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UNIFORM APPROXIMATION IN SEVERAL COMPLEX VARIABLES

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Introduction

Let K be a compact subset of the n-dimensional complex Euclidean space $\boldsymbol{C}^n$. Let $C(K)$ be the Banach algebra of all complex valued continuous functions defined on K, equipped with the sup-norm. There are several important closed subalgebras of $C(K)$. The problem of uniform approximation is to find the conditions that some of these subalgebras coincide with each other. Among these, we shall mainly deal with the problem for the subalgebra $H(K)$, the closure in $C(K)$ of the class of functions each of which is the restriction of a function holomorphic in a neighborhood of K. When $n=1$, this is the problem of rational approximation. When $n>1$, known results for $H(K)$, for the most part, were concerned with the case when K is the closure of a bounded domain with smooth boundary or a compact subset of a smooth real submanifold of $\boldsymbol{C}^n$.

The problem of finding the conditions under which $H(K)$ coincides with $C(K)$, when K is a compact subset of a smooth real submanifold of $\boldsymbol{C}^n$, originated with Wermer [13] and has been studied by several authors (Hörmander-Wermer [5], Nirenberg-Wells [7], Čirka [1], and Harvey-Wells [2]). The result of Hörmander-Wermer is the following:

Let M be a smooth real submanifold of $\boldsymbol{C}^n$ without complex tangent. Then, for every compact subset K of M, $H(K)=C(K)$ holds.

In this paper, we shall deal with the case when K is a subset of the zero set T of a nonnegative strictly plurisubharmonic function. Such a set T will be called a *totally real set*. (This use of terminology is supported by the fact that a smooth real submanifold M of $\boldsymbol{C}^n$ is a totally real set if and only if M has no complex tangents (Corollary of Proposition 6).) It is known that a totally real set is locally a subset of a totally real submanifold (Harvey-Wells [3]). Therefore, the local approximation theorem for totally real sets follows at once from the theorem of Hörmander-Wermer cited above. The main purpose of this paper is to establish the following global approximation theorem:

Let T be a totally real set. Then, for every compact subset K of T, $H(K)$

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$=C(K)$ *holds.*

The main tool that we shall make use of is the L^2-estimate for solutions of $\bar{\partial}$-problem due to Hörmander [4].

In order to establish the uniform approximation theorem for $H(K)$, we shall consider, in Section 2, some convexity condition on K that is a variant of the uniform H-convexity condition introduced by Čirka [1]. It will be proved in Section 5 that a compact totally real set satisfies a certain convexity condition of this kind.

In Section 3, we shall give a local representation of CR-submanifolds (Theorem 1). In Section 4, we shall define a totally real set and study some of its properties. Section 5 will be devoted to the proof of the main theorem (Theorem 2). The essential part of the proof consists of Lemma 4 and 6. In Section 6, we shall give an example of a smooth real submanifold M containing a set of points at which M has non zero complex tangents, while $H(K)=C(K)$ holds for every compact subset of M. To derive this example, we need to generalize a theorem due to Mergeljan. Section 7 is concerned with the problem of the (peak) interpolation for a nondegenerate analytic polyhedron or a strictly pseudoconvex domain (not necessarily with smooth boundary), as an application of the main theorem. This problem has been extensively discussed for a polydisk (cf. Stout [11]). In the last section, we shall prove an approximation theorem for CR-functions in some globally presented case. It seems to the author that the main difficulty in proving the approximation theorem for CR-functions in general form consists in the proof of the extension lemma corresponding to Lemma 6.

1. Notations and preliminaries

We denote by $\boldsymbol{C}^n$ the complex n-dimensional Euclidean space. When we must emphasize the complex coordinates $z=(z_1, \cdots, z_n)$, it will be denoted by $\boldsymbol{C}^n_z$. Similarly, $\boldsymbol{C}^k_{(z_1,\cdots,z_k)}$ or $\boldsymbol{R}^k_{(u_1,\cdots,u_k)}$ denotes the subspace with the coordinates $(z_1, \cdots, z_k)$ or $(u_1, \cdots, u_k)$ respectively. For any point z of $\boldsymbol{C}^n$, $|z|$ denotes the Euclidean norm of z. For a subset S of $\boldsymbol{C}^n$, we define the distnace function $d_S(z)=\inf\{|\zeta-z| : \zeta\in S\}$ and the ε-neighborhood $U_\varepsilon(S)=\{z: d_S(z)<\varepsilon\}$ of S. $B_n(a, r)$ denotes the n-dimensional ball $\{z\in\boldsymbol{C}^n: |z-a|<r\}$. If f is a continuous function defined on S, the sup- and L^2-norms are denoted by $||f||_S$ and $||f||_{L^2(S)}$ respectively.

Let U be an open subset of $\boldsymbol{C}^n$. For any real valued function $\rho\in C^2(U)$ and for any vector ξ of $\boldsymbol{C}^n$, we write

$$H[\rho; \xi](z) = \sum_{i,k} \frac{\partial^2\rho(z)}{\partial z_i \partial \bar{z}_k} \xi_i \bar{\xi}_k\,, \qquad z\in U\,.$$

If $H[\rho; \xi](z)>0$ holds for every nonzero ξ and for every point z of U, then ρ is called a *strictly plurisubharmonic* function in U. Let $\mathcal{P}(U)$ be the class of C^∞ functions nonnegative and strictly plurisubharmonic in U.

Lemma 1. *Let U be an open subset of $\boldsymbol{C}^n$. Let ρ and σ be strictly plurisubharmonic functions defined in U. Suppose that there exists a real number c such that*

$$G_0 = \{z \in U:\ \rho(z) < c\}$$

is relatively compact in U. Then, for any real number ε,

$$G_\varepsilon = \{z \in G_0:\ \sigma(z) < \varepsilon\}$$

is holomorphically convex. In particular, G_0 is holomorphically convex.

Proof. Set

$$u(z) = \frac{1}{c-\rho(z)} + \frac{1}{\varepsilon-\sigma(z)}.$$

Then, we have, for any vector $\xi \in \boldsymbol{C}^n$ and for any $z \in G_0$,

$$\begin{aligned} H[u;\ \xi](z) &= \frac{1}{[c-\rho(z)]^2} H[\rho;\ \xi](z) + \frac{2}{[c-\rho(z)]^3} \left| \sum_k \frac{\partial \rho(z)}{\partial z_k} \xi_k \right|^2 \\ &+ \frac{1}{[\varepsilon-\sigma(z)]^2} H[\sigma;\ \xi]z)) + \frac{2}{[\varepsilon-\sigma(z)]^3} \left| \sum_k \frac{\partial \sigma(z)}{\partial z_k} \xi_k \right|^2 . \end{aligned}$$

The right member is positive for nonzero vector ξ. If z is a point of ∂G_ε, then $\rho(z)=c$ or $\sigma(z)=\varepsilon$ holds. From this it follows that, for any real number a, the set $\{z \in G: u(z) \leq a\}$ is compact. This proves the lemma.

If $\omega = \sum_k \alpha_k(z) d\bar{z}_k$ is a C^∞ form of type (0, 1) defined in an open set U, then we write

$$|\omega(z)| = \sum_k |\alpha_k(z)|.$$

If V is any relatively compact open subset of U, then the sup- and L^2-norms of ω on V are denoted by $||\omega||_V$ and $||\omega||_{L^2(V)}$ respectively. The main tool which we shall make use of is the following theorem. (This is a special form of the theorem proved by Hrömander [4].)

Hörmander's theorem. *Let K be a compact subset of a bounded open set U. Suppose that ω is a C^∞ form of type (0, 1) satisfying $\bar\partial\omega = 0$ in U. Then, for every holomorphically convex open set G such that $K \subset G \Subset U$, there exists a C^∞ function u such that*

$$\bar\partial u = \omega \qquad \textit{and} \qquad ||u||_{L^2(G)} \leq \gamma ||\omega||_{L^2(G)}$$

where γ is a constant depending only on K.

We shall also make use of the following lemma due to Hörmander-Wermer [5].

Lemma 2. *Let u be a C^∞ function defined in a neighborhood of the closed ball $B=\overline{B_n(a, \varepsilon)}$. Then*

$$|u(a)| \leq \gamma_0\{\varepsilon^{-n}\|u\|_{L^2(B)}+\varepsilon\|\bar{\partial}u\|_B\}$$

holds, where γ_0 is an absolute constant.

Let K be a compact subset of $\boldsymbol{C}^n$. A subalgebra $\mathcal{A}$ of $C(K)$ is called a uniform algebra on K, if it is closed in $C(K)$, contains the constants, and separates the points of K. If $\mathcal{A}$ is a uniform algebra on K, then K is naturally immbedded in the maximal ideal space $M(\mathcal{A})$ of $\mathcal{A}$, and the Šilov boundary $\Gamma(\mathcal{A})$ of $\mathcal{A}$ is contained in K. We shall consider some uniform algebras on K. $A(K)$ is the algebra of functions in $C(K)$ which is holomorphic in the interior of K. $H(K)$ is the algebra of functions approximated uniformly on K by functions each holomorphic in a neighborhood of K. If $K\subset U$, $H(K, U)$ is the algebra of uniform limits on K of functions holomorphic in U. If $\{f_1, \cdots, f_m\}$ is a set of functions in $C(K)$ separating the points of K, $[f_1, \cdots, f_m; K]$ is the algebra of uniform limits of polynomials of $f_1, \cdots, f_m$. In particular, $[z_1, \cdots, z_n; K]$ is denoted by $P(K)$. Evidently, we have

$$P(K)=H(K, \boldsymbol{C}^n)\subset H(K, U)\subset H(K)\subset A(K)\subset C(K)\,.$$

2. Holomorphically convexity

Let K be a compact subset of $\boldsymbol{C}^n$. K is called a H-convex set, if the maximal ideal space $M(H(K))$ of $H(K)$ coincides with K. It is known that, if K is the intersection of holomorphically convex open sets containing K, then K is H-convex (cf. Rossi [9]). To establish the approximation theorem for $H(K)$, we need to impose a stronger convexity condition on K.

