i=1}^{r} C_i$ does not contain any cyclic chains.*

Proof. Note that one of C_i's is the cross-section S and the other components are contained in the fibres of p. Then the above assertions follow from 2.4. Q.E.D.

2.6. Let $V_0:=\mathrm{Spec}(k[x, y])$, and let F, G be as in 1.1. Let G_i $(1\leqslant i\leqslant n)$ be as in 1.9. Embed V_0 into $\boldsymbol{P}^2$ in a canonical way. Let $l_\infty:=\boldsymbol{P}^2-V_0$ and let $\bar{F}, \bar{G}, \bar{G}_j$ $(1\leqslant j\leqslant n)$ be the closures of F, G, G_j in $\boldsymbol{P}^2$, respectively. Let τ: $Z\rightarrow\boldsymbol{P}^2$ be a composition of the standard transformations of $\boldsymbol{P}^2$ with respect to triplets (P, F, G) (or (P, G, F)), where P runs over all points of $F\cap G$. Then we know that V' is embedded into Z as an open set. We may assume, by replacing W if necessary by a surface which is obtained from W by a succession of the quadratic transformations, that there exists a birational morphism φ: $W\rightarrow Z$ such that we have the following commutative diagram:

$$\begin{tikzcd}
 & W \arrow[r, "p"] \arrow[dl, "\varphi"'] & \boldsymbol{P}^1 \\
Z \arrow[dd, "\tau"] & V' \arrow[l, hook'] \arrow[u, hook, "\iota"'] \arrow[r, "q'"] \arrow[d, "\rho"] & U \arrow[u, hook, "\bar{\iota}"'] \\
 & V \arrow[d, "\pi"] & \\
\boldsymbol{P}^2 & V_0 \arrow[l, hook'] &
\end{tikzcd}$$

2.7. Let $P_1\in F\cap G$. Assume that F is nonsingular at P_1 but G is not. Then, in a neighborhood of $\tau^{-1}(P_1)$, $\tau^{-1}(F\cup G)$ has the configuration as in the Figure 1. With the notations of the Figure 1, we can show the following assertions:

(1) *A general fibre λ of p may intersect $\varphi'(E_N)$.*

(2) *$\varphi'(E_1), \cdots, \varphi'(E_{N-1})$ are contained in one and only one fibre of p.*

Indeed, $\lambda_{V'}=\lambda\cap V'$ is isomorphic to the affine line, and $\tau\varphi(\lambda-\lambda_{V'})$ lies on l_∞. Hence λ does not meet any of $\varphi(E_1), \cdots, \varphi(E_{N-1})$. This proves the second assertion. By the same reason, we have:

(3) *For* $1 \leqslant j \leqslant n$, $(\tau\varphi)'(\bar{G}_j)$ *is contained in a fibre of* p. *In particular,* $(\tau\varphi)'(\bar{G}_j)$ *is isomorphic to* $\boldsymbol{P}^1$.

2.8. We have the following:

Lemma. (1) *For* $1 \leqslant j \leqslant n$, G_j *has one place at infinity; every singular point of* G_j *is a one-place point.*

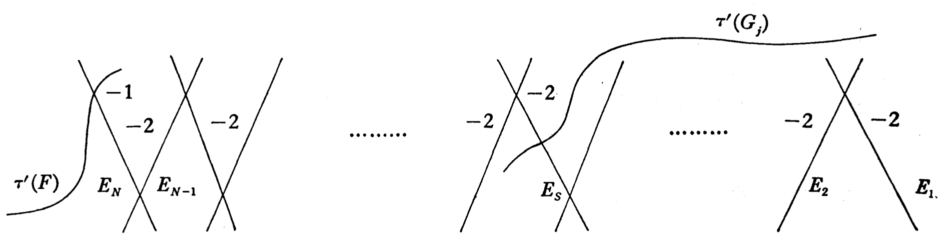
(2) *For distinct* i, j $(1 \leqslant i, j \leqslant n)$, $G_i \cap G_j = \phi$.

Proof. Note that if φ is not an isomorphism φ is a composition of quadratic transformations of Z with centers at a point on $\tau'(l_\infty)$ and its infinitely near points. Then, both assertions follow from 2.5 (cf. 2.7). Q.E.D.

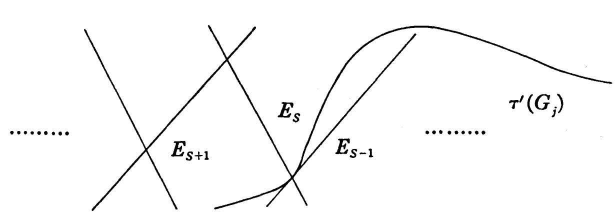
2.9. We prove the following:

Lemma. *For* $1 \leqslant j \leqslant n$, *the curve* G_j *is nonsingular.*

Proof. Note that if P is a singular point of G_j then P is a point of $F \cap G_j$ (cf. 2.7, (3)). Assume that $P \in F \cap G_j$. Then, in a neighborhood of $\tau^{-1}(P)$, $\tau^{-1}(F \cap G_j)$ must have the following configuration as in the Figure 1:



where $\varphi'(E_1), \cdots, \varphi'(E_{N-1})$ and $(\tau\varphi)'(G_j)$ belong to the same fibre of p. Note that $(\tau\varphi)'(G_j)$ intersects $\varphi'(E_s)$ transversally in one point if $N \geqslant s+1$ (cf. Lemma 2.4). Assume that $\nu_b \geqslant 2$ and $\nu_{b+1} = \cdots = \nu_s = 1$. (For the notations, see 1.3.) Such b exists because we assume that P is a singular point of G_j. Then $N \geqslant s+1$, and it is not hard to see that $s = b+1$, and that we have the configuration:



where $\tau'(G_j)$ touches E_{s-1} with $(\tau'(G_j) \cdot E_{s-1}) = \nu_b - 1$. This contradicts Lemma 2.4, (3). Therefore, the curve G_j is nonsingular. Q.E.D.

2.10. Now we can prove:

Theorem. *Assume that V has only isolated singularities. Then A has a nonzero locally nilpotent k-derivation if and only if we have $g\in k[y]$ after a suitable change of coordinates x, y of $k[x, y]$.*

Proof. Assume that $g\in k[y]$ after a suitable change of coordinates x, y of $k[x, y]$. Then $D=g\frac{\partial}{\partial x}$ is a nonzero locally nilpotent k-derivation on A. We prove the converse. With the notations of 2.1~2.9, G_j $(1\leqslant j\leqslant n)$ is a nonsingular rational curve with one place at infinity (cf. 2.8, (1) and 2.9). [Note that G_j is a rational curve because $(\tau\varphi)'(\bar{G}_j)$ is a component of a fibre of p (cf. 2.4).] Hence, G_j is isomorphic to the affine line A^1. By virtue of the Embedding Theorem of Abhyankar-Moh (cf. [1], [5]), we may assume that $g_1=y$ after a suitable change of coordinates x, y of $k[x, y]$. Then, for $2\leqslant j\leqslant n$, g_j is written in the form: $g_j=c_j+yh_j$ with $c_j\in k$ and $h_j\in k[x, y]$ because $G_j\cap G_1=\phi$ (cf. 2.8, (2)). On the other hand, the fact that G_j has only one place at infinity implies that the curve $g_j=\alpha$ on A^2 is irreducible for every $\alpha\in k$ (cf. [5]). Therefore, h_j is a constant $\in k$. Thus, $g\in k[y]$. Q.E.D.

2.11. We know by [4; Theorem 1] that A is isomorphic to a polynomial ring over k if and only if A satisfies the following conditions:

(1) A is a unique factorization domain,

(2) $A^*=k^*$,

(3) A has a nonzero locally nilpotent k-derivation.

The condition (1) above can be described as follows:

Lemma. *Assume that A satisfies the conditions* (2) *and* (3) *above. We may assume that $g\in k[y]$ after a suitable change of coordinates x, y of $k[x, y]$. Write: $f(x, y)=a_0(y)+a_1(y)x+\cdots+a_r(y)x^r$ with $a_i(y)\in k[y]$ $(0\leqslant i\leqslant r)$. Then A is a unique factorization domain if and only if $a_1(y)$ is a unit modulo $gk[x, y]$ and $a_i(y)$ is nilpotent modulo $gk[x, y]$ for $2\leqslant i\leqslant r$.*

Proof. Assume that A is a unique factorization domain. With the notations of 1.9, $a=0$ because every G_j $(1\leqslant j\leqslant n)$ is nonsingular and $G_i\cap G_j=\phi$ if $i\neq j$. By virtue of 1.13, we have: $e=n$. Theorem 1.10 then implies that every G_i intersects F transversally. This is easily seen to be equivalent to the condition on $f(x, y)$ in the above statement. The "if" part of Lemma will be clear by the above argument and Theorem 1.10.

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References

[1] S.S. Abhyankar and T.T. Moh: *Embeddings of the line in the plane*, J. Reine Angew. Math. **276** (1976), 148–166.

[2] M. Artin: *Some numerical criteria for contractibility of curves on algebraic surfaces*, Amer. J. Math. **84** (1962), 485–496.

[3] M.H. Gizatullin: *On affine surfaces that can be completed by a nonsingular rational curve*, Izv. Akad. Nauk SSSR, Ser. Mat. **34** (1970), 778–802; Math. USSR-Izvestija, **4** (1970), 787–810.

[4] M. Miyanishi: *An algebraic characterization of the affine plane*, J. Math. Kyoto Univ. **15** (1975), 169–184.

[5] M. Miyanishi: *Analytic irreducibility of certain curves on a nonsingular affine rational surface*, to appear in the Proceedings of the International Symposium on Algebraic Geometry, Kyoto, 1977.

[6] M. Miyanishi and Y. Nakai: *Some remarks on strongly invariant rings*, Osaka J. Math. **12** (1975), 1–17.

[7] M. Nagata: Lectures on the fourteenth problem of Hilbert, Tata Institute of Fundamental Research, 1965, Bombay.

[8] P. Russell: *Simple birational extension of two-dimensional affine rational domains*, Compositio Math. **33** (1976), 197–208.

[9] A. Sathaye: *On linear planes*, Proc. Amer. Math. Soc. **56** (1976), 1–7.