Let $\delta(z)$ be a nonnegative continuous function defined in an open subset U of $\boldsymbol{C}^n$. A compact subset K of U is said to be in the class $\mathfrak{K}(\delta)$, if we can find constants η and ε_0 so that, for every ε, $0<\varepsilon<\varepsilon_0$, there exists a holomorphically convex open set $G_\varepsilon \Subset U$ satisfying

$$U_\varepsilon(K)\subset G_\varepsilon\subset\{z\in U\colon \delta(z)<\eta\varepsilon\}\,. \tag{1}$$

K is called a δ-*convex set,* if, in addition to (1), the condition

$$K=\bigcap_{\varepsilon>0} G_\varepsilon \tag{2}$$

is satisfied. A δ-convex set is H-convex by definition. When $\delta(z)$ is the distance function $d_K(z)$, (1) implies (2). The d_K-convexity is nothing but the uniform H-convexity introduced by Čirka [1]. We shall give some examples of δ-convex sets.

EXAMPLE 1. Let G be a bounded strictly pseudoconvex domain defined by a strictly plurisubharmonic function σ in an open set U containing $\bar{G}$: $G=\{z\in U: \sigma(z)<0\}$. Suppose that $d\sigma$ does not vanish on ∂G. Let V be a relatively compact open subset of U containing $\bar{G}$. Then, there exist positive constants c_1 and c_2 such that

$$c_1 d_G(z)<\sigma(z)<c_2 d_G(z)\,, \qquad z\in V\backslash\bar{G}\,.$$

We choose ε_0 so that the open set $\{z\in V: \sigma(z)<c_2\varepsilon_0\}$ is relatively compact in V. It follows from Lemma 1 that, for any ε, $0<\varepsilon<\varepsilon_0$, the open set

$$G_\varepsilon=\{z\in V: \sigma(z)<c_2\varepsilon\}$$

is holomorphically convex. Setting $\eta=c_1^{-1}c_2$, we have

$$U_\varepsilon(G)\subset G_\varepsilon\subset\{z\in V: d_G(z)<\eta\varepsilon\}\,.$$

Therefore, $\bar{G}$ is d_G-convex. Moreover, if we set $\delta(z)=\max\{\sigma(z), 0\}$, then G is also a δ-convex set.

EXAMPLE 2. Let U be an open subset of $\boldsymbol{C}^n$. Let f_ν, $1\leq\nu\leq s$, $s\leq n$, be functions in $C^\infty(U)$. Suppose that f_ν are holomorphic in $z_{s+1},\cdots,z_n$ in U and that

$$\det\left[\frac{\partial f_\nu}{\partial \bar{z}_k};\ \begin{matrix}\nu=1,\cdots,s\\ k=1,\cdots,s\end{matrix}\right]\neq 0 \qquad \text{on} \qquad U\,.$$

Let K be a d_k-convex compact subset of U, and let K^* be the graph of $(f_1,\cdots,f_s)$ on K:

$$K^*=\{(z,w)\in\boldsymbol{C}^{n+s}: w_\nu=f_\nu(z_1,\cdots,z_n),\ 1\leq\nu\leq s\}\,.$$

If G_ε is a holomorphically convex open subset of U in the definition of the d_k-convexity of K, then the open set

$$V_\varepsilon=\{(z,w)\in\boldsymbol{C}^{n+s}: \sum_{\nu=1}^{s}|w_\nu-f_\nu(z)|^2<\varepsilon^2,\ z\in G_\varepsilon\}$$

is holomorphically convex (cf. Sakai [10]). We can choose a positive constant c so that, for every sufficiently small $\varepsilon>0$,

$$U_\varepsilon(K^*)\subset V_{c\varepsilon}\subset\{(z,w)\in\boldsymbol{C}^{n+s}: d_{K^*}(z,w)<c\varepsilon\}\,.$$

holds. Therefore, K^* is d_{K^*}-convex.

EXAMPLE 3. A real C^∞ submanifold M of $\boldsymbol{C}^n$ is said to be finite, if M is a manifold with boundary and if ∂M is a real C^∞ submanifold of $\boldsymbol{C}^n$. If M is a compact or finite C^∞ submanifold of $\boldsymbol{C}^n$ and has no complex tangents, then M is d_M convex. This is derived from the fact that $d_M(s)^2$ is strictly plurisubharmonic in a neighborhood of M (cf. Hörmander-Wermer [5]). More generally, if ρ is a function in $\mathcal{P}(U)$ and if the zero set K of ρ is compact, then K is δ-convex, where δ is the function defined by

$$\delta(z)^2 = \sum_k \left|\frac{\partial \rho(z)}{\partial z_k}\right|^2$$

(see Lemma 5, Section 5).

Let δ be a nonnegative continuous function in U and m a positive integer. A function $F \in C^\infty(U)$ is said to be in the class $\mathfrak{M}_m(U, \delta)$, if, for any relatively compact open subset V of U, there exists a constant c such that

$$|\bar{\partial} F(z)| \leq c\delta(z)^m, \qquad z \in V.$$

Proposition 1. *Let K be a compact subset of U in the class $\mathfrak{K}(\delta)$. If $F \in \mathcal{M}_{n+1}(U, \delta)$, then $F|_K$ belongs to $H(K)$.*

Proof. Set $\omega = \bar{\partial} F$ in U. Let ε_0 and η be the constants in (1), and let ε be an arbitrary number with $0 < \varepsilon < \varepsilon_0$. We use the notation γ for unspecific constants that are independent of ε. By Hörmander's theorem, we can find a function $u \in C^\infty(G_\varepsilon)$ such that

$$\bar{\partial} u_\varepsilon = \omega \qquad \text{and} \qquad \|u_\varepsilon\|_{L^2(G_\varepsilon)} \leq \gamma \|\omega\|_{L^2(G_\varepsilon)}.$$

Let z be an arbitrary point of K. The ball $B = B_n(z, \varepsilon)$ is contained in G_ε. It follows from Lemma 2 that

$$|u_\varepsilon(z)| \leq \gamma_0 \{\varepsilon^{-n} \|u_\varepsilon\|_{L^2(G_\varepsilon)} + \varepsilon \|\omega\|_{G_\varepsilon}\}.$$

Since $F \in \mathcal{M}_{n+1}(U, \delta)$, (1) yields $\|\omega\|_{G_\varepsilon} < \gamma \varepsilon^{n+1}$. Therefore, we obtain $|u_\varepsilon(z)| < \gamma\varepsilon$. We set $F_\varepsilon = F - u_\varepsilon$ in G_ε. Then F_ε is holomorphic in G_ε. For every z of K, we have

$$|F(z) - F_\varepsilon(z)| = |u_\varepsilon(z)| < \gamma\varepsilon,$$

which proves the proposition.

Let G be a bounded domain in $\boldsymbol{C}^n$. Let $A^\infty(\bar{G})$ denote the closure in $C(\bar{G})$ of the class of functions of $A(\bar{G})$ each of which can be extended as a C^∞ function in a neighborhood of $\bar{G}$.

Corollary. *If G is a bounded domain such that $\bar{G}$ is d_G-convex, then $A^\infty(\bar{G}) = H(\bar{G})$ holds.*

Proof. Let F be a C^∞ function defined in a neighborhood U of $\bar{G}$ and holomorphic in G. Since $\bar{\partial}F=0$ in $\bar{G}$, we have $F\in\mathcal{M}_{n+1}(U, d_G)$. Therefore, $F|_{\bar{G}}$ belongs to $H(\bar{G})$.

A C^∞ map $\phi=(\phi_1, \cdots, \phi_n)$ of an open subset U of $\boldsymbol{C}^n$ into $\boldsymbol{C}^n$ is said to be in the class $\tilde{\mathcal{M}}_m(U, \delta)$, if every ϕ_k is in $\mathcal{M}_m(U, \delta)$.

Proposition 2. *Let δ be a nonnegative continuous function defined in an open set U and K a compact subset of U in the class $\mathfrak{K}(\delta)$. Let ϕ be a map in $\tilde{\mathcal{M}}_{n+1}(U, \delta)$. If f' is a function in $H(\phi(K))$, then $f=f'\circ\phi$ is in $H(K)$.*

Proof. There exists a sequence of functions $\{g'_\nu\}$ each holomorphic in a neighborhood U'_ν of $K'=\phi(K)$ such that f' is the uniform limit of $\{g'_\nu\}$ on K'. Set $g_\nu=g'_\nu\circ\phi$ and $U_\nu=\phi^{-1}(U'_\nu)$. Since $\phi\in\tilde{\mathcal{M}}_{n+1}(U, \delta)$ we have $g_\nu\in\mathcal{M}_{n+1}(U_\nu, \delta)$. Therefore, by Pıoposition 1, we have $g_\nu|_L\in H(K)$. Since f is the uniform limit of $\{g_\nu\}$ on K, we have $f\in H(K)$, as required.

Corollary. *Let ϕ be a diffeomorphism in $\tilde{\mathcal{M}}_{n+1}(U, \delta)$. Set $\delta'=\delta\circ\phi^{-1}$. If K is in $\mathfrak{K}(\delta)$ and if $K'=\phi(K)$ is in $\mathfrak{K}(\delta')$, then $H(K)$ and $H(K')$ are isomorphic as uniform algebras.*

Proof. The inverse map ϕ^{-1} is in $\tilde{\mathcal{M}}_{n+1}(U', \delta')$, where $U'=\phi(U)$. Hence, we have the Corollary.

3. CR-submanifolds

Let M be a real C^∞ submanifold of $\boldsymbol{C}^n$. We denote by $T_z(M)$ the real tangent space of M at z. We say that M has the complex rank r at z, if the complex tangent space

$$CT_z(M) = T_z(M)\cap iT_z(M)$$

has the complex dimension r. M is called a *CR-submanifold* of complex rank r, if it has a constant complex rank r at every point, which will be denoted by $r(M)$.

The following lemma gives an example of a CR-submanifold of $\boldsymbol{C}^n$.

Lemma 3. *Let $f=(f_1, \cdots, f_n)$ be a C^∞ imbedding of an subset N of $\boldsymbol{R}^d_w$ into $\boldsymbol{C}^n_w$. Then, the image $M=f(N)$ is a CR-submanifold of $\boldsymbol{C}^n_w$ of complex rank r, if and only if*

$$\operatorname{rank}\begin{bmatrix}\dfrac{\partial f_k}{\partial x_\nu}\ ; & \begin{matrix}k=1, \cdots, n\\ \nu=1, \cdots, d\end{matrix}\end{bmatrix} = d-r$$

holds at every point of N.

Proof. Let x^0 be any point of N and set $\alpha=f(x^0)$. M has a nonzero complex tangent at α, if and only if there exist two nonzero vectors $t, s\in\boldsymbol{R}^d$ such that

(1) $$\sum_\nu \frac{\partial f_k}{\partial x_\nu}(x^0)t_\nu = i\sum_\nu \frac{\partial f_k}{\partial x_\nu}(x^0)s_\nu\,, \qquad 1\leq k\leq n\,.$$

If $t=0$, then (1) implies that

$$\sum_\nu \frac{\partial u_k}{\partial x_\nu}(x^0)s_\nu = 0\,, \qquad \sum_\nu \frac{\partial v_k}{\partial x_\nu}(x^0)s_\nu = 0\,, \qquad 1\leq k\leq n\,,$$

where $f_k=u_k+iv_k$. Since $\mathrm{rank}\left[\frac{\partial u_k}{\partial x_\nu}, \frac{\partial v_k}{\partial x_\nu};\ \begin{matrix} k=1,\cdots,n \\ \nu=1,\cdots,d\end{matrix}\right]=d$, we have $s=0$. Set $\xi=t-is$. Then ξ is nonzero solution of

(2) $$\sum_\nu \frac{\partial f_k}{\partial x_\nu}(x^0)\xi_\nu = 0\,, \qquad 1\leq k\leq n\,,$$

if and only if t and s are nonzero and satisfy (1). Thus, the complex dimension of $CT_\alpha(M)$ coincides with the number of linearly independent complex vectors ξ satisfying (2). This proves the lemma.

Let M be a CR-submanifold of complex rank r. We say that M is *holomorphic* if, for every point z of M, there exists a neighborhood U_z of z in $\boldsymbol{C}^n$ such that $M\cap U_z$ is represented as a real C^∞ parametric family of complex submanifolds of $\boldsymbol{C}^n$ of complex dimension r. If $r(M)=0$, then M is trivially holomorphic. A CR-submanifold of positive rank is not necessarily holomorphic. For example, the hypersphere S^{2n-1} in $\boldsymbol{C}^n$, $n>1$, is a CR-submanifold of complex rank $n-1$. However, S^{2n-1} can contain no complex submanifolds of $\boldsymbol{C}^n$ of positive dimensions. To see this, we suppose that S^{2n-1} contains a complex submanifold X of $\boldsymbol{C}^n$. We may assume that X contains the point $z^0=(1,0,\cdots,0)$. Then the function $f(z)=\frac{1}{2}(1+z_1)$ induces a holomorphic function F on X. $|F|$ attains its maximal value 1 in X only at z^0. It follows from the maximal modulus principle that X reduces to $\{z^0\}$.

We shall now give a local representation of holomorphic CR-submanifolds. For simplicity, we use the abbreviations

$$\begin{aligned} &u' = (u_1,\cdots,u_t)\,, \qquad v' = (v_1,\cdots,v_t)\,, \\ &w'' = (w_{t+1},\cdots,w_{t+r}) \qquad \text{and} \qquad w''' = (w_{t+r+1},\cdots,w_n)\,, \end{aligned}$$

where $w_\nu=u_\nu+iv_\nu$ and $0\leq t\leq t+r\leq n$.

Let V be an open subset of $\boldsymbol{C}^n_w$. Suppose that $N=V\cap(\boldsymbol{R}^t_{u'}\times\boldsymbol{C}^r_{w''})$ is not empty. If ϕ is a diffeomorphism of V into $\boldsymbol{C}^n_z$ which is in $\tilde{\mathscr{M}}_1(V, d_N)$, then $\phi|_N$ is holomorphic in w'' on N. Therefore, $M=\phi(N)$ is a holomorphic CR-submanifold of $\boldsymbol{C}^n_z$.

Conversely, we have the following theorem.

Theorem 1. *Let M be a holomorphic CR-submanifold of $\boldsymbol{C}^n_z$. For any* *positive integer m and for any point z of M, there exist a neighborhood U_z of z in C^n_z, a neighborhood V of the origin of C^n_w, and a C^∞ diffeomorphism ϕ of V onto U_z satisfying the following conditions*:

(i) $N=\phi^{-1}(M\cap U_z)$ *is an open subset of* $\boldsymbol{R}^t_{u'}\times \boldsymbol{C}^r_{w''}$, *where* $r=r(M)$ *and* $t=\dim_R M-2r$;

(ii) *every component* ϕ_k *of* ϕ *is holomorphic in* $w_{t+1},\cdots,w_n$;

(iii) $\phi\in\tilde{\mathcal{M}}_m(V,d_N)$.

Proof. Since M is holomorphic, we can choose a neighborhood U'_z of z, a neighborhood N' of the origin in $\boldsymbol{R}^t_{u'}\times \boldsymbol{C}^r_{w''}$, and a C^∞ map ψ of N' into $\boldsymbol{C}^n$ with $\psi(0)=z$, satisfying the following conditions:

(a) $\psi(N')=M\cap U'_z$;

(b) every component ψ_k of ψ is holomorphic in w'' on N'.

By Lemma 3, we can assume that

$$\text{(c)}\qquad \det\left[\frac{\partial\psi_k}{\partial u_\nu};\ \begin{matrix}k=1,\cdots,t+r\\ \nu=1,\cdots,t+r\end{matrix}\right]\neq 0 \qquad \text{on} \qquad N'.$$

We define the function $\tilde{\psi}\in C^\infty(\Omega)$, $\Omega=N'\times \boldsymbol{R}^t_{v'}\times \boldsymbol{C}^{n-r-t}_{w'''}$, by

$$\tilde{\psi}(w_1,\cdots,w_n)=\psi_k(u',w'')+i\sum_{\nu=1}^{t}\frac{\partial\psi_k}{\partial u_\nu}(u',w'')v_\nu$$
$$+\cdots+\frac{i^m}{m!}\sum_{\nu_1,\cdots,\nu_m=1}^{t}\frac{\partial^m\psi_k}{\partial u_{\nu_1}\cdots\partial u_{\nu_m}}(u',w'')v_{\nu_1}\cdots v_{\nu_m}.$$

It follows at once from (b) that $\dfrac{\partial\tilde{\psi}_k}{\partial\overline{w}_\mu}\equiv 0$ for $\mu=t+1,\cdots,n$. For $\mu=1,\cdots,t$, we have

$$\begin{aligned}
&\frac{i^p}{p!}\sum_{\nu_1,\cdots,\nu_p}\frac{\partial^{p+1}\psi_k}{\partial u_{\nu_1}\cdots\partial u_{\nu_p}\partial\overline{w}_\mu}v_{\nu_1}\cdots v_{\nu_p}\\
&\qquad+\frac{i^{p+1}}{(p+1)!}\sum_{\nu_1,\cdots,\nu_{p+1}}\frac{\partial^{p+1}\psi_k}{\partial u_{\nu_1}\cdots\partial u_{\nu_{p+1}}}\frac{\partial}{\partial\overline{w}_\mu}(v_{\nu_1}\cdots v_{\nu_{p+1}})\\
&=\frac{i^p}{2p!}\sum_{\nu_1,\cdots,\nu_p}\frac{\partial^{p+1}\psi_k}{\partial u_{\nu_1}\cdots\partial u_{\nu_p}\partial u_\mu}v_{\nu_1}\cdots v_{\nu_p}\\
&\qquad+\frac{i^{p+1}}{(p+1)!}\frac{i}{2}\sum_{\nu_1,\cdots,\nu_{p+1}}\frac{\partial^{p+1}\psi_k}{\partial u_{\nu_1}\cdots\partial n_{\nu_{p+1}}}\frac{\partial}{\partial v_\mu}(v_{\nu_1}\cdots v_{\nu_{p+1}})\\
&=\frac{i^p}{2p!}\sum_{\nu_1,\cdots,\nu_p}\frac{\partial^{p+1}\psi_k}{\partial u_\mu\partial u_{\nu_1}\cdots\partial u_{\nu_p}}v_{\nu_1}\cdots v_{\nu_p}\\
&\qquad-\frac{i^p}{2p!}\sum_{\mu_1,\cdots,\mu_p}\frac{\partial^{p+1}\psi_k}{\partial u_\mu\partial u_{\mu_1}\cdots\partial u_{\mu_p}}v_{\mu_1}\cdots v_{\mu_p}\\
&=0.
\end{aligned}$$

Therefore, we have

$$\frac{\partial\tilde{\psi}_k}{\partial\bar{w}_\mu}=\frac{i^m}{m!}\sum_{\nu_1,\cdots,\nu_m}\frac{\partial^{m+1}\psi_k}{\partial\bar{w}_\mu\partial u_{\nu_1}\cdots\partial u_{\nu_m}}(u',w'')v_{\nu_1}\cdots v_{\nu_m}.$$

From this it follows that $\tilde{\psi}\in\tilde{\mathcal{M}}_m(\Omega, d_N)$.

Let us now define the C^∞ map $\phi=(\phi_1,\cdots,\phi_n)$ of Ω into C^n by

$$\phi_k(w_1,\cdots,w_n)=\begin{cases}\tilde{\psi}_k(w) & (1\leq k\leq t+r)\\ \tilde{\psi}_k(w)+w_k & (t+r+1\leq k\leq n).\end{cases}$$

Then ϕ is clearly in the class $\tilde{\mathcal{M}}_m(\Omega, d_N)$, and hence, for any point w of N' and for any index ν, $1\leq\nu\leq t+r$, we have $\dfrac{\partial\tilde{\psi}_k}{\partial w_\nu}(w)=\dfrac{\partial\tilde{\psi}_k}{\partial u_\nu}(w)=\dfrac{\partial\psi_k}{\partial u_\nu}(w)$. Therefore, the Jacobian of ϕ at w is

$$\begin{aligned}J(w)&=\left|\det\begin{bmatrix}\dfrac{\partial\phi_k(w)}{\partial w_\nu}\,; & \begin{matrix}k=1,\cdots,n\\ \nu=1,\cdots,n\end{matrix}\end{bmatrix}\right|^2\\ &=\left|\det\begin{bmatrix}\dfrac{\partial\tilde{\psi}_k(w)}{\partial u_\nu}\,; & \begin{matrix}k=1,\cdots,t+r\\ \nu=1,\cdots,t+r\end{matrix}\end{bmatrix}\right|^2\\ &=\left|\det\begin{bmatrix}\dfrac{\partial\psi_k(w)}{\partial u_\nu}\,; & \begin{matrix}k=1,\cdots,t+r\\ \nu=1,\cdots,t+r\end{matrix}\end{bmatrix}\right|^2.\end{aligned}$$

It follows from (c) that the last member does not vanish on N' and hence in a neighborhood of N' in C^n_w. Therefore, we can find a neighborhood V of the origin in C^n_w such that ϕ is diffeomorphic in V. Setting $N=N'\cap V$ and $U_z=\phi(V)$, we have the theorem.

4. Totally real sets

A subset T of C^n is called a *totally real set,* if there exist an open subset U of C^n containing T and a function ρ in $\mathcal{P}(U)$ such that

$$T=\{z\in U:\ \rho(z)=0\}.$$

ρ is then called a *defining function* of T. We note that if T is totally real, then $\rho(z)=d\rho(z)=0$ holds for $z\in T$.

Proposition 3. *If every point z of T has a neighborhood U_z in C^n such that $T\cap U_z$ is totally real, then T is totally real.*

Proof. We can find a locally finite open covering $\{U_\nu\}$ of T such that, in each U_ν, there exists a function $\rho_\nu\in\mathcal{P}(U_\nu)$ satisfying $T\cap U_\nu=\{z\in U_\nu:\ \rho_\nu(z)=0\}$. Let $\{\lambda_\nu\}$ be a partition of unity subordinate to $\{U_\nu\}$. We set

$$\rho(z)=\sum_\nu\lambda_\nu(z)\rho_\nu(z)\,,\qquad z\in U=\bigcup_\nu U_\nu\,.$$

For any vector ξ of C^n, we have

$$H[\rho;\xi]=\sum_\nu \lambda_\nu H[\rho_\nu;\xi]+\sum_\nu \rho_\nu H[\lambda_\nu;\xi] \\ +\sum_{\nu,i,k}\left(\frac{\partial\lambda_\nu}{\partial z_i}\frac{\partial\rho_\nu}{\partial \bar z_k}+\frac{\partial\lambda_\nu}{\partial \bar z_k}\frac{\partial\rho_\nu}{\partial z_i}\right)\xi_i\bar\xi_k .$$

If $\xi\neq 0$, the first sum is positive. The second and the third sums vanish on T, since $\rho_\nu(z)=d\rho_\nu(z)=0$ for $z\in T\cap U_\nu$. Therefore, we can find a neighborhood V of T so that ρ is a defining function of T in $\mathcal{P}(U)$.

Proposition 4. *Let T be a totally real set defined by $\rho\in\mathcal{P}(U)$. If f_ν $(1\leq\nu\leq t)$ are functions in $C^\infty(U)$, then*

$$T_1=\{z\in T: f_\nu(z)=0\,,\ 1\leq\nu\leq t\}$$

is a totally real set.

Proof. Set $\rho_1(z)=\rho(z)+\sum_\nu|f_\nu(z),|^2$, $z\in U$. Then we have

$$H[\rho_1;\xi]=H[\rho;\xi]+\sum_\nu|\sum_i\frac{\partial f_\nu}{z\partial_i}\xi_i|^2+\sum_\nu|\sum_i\frac{\partial \bar f_\nu}{\partial z_i}\xi_i|^2 \\ +\sum_\nu f_\nu\sum_{i,k}\frac{\partial^2\bar f_\nu}{\partial z_i\partial\bar z_k}\xi_i\bar\xi_k+\sum_\nu \bar f_\nu\sum_{i,k}\frac{\partial^2 f_\nu}{\partial z_i\partial\bar z_k}\xi_i\bar\xi_k .$$

For any nonzero vector ξ, the right member is positive at every point of T_1. Hence there exists a neighborhood V of T_1 such that ρ_1 is in $\mathcal{P}(V)$.

Corollary. *Let T be a totally real set defined by $\rho\in\mathcal{P}(U)$. If V is a relatively compact open subset of U with the smooth boundary, then $T\cap\bar V$ is a totally real set.*

Proof. We can choose a C^∞ function $\lambda(z)$ such that $\lambda(z)=0$ for $z\in\bar V$ and $\lambda(z)>0$ for $z\in U\cap\bar V$. Since $T\cap\bar V=\{z\in T: \lambda(z)=0\}$, $T\cap\bar V$ is totally real.

Proposition 5. *If T_1 and T_2 are totally real sets in C^n_z and C^m_w respectively, then $T=T_1\times T_2$ is totally real in $C^n_z\times C^w_m$.*

Proof. Let ρ_ν be defining functions of T_ν respectively. Then, $\rho(z,w)=\rho_1(z)+\rho_2(w)$ is a defining function of T.

Let $f_\nu(z_\nu)$ be holomorphic functions in open subsets U_ν of $C^1_{z_\nu}$ $(1\leq\nu\leq N)$ respectively. Set $T_\nu=\{z_\nu\in U_\nu: |f_\nu(z_\nu)|=1\}$, $1\leq\nu\leq N$. Suppose that every $f'_\nu(z_\nu)$ does not vanish on T. Then every T_ν is a totally real set in $C^1_{z_\nu}$ defined by $\rho_\nu(z_\nu)=(|f_\nu(z_\nu)|^2-1)^2$. Therefore, $T=T_1\times\cdots\times T_N$ is a totally real set in C^N.

Proposition 6. *Let δ be a nonnegative continuous function defined in an open set U and ϕ a diffeomorphism in $\tilde{\mathcal{M}}_1(U,\delta)$. Let T be a subset of the zero set* *of δ (assumed to be nonempty). Then T is totally real if and only if $T'=\phi(T)$ is totally real.*

Proof. Set $U'=\phi(U)$ and $\delta'=\delta\circ\phi^{-1}$. Since ϕ^{-1} is in $\tilde{\mathcal{M}}_1(U', \delta')$, it suffices to prove that if T' is totally real then so is T. There exists an open set V' such that $T'\subset V'\subset U'$ and a function $\rho'\in\mathcal{P}'(V')$ such that $T'=\{w\in V': \rho'(w)=0\}$. We write ϕ as $w_\nu=\phi_\nu(z)$, $1\leq\nu\leq n$. Then, for any point $z\in V=\phi^{-1}(V')$ and for any vector $\xi\in \boldsymbol{C}^n$, we have

$$H[\rho;\xi](z)=H[\rho';\eta_z](w)+\sum_{i,k}\varepsilon_{ik}(z)\xi_i\bar{\xi}_k,$$

where η_z is the vector $(\eta_1(z), \cdots, \eta_n(z))$ defined by

$$\eta_\nu(z)=\sum_k\frac{\partial\phi_\nu}{\partial z_k}(z)\xi_k, \qquad 1\leq\nu\leq n,$$

and

$$\begin{aligned}\varepsilon_{ik}=&\sum_{\nu,\mu}\frac{\partial^2\rho'}{\partial\bar{w}_\nu\partial\bar{w}_\mu}\frac{\partial\bar{\phi}_\nu}{\partial z_i}\frac{\partial\bar{\phi}_\mu}{\partial\bar{z}_k}+\sum_{\nu,\mu}\frac{\partial^2\rho'}{\partial\bar{w}_\nu\partial w_\mu}\frac{\partial\bar{\phi}_\nu}{\partial z_i}\frac{\partial\phi_\mu}{\partial\bar{z}_k}\\&+\sum_{\nu,\mu}\frac{\partial^2\rho'}{\partial w_\nu\partial w_\mu}\frac{\partial\phi_i}{\partial z_i}\frac{\partial\phi_\mu}{\partial\bar{z}_k}+\sum_\nu\frac{\partial\rho'}{\partial w_\nu}\frac{\partial^2\phi_\nu}{\partial z_i\partial\bar{z}_k}+\sum_\nu\frac{\partial\rho'}{\partial\bar{w}_\nu}\frac{\partial^2\bar{\phi}_\nu}{\partial z_i\partial\bar{z}_k}\end{aligned}$$

Since $\phi\in\tilde{\mathcal{M}}_1(V,\delta)$, the first three sums of the last expression vanish on T. Since $d\rho'=0$ on T, the other terms vanish on T. Therefore, we have $\varepsilon_{ik}(z)=0$ for $z\in T$. If $\xi\neq0$, then we have $\eta_z\neq0$, since $\det\left[\dfrac{\partial\phi_\nu}{\partial z_k}\right]\neq0$ on T. Thus we can find a neighborhood V of T such that $\rho\in\mathcal{P}(V)$.

Corollary. *Let M be a real C^∞ submanifold of $\boldsymbol{C}^n$. Then M is totally real if and only if $r(M)=0$.*

Proof. Suppose that $r(M)=0$. Then, for any point z^0 of M, there exists a neighborhood U of z^0 such that $M\cap U$ is mapped by a diffeomorphism $\phi\in\tilde{\mathcal{M}}_1(U, d_{M\cap U})$ onto an open subset N of the real subspace $\boldsymbol{R}^d_{u'}$ of $\boldsymbol{C}^n_w$, where $d=\dim_R M$ and $u'=(u_1, \cdots, u_d)$. Since $\boldsymbol{R}^d_{u'}$, is clearly a totally real set in $\boldsymbol{C}^n_w$, we can assume that N is totally real by Corollary of Proposition 4. It follows from Proposition 3 and 6 that M is totally real.

Conversely, we suppose that M is totally real. For every point z of M there exist a neighborhood U_z and a real submanifold M'_z with $r(M'_z)=0$ such that $M\cap U_z\subset M'_z$ (Harvey-Wells [3]). Since $T_z(M\cap U_z)\subset T_z(M'_z)$ and $iT_z(M\cap U_z)\subset iT_z(M'_z)$, we have $CT_z(M\cap U_z)=\{0\}$ as required.

It should be noted that some of the properties of totally real sets has been studied in Harvey-Wells [2], [3]. We note also that the necessary part of Proposition 6 is due to Hörmander-Wermer [5]. Our proof is a different one.

5. Uniform approximation on totally real sets

The purpose of this section is to prove the following theorem.

Theorem 2. *Let T be a totally real set in $\boldsymbol{C}^n$. Then, for every compact subset K of T, $H(K)=C(K)$ holds.*

We begin by proving the following lemmas. We write

$$|\operatorname{grad} \rho(z)| = \Big(\sum_k \Big|\frac{\partial \rho}{\partial z_k}(z)\Big|^2\Big)^{1/2}.$$

Lemma 4. *Let T be a totally real set defined by $\rho \in \mathcal{P}(U)$. Then, there exists an open set V such that $T \subset U \subset V$ and that $u(z) = |\operatorname{grad} \rho(z)|^2$ is in $\mathcal{P}(V)$. (T is contained in the zero set of u.)*

Proof. For any vector ξ of $\boldsymbol{C}^n$, we have

$$H[u; \xi] = \sum_k \Big(\sum_{i,j} \frac{\partial^3 \rho}{\partial z_i \partial \bar{z}_j \partial z_k} \xi_i \bar{\xi}_j\Big) \frac{\partial \rho}{\partial \bar{z}_k} + \sum_k \Big(\sum_{i,j} \frac{\partial^3 \rho}{\partial z_i \partial \bar{z}_j \partial \bar{z}_k} \xi_i \bar{\xi}_j\Big) \frac{\partial \rho}{\partial z_k}$$
$$+ \sum_k \Big|\sum_i \frac{\partial^2 \rho}{\partial z_i \partial z_k} \xi_i\Big|^2 + \sum_k \Big|\sum_i \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_k} \xi_i\Big|^2 .$$

The first and the second sums vanish on T, since $d\rho(z)=0$ for $z \in T$. The fourth sum is positive for any nonzero ξ, since the matrix $\left[\frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_k}\right]$ is nonsingular. Therefore, we can find a neighborhood V of T such that $H[u; \xi] > 0$ for any $z \in V$ and for any nonzero ξ.

Lemma 5. *Let K be a compact totally real set defined by $\rho \in \mathcal{P}(U)$. Set $\delta(z) = |\operatorname{grad} \rho(z)|$. Then, K is δ-convex.*

Proof. By Lemma 4, there exists an open set V such that $K \subset V \subset U$ and that δ^2 is in $\mathcal{P}(V)$. Since K is the zero set of ρ, we can choose a positive constant c so that the open set

$$G_0 = \{z \in V\colon \rho(z) < c\}$$

is relatively compact in V. Since $d\rho(z)=0$ for $z \in K$, there exists a positive constant η such that

$$\delta(z) < \eta d_K(z), \qquad z \in G_0 .$$

It follows from Lemma 1 that, for any positive number ε, the open set

$$G_\varepsilon = \{z \in G_0\colon \delta(z) < \eta\varepsilon\}$$

is holomorphically convex. Setting $\varepsilon_0 = \operatorname{dist}(K, G_0)$, we have, foa rny ε, $0 < \varepsilon < \varepsilon_0$,

$$U_\varepsilon(K)\subset G_\varepsilon\subset\{z\in V:\ \delta(z)<\eta\varepsilon\}\ .$$

Since c can be chosen arbitrarily small, we have $K=\bigcap_{\varepsilon>0} G$, which completes the lemma.

Lemma 6. *Let T be a totally real set defined by $\rho\in\mathcal{P}(U)$. For any positive integer m and for any function $f\in C^\infty(U)$, there exists a function $F\in\mathcal{M}_m(U, |\mathrm{grad}\,\rho|)$ such that $F(z)=f(z)$ for $z\in T$.*

Proof. The system of equations

$$\frac{\partial f}{\partial\bar z_i}+\sum_\alpha g_\alpha(z)\frac{\partial^2\rho}{\partial z_\alpha\partial\bar z_i}(z)=0\,,\qquad z\in U\,,\qquad 1\leq i\leq n \tag{1}$$

has a unique solution (g_α), $g_\alpha\in C^\infty(U)$, since the matrix $\left[\dfrac{\partial^2\rho}{\partial z_j\partial\bar z_k}\right]$ is nonsingular. Differentiating (1) by $\bar z_j$, we have

$$\frac{\partial g_\alpha}{\partial\bar z_j}\frac{\partial^2\rho}{\partial z_\alpha\partial\bar z_i}=-\frac{\partial^2 f}{\partial\bar z_i\partial\bar z_j}-\sum_\alpha g_\alpha\frac{\partial^3\rho}{\partial z_\alpha\partial\bar z_i\partial\bar z_j}\,.$$

Since the right member is symmetric with respect to i and j, we have

$$\sum_\alpha\frac{\partial g_\alpha}{\partial\bar z_j}\frac{\partial^2\rho}{\partial z_\alpha\partial\bar z_i}=\sum_\alpha\frac{\partial g_\alpha}{\partial\bar z_i}\frac{\partial^2\rho}{\partial z_\alpha\partial\bar z_j}\,. \tag{2}$$

For every α, the system of equations

$$\frac{\partial g_\alpha}{\partial\bar z_j}+\sum_\beta g_{\alpha\beta}\frac{\partial^2\rho}{\partial z_\beta\partial\bar z_j}=0\,,\qquad 1\leq j\leq n \tag{3}$$

has a unique solution $(g_{\alpha\beta})$, $g_{\alpha\beta}\in C^\infty(U)$. Substituting (3) to (2), we have

$$\sum_{\alpha,\beta}g_{\alpha\beta}\frac{\partial^2\rho}{\partial z_\alpha\partial\bar z_i}\frac{\partial^2\rho}{\partial z_\beta\partial\bar z_j}=\sum_{\alpha,\beta}g_{\alpha\beta}\frac{\partial^2\rho}{\partial z_\alpha\partial\bar z_j}\frac{\partial^2\rho}{\partial z_\beta\partial\bar z_i}\,,\qquad 1\leq i,\ j\leq n\,,$$

or equivalently,

$$\sum_{\alpha,\beta}(g_{\alpha\beta}-g_{\beta\alpha})\frac{\partial^2\rho}{\partial z_\alpha\partial\bar z_i}\frac{\partial^2\rho}{\partial z_\beta\partial\bar z_j}=0\,,\qquad 1\leq i,\ j\leq n\,,$$

which implies that $g_{\alpha\beta}=g_{\beta\alpha}$.

Suppose that, for a multi-index $J=(j_1,\cdots,j_p)$, g_J is already defined. Let $J\alpha$ denotes the multi-index $(j_1,\cdots,j_p,\alpha)$. Then $(g_{J\alpha})$ will be defined as a unique solution of the system of equations

$$\frac{\partial g_J}{\partial\bar z_j}+\sum_\alpha g_{J\alpha}\frac{\partial^2\rho}{\partial z_\alpha\partial\bar z_j}=0\,,\qquad 1\leq j\leq n\,. \tag{4}$$

Thus we define g_J for all multi-indices J inductively.
We shall now prove the symmetry of g_J with respect to J by induction. Suppose that g_J are symmetric for all multi-indices J of length p. Fix any positive number ν, $1 \leq \nu \leq p$, and for any indices α, β, we write

$$I = (j, \cdots, \hat{j}_\nu, \cdots, j_p), \qquad J' = (j_1, \cdots, \overset{\nu}{\alpha}, \cdots, j_p)$$

and

$$J'' = (j_1, \cdots, \overset{\nu}{\beta}, \cdots, j_p),$$

where $\hat{j}_\nu$ means that j_ν shall be omitted and $\overset{\nu}{\alpha}$ (or $\overset{\nu}{\beta}$) in J' (or J'' resp.) means that α (or β resp.) shall be posed at ν-th position. By the assumption of induction, we have

$$\frac{\partial g_I}{\partial \bar{z}_i} + \sum_\alpha g_{J'} \frac{\partial^2 \rho}{\partial z_\alpha \partial \bar{z}_i} = 0, \qquad 1 \leq i \leq n. \tag{5}$$

Differen tiating (5) by $\bar{z}_j$ and using the argument analogous to one in the case of $p=1$, we have

$$\sum_{\alpha,\beta} (g_{J'\beta} - g_{J''\alpha}) \frac{\partial^2 \rho}{\partial z_\alpha \partial \bar{z}_i} \frac{\partial^2 \rho}{\partial z_\beta \partial \bar{z}_j} = 0, \qquad 1 \leq i, j \leq n.$$

Therefore, we obtain $g_{J'\beta} = g_{J''\alpha}$, which implies the symmetry of g_J with respect to all multi-indices J of length $p+1$.

Now, we define the function $F \in C^\infty(U)$ by

$$F(z) = f(z) + \sum_{p=1}^{m} \frac{1}{p!} \sum_{j_1, \cdots, j_p} g_{j_1 \cdots j_p}(z) \frac{\partial \rho(z)}{\partial z_{j_1}} \cdots \frac{\partial \rho(z)}{\partial z_{j_p}}. \tag{6}$$

Since $d\rho(z) = 0$ for $z \in T$, we have $F(z) = f(z)$ for $z \in T$. Differentiating (6) by $\bar{z}_k$, we have

$$\begin{aligned} \frac{\partial F}{\partial \bar{z}_k} = \frac{\partial f}{\partial \bar{z}_k} &+ \sum_J g_j \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} + \sum_{p=1}^{m-1} \frac{1}{p!} \sum_{j_1, \cdots, j_p} \frac{\partial g_{j_1 \cdots j_p}}{\partial \bar{z}_k} \frac{\partial \rho}{\partial z_{j_1}} \cdots \frac{\partial \rho}{\partial z_{j_p}} \\ &+ \sum_{p=1}^{m-1} \frac{1}{(p+1)!} \sum_{j_1, \cdots, j_{p+1}} g_{j_1 \cdots j_{p+1}} \frac{\partial}{\partial \bar{z}_k} \left(\frac{\partial \rho}{\partial z_{j_1}} \cdots \frac{\partial \rho}{\partial z_{j_{p+1}}} \right) \\ &+ \frac{1}{m!} \sum_{j_1, \cdots, j_m} \frac{\partial g_{j_1 \cdots j_m}}{\partial \bar{z}_k} \frac{\partial \rho}{\partial z_{j_1}} \cdots \frac{\partial \rho}{\partial z_{j_m}}. \end{aligned}$$

By the way of construction of $g_{j_1 \cdots j_p}$, we have

$$\begin{aligned} &\frac{1}{(p+1)!} \sum_{j_1, \cdots, j_{p+1}} g_{j_1 \cdots j_{p+1}} \frac{\partial}{\partial \bar{z}_k} \left(\frac{\partial \rho}{\partial z_{j_1}} \cdots \frac{\partial \rho}{\partial z_{j_{p+1}}} \right) \\ &\quad = \frac{1}{p!} \sum_{i_1, \cdots, i_p} \sum_\alpha g_{i_1 \cdots i_p \alpha} \frac{\partial^2 \rho}{\partial z_\alpha \partial \bar{z}_k} \frac{\partial \rho}{\partial z_{i_1}} \cdots \frac{\partial \rho}{\partial z_{i_p}} \end{aligned}$$

$$= -\frac{1}{p!}\sum_{i_1,\cdots,i_p}\frac{\partial g_{i_1\cdots i_p}}{\partial z_k}\frac{\partial\rho}{\partial z_{i_1}}\cdots\frac{\partial\rho}{\partial z_{i_p}}.$$

Thus, we obtain

$$\frac{\partial F}{\partial z_k} = \frac{1}{m!}\sum_{j_1,\cdots,j_m}\frac{\partial g_{j_1\cdots j_m}}{\partial z_k}\frac{\partial\rho}{\partial z_{j_1}}\cdots\frac{\partial\rho}{\partial z_{j_m}},$$

which implies that F belongs to $\mathcal{M}_m(U, |\mathrm{grad}\,\rho|)$. The lemma is proved.

Proof of Theorem 2. If K is a compact subset of T, then there exists an open set V, $K\subset V\subset U$, with the smooth boundary. It follows from Corollary of Proposition 2 that $T\subset\bar{V}$ is totally real. Since $H(T\cap\bar{V})|_K\subset H(K)\subset C(K)=C(T\cap\bar{V})|_K$, it is sufficient to prove the theorem in the case when K is totally real. Let $\rho\in P(U)$ be a defining function of K, and set $\delta=|\mathrm{grad}\,\rho|$. Then, by Lemma 5, K is δ-convex. Let f be an arbitrary function in $C^\infty(U)$. Then, by Lemma 6, $f|_K$ has a C^∞ extension F in $\mathcal{M}_{n+1}(U,\delta)$. It follows from Proposition 1 that $f|_K=F|_K\in H(K)$, which proves the theorem.

Theorem 2, when T is a C^k totally real submanifold M of $\boldsymbol{C}^n$, was proved by Hörmander-Wermer [5] for $2k\geq\dim_R M+2$, and by Harvey-Wells [2] for $k=1$, Nirenberg-Wells [7] proved a corresponding result when M is a C^∞ totally real submanifold of a complex manifold.

Theorem 2 implies that any compact subset K of a totally real set T is H-convex. This fact is due to a strong convexity of T. (K is contained in a compact totally real set K_0. K_0 is δ-convex, $\delta=|\mathrm{grad}\,\rho,|$, (Lemma 5), and K is $\mathcal{O}_{K_0}$-convex (Harvey-Wells [2]).)

6. A theorem of Mergeljan

Let f be a real valued continuous function defined on the closed unit disk D in $\boldsymbol{C}^1$. For every real number u, the level set $\{z\in D: f(z)=u\}$ will be denoted by L_u. We consider a uniform algebra

$$B = \{g\in C(D):\ |g_{L_u}\in A(L_u) \text{ for every } u\in f(D)\}.$$

Mergeljan [6] proved the following theorem.

Theorem of Mergeljan. *If, for every $u\in f(D)$, L_u does not divide the plane, then $[z,f;D]=B$ holds. In particular, if every L_u has no interior points in addition, then $[z,f;D]=C(D)$ holds.*

We shall generalize this theorem to the higher dimensional case. Let K be a compact subset of $\boldsymbol{C}^n_z$ and T a totally real subset of $\boldsymbol{C}^m_w$. Let $f=(f_1,\cdots,f_m)$ be a continuous map of K into T. For any point w of $f(K)$, we set $L_w=\{z\in K: f(z)=w\}$. We consider a uniform algebra

$$\mathcal{B}=\{g\in C(K)\colon g|_{L_w}\in A(L_w) \text{ for every } w\in f(K)\}\ .$$

If g is in $\mathcal{A}=[z_1, \dots, z_n, f_1, \cdots, f_m; K]$, then, for every $w\in f(K)$, $g|_{L_w}\in P(L_w)$, and therefore we have $\mathcal{A}\subset\mathcal{B}$. If $\mathcal{A}=\mathcal{B}$ holds, then we have $P(L_w)=A(L_w)$ for every $w\in f(K)$.

Theorem 3. *Suppose that there exists a polynomially convex compact set T_0 such that $f(K)\subset T_0\subset T$. If $P(L_w)=A(L_w)$ holds for every $w\in f(K)$, then we have $\mathcal{A}=\mathcal{B}$. In particular, if $P(L_w)=C(L_w)$ holds for every $w\in f(K)$, then $\mathcal{A}=C(K)$.*

Proof. Fix an arbitrary function g in B and an arbitrary positive number ε. Let α be any point of $f(K)$. Since $P(L_\alpha)=A(L_\alpha)$, we can find a polynomial $P_\alpha(z)$ such that $\|g-P_\alpha\|_{L\alpha}<\varepsilon/2$. By the continuity of $g-P_\alpha$, there exists a positive number δ such that $\|g-P_\alpha\|_{L_w}<\varepsilon$ for every L_w contained in the δ-neighborhood of L_α in C_z^n. By the continuity of f, we can find a positive number η such that L_w is contained in the δ-neighborhood of L_α for every $w\in f(K)$ with $|w-\alpha|<\eta$. Thus we have $\|g-P_\alpha\|_{L_w}<\varepsilon$ for every $w\in f(K)$ with $|w-\alpha|<\eta$. By the compactness of $f(K)$, we can choose a finite open covering $\{V_\nu\}_{\nu=1}^N$ of $f(K)$ in C_w^m and a set of polynomials $\{P_\nu(z)\}_{\nu=1}^N$ so that, for every ν and for every $w\in f(K)\cap V_\nu$, $\|g-P_\nu\|_{L_w}<\varepsilon$ holds. Let $\{\lambda_\nu(w)\}$ be a partition of unity subordinate to $\{V_\nu\}$. Since T is totally real and T_0 is polynomially covenx, we have $P(T_0)=C(T_0)$ by Theorem 2 and Oka's approximation theorem. Therefore, for every ν, there exists a polynomial $Q_\nu(w)$ in w such that

$$\|\lambda_\nu-Q_\nu\|_{T_0}<\varepsilon[\sum_\mu \|P_\mu\|_K]^{-1}\ .$$

We set $h(z)=\sum_\nu Q_\nu(f(z))P_\nu(z)$. Then h belongs to $\mathcal{A}$.

Let z be an arbitrary point of K. We denote by Λ the set of indices ν for which the point $f(z)$ belongs to V_ν. Since $\lambda_\nu(f(z))=0$ for any $\nu\notin\Lambda$, we have

$$\begin{aligned}|g(z)-h(z)| &\leq \sum_{\nu\in\Lambda}\lambda_\nu(f(z))|g(z)-P_\nu(z)| \\ &\quad+\sum_{\nu=1}^N|\lambda_\nu(f(z))-Q_\nu(f(z))|\,|P_\nu(z))| \\ &<\varepsilon\sum_{\nu\in\Lambda}\lambda_\nu(f(z))+\varepsilon(\sum_\mu\|P_\mu\|_K)^{-1}\sum_\nu|P_\nu(z)|<2\varepsilon\ .\end{aligned}$$

From this it follows that g belongs to $\mathcal{A}$.

Corollary. *Suppose that f_k are real valued continuous functions defined on K, $1\leq k\leq m$. If $P(L_u)=A(L_u)$ holds for every $u\in f(K)$, then we have $\mathcal{A}=\mathcal{B}$. In particular, if $P(L_u)=C(L_u)$ holds for every $u\in f(K)$, then we have $\mathcal{A}=C(K)$.*

Proof. We canconsider $f=(f_1, \cdots, f_m)$ as a continuous map of K into a real subspace $\boldsymbol{R}_u^m$ of $\boldsymbol{C}_w^m$. $\boldsymbol{R}_u^m$ is a totally real set in $\boldsymbol{C}_w^m$. For a sufficiently large polydisk D_0, $T_0=D_0\cap \boldsymbol{R}^m_u$ is a polynomially convex set containing $f(K)$. Thus, all the conditions of Theorem 5 are satisfied.

As an application of this corollary, we give an example to show that a compact set K satisfying $H(K)=C(K)$ is not necessarily a subset of a totally real set.

Let $K=\prod_{\nu=1}^{n} K_\nu$ be a compact subset of $\boldsymbol{C}^n_z$ and let f_ν be real valued continuous functions defined on K_ν respectively. Suppose that, for every $u_\nu\in f_\nu(K_\nu)$, the level set $L_{u_\nu}=\{z_\nu\in K_\nu: f_\nu(z_\nu)=u_\nu\}$ has no interior points and does not deivide $\boldsymbol{C}^1_{z_\nu}$. Then, we havve $P(L_{u_\nu})=C(L_{u_\nu})$. Set $f(z)=(f_1(z_1), \cdots, f_n(z_n))$, $z\in K$. Then for every vector $u\in f(K)$, we have $L_u=\prod_\nu L_{u_\nu}$, and therefore $P(L_u)=C(L_u)$. This follows from Stone-Weierstrass's theorem, since the totality of polynomials that are real valued on K separates the points of K. It follows from Corollary of Theorem 3 that $\mathcal{A}=[z_1, \cdots, z_n, f_1, \cdots, f_n; K]$ coincides with $C(K)$. Set

$$K^* = \{(z, f(z))\in \boldsymbol{C}^{2n}_{(z,w)}:\ w_\nu=f_\nu(z_\nu)\,,\ 1\leq\nu\leq n\}\ .$$

The projection of $\boldsymbol{C}^{2n}_{(z,w)}$ onto $\boldsymbol{C}^n_z$ induces isomorphisms of $P(K^*)$ onto $\mathcal{A}$ and of $C(K^*)$ onto $C(K)$. Thus we have $P(K^*)=C(K^*)$ $(=H(K^*))$.

However, K^* is not necessarily totally real. We consider a simple case when $f_\nu(z_\nu)=q_\nu(x_\nu)$, where $q_\nu(x)$ are C^∞ functions defined on an open interval $I=(-2, 2)$ of a real variable x. Suppose that, for every ν, and for every $s\in q_\nu(I)$, the level set $\{x\in I: q_\nu(x)=s\}$ is a discrete set. Set $K_\nu=\{z_\nu\in \boldsymbol{C}^1_{z_\nu}: |z_\nu|\leq 1\}$. Then, we have $H(K^*)=C(K^*)$ by the argument above. Set

$$M = \{(z, f(z))\in \boldsymbol{C}^{2n}_{(z,w)}:\ w_\nu=f_\nu(z_\nu),\ x_\nu\in I,\ 1\leq\nu\leq n\}\ .$$

Then, M is a C^∞ real submanifold of $\boldsymbol{C}^{2n}_{(z,w)}$. It follows from Lemma 3 that the complex rank of M at the point $(z^0, f(z^0))$ of M is given by $n-\operatorname{rank}\left[\frac{\partial f_\nu}{\partial z_k}(z^0)\right]$. We impose an additional assumption that every $q'_\nu(x)$ has an isolated zero at $x=0$. Set

$$E_0 = \{z\in \boldsymbol{C}^n_z:\ x_\nu=0\,,\ 1\leq\nu\leq n\}\ ,$$

and

$$E = \{(z, f(z))\in \boldsymbol{C}^{2n}_{(z,w)}:\ z\in E_0\}\ .$$

Since

$$\operatorname{rank}\left[\frac{\partial f_\nu}{\partial z_k}(z)\right] = \operatorname{rank}\begin{bmatrix} q'_1(0) & & 0 \\ & \ddots & \\ 0 & & q'_u(0)\end{bmatrix} = 0$$

at every point $z\in E_0$, M has the complex rank n at every point of E. Thus, K^*

can not be a subset of a totally real set in $C^{2n}_{(w,z)}$. We remark that E is an n-dimensional totally real subspace of $C^{2n}_{(w,z)}$.

7. Interpolation sets

Let G be a bounded domain in C^n. A closed subset K of the Šilov boundary of $A(\bar{G})$ is called an *interpolation set for* $A(\bar{G})$, if $A(\bar{G})|_K=C(K)$. If, for every function f in $C(K)$, there exists a function F in $A(\bar{G})$ such that $F(z)=f(z)$ for $z\in K$ and $|F(z)|<\|f\|_K$ for $z\in\bar{G}\backslash K$, then K is called a *peak interpolation set for* $A(\bar{G})$. It is known that an interpolation set K is a peak interpolation set if and only if K is a *peak set for* $A(\bar{G})$, that is, there exists a function f in $A(\bar{G})$ such that $f(z)=1$ for $z\in K$ and $|f(z)|<1$ for $z\in\bar{G}\backslash K$.

When $\bar{G}$ is a compact polydisk D^n, the Šilov boundary of $A(D^n)$ is the n-dimensional torus T^n. In [11], Stout proved that a closed subset K of T^n is a peak interpolation set if and only if K is the zero set of a function in $A(D^n)$. We shall consider the case when G is an analytic polyhedron or a strictly pseudoconvex domain.

We use the following lemma known in the theory of uniform algebras (cf. Stout [12], Chap. 4).

Lemma 7. *Suppose that K satisfies the following conditions:*
(i) *$\bar{G}\backslash K$ is simply connected;*
(ii) *there exists a function $h\in A(\bar{G})$ such that*

$$K=\{z\in\bar{G}:\ h(z)=0\}\ .$$

Then, K is a peak set for $A(\bar{G})$, and $A(\bar{G})|_K$ is a closed subalgebra of $C(K)$.

Theorem 4. *Let G be a bounded analytic polyhedron in C^n defined by*

$$G=\{z\in U:\ |f_\nu(z)|<1,\ 1\leq\nu\leq n\}\ ,$$

where U is an open set containing $\bar{G}$, and f_ν are functions holomorphic in U. Suppose that $\det\left[\dfrac{\partial f_\nu}{\partial z_k}\right]$ *has no zeros on the set*

$$\Gamma=\{z\in U:\ |f_\nu(z)|=1,\ 1\leq\nu\leq n\}\ .$$

If K is a closed subset of Γ satisfying the condition (i) *and* (ii) *of Lemma 7, then K is a peak interpolation set for $A(\bar{G})$.*

Proof. It suffices to prove that $A(\bar{G})|_K$ is dense in $C(K)$, by Lemma 7. Choose a positive number $r>1$ so that the open set

$$V=\{z\in U:\ |f_\nu(z)|<r,\ 1\leq\nu\leq n\}$$

is relatively compact in U. We first prove that K is $\mathcal{O}_V$-convex. If z^0 is a point of $V\backslash\bar{G}$, then we can find a function g holomorphic in V suhc that $|g(z^0)|>||g||_G$, since $\bar{G}$ is $\mathcal{O}_V$-convex. If z_0 is a point of $\bar{G}\backslash K$, we have $|h(z^0)|>0$ by the condition (ii) of Lemma 7. There exists a function $\tilde{h}$ holomorphic in a neighborhood of $\bar{G}$ such that $||h-\tilde{h}||_G<|h(z^0)|/4$ (cf. Petrosjan [8]). Since $\bar{G}$ is $\mathcal{O}_V$-convex, we can find a function g holomorphic in V such that $||\tilde{h}-g||_G<|h(z^0)|/4$. Therefore, we have $|g(z^0)|>||g||_K$. Thus, K is $\mathcal{O}_V$-convex. From this it follows that $H(K, V)=H(K)$. Since $H(K, V)$ is contained in the closure of $A(\bar{G})|_K$ in $C(K)$ and since $A(\bar{G})|_K$ is closed in $C(K)$, it remains to prove that $H(K)=C(K)$.

We consider the function

$$\rho(z)=\sum_{\nu}(|f_\nu(z)|^2-1)^2$$

in U. Then we have

$$H[\rho;\xi](z)=2\sum_{\nu}(|f_\nu(z)|^2-1)|\sum_{k}\frac{\partial f_\nu(z)}{\partial z_k}\xi_k|^2$$
$$+2\sum_{\nu}|f_\nu(z)|^2|\sum_{k}\frac{\partial f_\nu(z)}{\partial z_k}\xi_k|^2 .$$

Since $\det\left[\frac{\partial f_\nu}{\partial z_k}\right]\neq 0$ and $|f_\nu(z)|=1$ on Γ, ρ is in $\mathcal{P}(U_1)$ for a neighborhood U_1 of Γ, and therefore Γ is totally real. Thus the theorem follows from Theorem 2.

In the next place, we consider the case of strictly pseudoconvex domains (not necessarily with smooth boundaries).

Theorem 5. *Let G be a bounded strictly pseudoconvex domain in $\boldsymbol{C}^n$ defined by*

$$G=\{z\in U:\ \sigma(z)<0\} ,$$

where U is an open set containing $\bar{G}$ and σ is strictly plurisubharmonic in U. Let K be a closed subset of ∂G satisfying the condition (i) *of Lemma* 7 *and the following condition*:

(ii)′ *there exists a function h holomorphic in U such that $K=\{z\in\bar{G}: h(z)=0\}$ and dh has no zero on K.*

Then K is a peak interpolation set for $A(\bar{G})$.

Proof. Since the condition (ii)′ is stronger than (ii) of Lemma 7, it suffices to prove that $A(\bar{G})|_K$ is dense in $C(K)$. We can assume that U is holomorphically convex and the zero set X of h in U is a closed submanifold of U.

Let z be any point of K. Then there exist a neighborhood U_z of z with $U_z\subset U$ and a local complex coordinate $\zeta=(\zeta_1, \cdots, \zeta_n)$ in U_z such that $X\cap U_z=\{\zeta\in U_z: \zeta_n=0\}$. Let $\tilde{\sigma}(\zeta_1, \cdots, \zeta_{n-1})$ be the restriction of σ to $X\cap U_z$ and set

$$\rho(\zeta)=\tilde{\sigma}(\zeta_1, \cdots, \zeta_{n-1})+|\zeta_n|^2 , \qquad \zeta\in U_z .$$

Then ρ is a function in $\mathcal{P}(U_z)$ and satisfies

$$K \cap U_z = \{\zeta \in U_z \colon \rho(\zeta) = 0\} .$$

It follows from Proposition 1 that K is totally real. Thus we have $H(K) = C(K)$.

Since $\tilde{\sigma}$ is a strictly plurisubharmonic function on X and since $K = \{\zeta' \in X \colon \tilde{\sigma}(\zeta') \leq 0\}$, K is $\mathcal{O}_X$-convex. Since X is the zero set of h in U, K is $\mathcal{O}_U$-convex. Thus we can conclude that $A(\bar{G})|_K = C(K)$, by using an argument similar to one used in the proof of Theorem 4.

We remark that, when $n=1$, the condition (ii)$'$ reduces our problem to a very simple one, since then K is a finite point set. When $n>1$, it is not trivial. We consider, for example, the case when G is the unit ball in $\boldsymbol{C}^z_n$ and $h(z) = \sum_k z_k^2 - 1$. Then, all the conditions of Theorem 5 are satisfied for $K = \{(x_1, \cdots, x_n) \in \boldsymbol{R}^n_x \colon \sum_k x_k^2 = 1\}$. In this case, K is an $(n-1)$-diemnsional totally real submanifold of $\boldsymbol{C}^n_z$.

8. CR-functions

Let T be a totally real set in $\boldsymbol{C}^n_z$ and G a holomorphically convex open subset of $\boldsymbol{C}^m_w$. The projections of $\boldsymbol{C}^N = \boldsymbol{C}^n_z \times \boldsymbol{C}^m_w$ onto $\boldsymbol{C}^n_z$ and $\boldsymbol{C}^m_w$ are denoted by π_1 and π_2 respectively. Let K be a compact set of $\boldsymbol{C}^N$. For any z of $\boldsymbol{C}^n_z$, we set $K_z = \pi_1^{-1}(z) \cap K$ and $K'_z = \pi_2(K_z)$.

Theorem 6. *Let K be a compact subset of $\boldsymbol{C}^N$ satisfying the following conditions:*

(i) $\pi_1(K) = T$;

(ii) *for every $z \in T$, there exists a complex analytic subvariety X_z of G such that K'_z is a $\mathcal{O}_{X_z}$-convex compact subset of X_z.*

If f is a C^∞ function defined in a neighborhood U of K in $\boldsymbol{C}^N$ which is holomorphic in w on $X_z \cap U$ for every $z \in T$, then f belongs to $H(K)$.

Proof. Let α be any point of T. By the compactness of K_α, for every number $\delta > 0$, there exists a $\eta > 0$ such that K_z is contained in the δ-neighborhood of K_α in $\boldsymbol{C}^N$, for every $z \in T$ with $|z - \alpha| < \eta$. Since K'_α is $\mathcal{O}_{X_\alpha}$-convex, and since X_α is a closed subvariety of G, there exists a function $g_\alpha(w)$ holomorphic in G such that

$$|f(\alpha, w) - g_\alpha(w)| < \varepsilon/2 , \qquad w \in K'_\alpha .$$

By the continuity of $f - g_\alpha$, there exists $\eta > 0$ such that

$$|f(z, w) - g_\alpha(w)| < \varepsilon , \qquad w \in K'_z$$

holds for every $z\in T$ with $|z-\alpha|<\eta$.

By the compactness of T, we can find a finite open covering $\{V_\nu\}_{\nu=1}^N$ of T in C_z^n and a set of functions $\{g_\nu(w)\}_{\nu=1}^N$ each holomorphic in G and satisfying

$$|f(z, w)-g_\nu(w)|<\varepsilon, \qquad z\in T\cap V_\nu, \qquad w\in K'_z.$$

Choose a partition of unity $\{\lambda_\nu(z)\}$ subordinate to $\{V_\nu\}$. Since T is totally real, there exists a set of functions $\{h_\nu(z)\}$ each holomorphic in a neighborhood V of T in C_z^n satisfying

$$|h_\nu(z)-\lambda_\nu(z)|<\varepsilon[\sum_\mu \|g_\mu\|_{\pi_2(K)}]^{-1}, \qquad z\in T.$$

We set $f_\varepsilon(z, w)=\sum_\nu h_\nu(z)g_\nu(w)$.

Let (z, w) be any point of K. We denote by Λ the set of indices ν for which z belongs to V_ν. Then we have

$$\begin{aligned}|f(z, w)-f_\varepsilon(z, w)| &\leq \sum_{\nu\in\Lambda}\lambda_\nu(z)|f(z, w)-g_\nu(w)|\\ &+\sum_{\nu=1}^N|\lambda_\nu(z)-h_\nu(z)|\,|g_\nu(w)|\\ &<\varepsilon\sum_\nu\lambda_\nu(z)+\varepsilon\sum_\nu[\sum_\lambda\|g_\mu\|_{\pi_2(K)}]^{-1}|g_\nu(w)|<2\varepsilon.\end{aligned}$$

Since $f_\varepsilon(z, w)$ is holomorphic in the open set $V\times G$, we have $f|_K\in H(K)$, as required.

Let M be a holomorphic CR-submanifold of C^N. A C^∞ function f defined on M is called a *CR-function,* if f is holomorphic with respect to the complex coordinates in M. Let K be a compact subset of M. We denote by $CR(K)$ the closure in $C(K)$ of the class of functions each of which is the restriction of a CR-function defined on a neighborhood of K in M. If $r(M)=0$, then $CR(K)$ trivially coincides with $C(K)$. Let T be a totally real submanifold of C_z^n and G a holomorphically convex compact subset of C_w^m. If M is a closed real submanifold of $T\times G$ such that, for every $z\in T$, $M'_z=\pi_2(M_z)$ is an r-dimensional closed complex analytic submanifold of G, then M is a CR-submanifold of C^N of complex rank r. In this case, we have the following corollary.

Corollary. *Let K be a compact subset of M such that every K'_z is $\mathcal{O}_{M'_z}$-convex. Then we have $H(K)=CR(K)$.*

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