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ON THE ALEXANDER POLYNOMIALS OF COBORDANT LINKS

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R.H. Fox and J.W. Milnor in [4] showed that the Alexander polynomial of a slice knot is of the form $f(t)f(t^{-1})$ for an integral polynomial $f(t)$ with $|f(1)|=1$. This clearly implies that the Alexander polynomials of cobordant knots are identical up to the polynomials of the form $f(t)f(t^{-1})$. The purpose of this paper is to generalize this property to that of arbitrary cobordant links. On the basis of the work done by K. Reidemeister, H.G. Shumann and W. Burau, R.H. Fox defined the μ-variable Alexander polynomial $A^0(t_1, \cdots, t_\mu)$ of a link L^μ with μ components. (cf. R.H. Fox [3], G. Torres [9].) One difficulty in our study is that using this definition the polynomial $A^0(t_1, \cdots, t_\mu)$ vanishes for many links. For example, any decomposable link (that is, a link separated into two sublinks by a 2-sphere within a 3-sphere) has $A^0(t_1, \cdots, t_\mu)=0$. To avoid this difficulty we shall re-define the Alexander polynomial $A(t_1, \cdots, t_\mu)$ so that it is *always a non-zero polynomial.* To measure the difference between $A_0(t_1, \cdots, t_\mu)$ and $A(t_1, \cdots, t_\mu)$, we will also introduce a numerical invariant $\beta(L^\mu)$ with $0\leq\beta(L^\mu)\leq\mu-1$ such that

$$A^0(t_1, \cdots, t_\mu) = \begin{cases} A(t_1, \cdots, t_\mu) & \text{if } \beta(L_\mu)=0 \\ 0 & \text{if } \beta(L^\mu)\neq 0\,. \end{cases}$$

A link is the disjoint union of piecewise-linearly embedded, oriented 1-spheres in the oriented 3-sphere S^3. Two links L_0 and L_1 with μ components are *PL cobordant,* if there exist mutally disjoint, piecewise-linearly embedded proper annuli $F_1, \cdots, F_\mu$ in $S^3\times[0, 1]$ spanning $S^3\times 0$ and $S^3\times 1$ such that $(F_1\cup\cdots\cup F_\mu)\cap S^3\times 0=L_0\times 0$ and $(F_1\cup\cdots\cup F_\mu)\cap S^3\times 1=(-L_1)\times 1$, where $-L_1$ is L_1 with orientation reversed. If the annuli $F_1, \cdots, F_\mu$ are locally flat, then the links L_0 and L_1 are simply said to be *cobordant.* A link that is cobordant to the trivial link is called *a slice link in the strong sense.* (cf. R.H. Fox [3].) For (PL) cobordant links L_i, $i=0, 1$ with μ components the Alexander polynomials $A_i(t_1, \cdots, t_\mu)$ of L_i should be chosen to be the Alexander polynomials associated with meridian bases of $H_1(S^3-L_i; Z)$ consistent through the cobordism annuli $F_1, \cdots, F_\mu$.

Our main results are as follows:

Theorem A. *The integer $\beta(L)$ is the invariant of links that are PL cobordant to the link L.*

Theorem B. *For cobordant links L_i, $i=0, 1$, with μ components, there exist two integral polynomials $F_i(t_1, \cdots, t_\mu)$, $i=0, 1$ with $|F_i(1, \cdots, 1)|=1$ such that $A_0(t_1, \cdots, t_\mu)F_0(t_1, \cdots, t_\mu)F_0(t_1^{-1}, \cdots, t_\mu^{-1}) \doteq$*) $A_1(t_1, \cdots, t_\mu)F_1(t_1, \cdots, t_\mu)F_1(t_1^{-1}, \cdots, t_\mu^{-1})$.*

Our proof of Theorem B is based on the Blanchfield duality theorem [1].

Corollary 1. *For PL cobordant links L_i, $i=0, 1$, with μ components, there exist two integral polynomials $F_i(t_1, \cdots, t_\mu)$ with $|F_i(1, \cdots, 1)|=1$ and (integral) knot polynomials $p_1(t_1), \cdots, p_\mu(t_\mu)$ such that $A_0(t_1, \cdots, t_\mu)F_0(t_1, \cdots, t_\mu)F_0(t_1^{-1}, \cdots, t_\mu^{-1}) \doteq A_1(t_1, \cdots, t_\mu)F_1(t_1, \cdots, t_\mu)F_1(t_1^{-1}, \cdots, t_\mu^{-1})p_1(t_1)\cdots p_\mu(t_\mu)$.* [Note that L_0 is cobordant to a link L_1' each component of which is obtained from a component of L_1 by tying a knot in a small 3-cell.]

Corollary 2. *The Alexander polynomial $A(t_1, \cdots, t_\mu)$ of a slice link L with μ components in the strong sense necessarily satisfies $A(t_1, \cdots, t_\mu) \doteq F(t_1, \cdots, t_\mu) \times F(t_1^{-1}, \cdots, t_\mu^{-1})$, $|F(1, \cdots, 1)|=1$, and $\beta(L)=\mu-1$.*

Note that we are dealing with Problem 26 of R.H. Fox [3]. As far as the author knows, this corollary has not been deduced before, but one-variable analogy of this corollary is already known. (See A. Kawauchi [5], K. Murasugi [8].)

As a simple application, the classical Alexander polynomial $A^0(t_1, \cdots, t_\mu)$ of a slice link L with μ components in the strong sense is 0 if $\mu \geq 2$, since $\beta(L)=\mu-1>0$.

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Throughout the paper, spaces are considered in the piecewise linear category.

1. Preliminaries and precise definitions

Let Λ be the integral group ring $Z[t_1, \cdots, t_\mu]$ of the free abelian multiplicative group $\langle t_1, \cdots, t_\mu \rangle$ generated by $t_1, \cdots, t_\mu$. Consider a finitely generated Λ-module $\mathfrak{M}$ and let P be an $m \times n$ presentation matrix of $\mathfrak{M}$, that is, a matrix representing a homomorphism $\Lambda^m \to \Lambda^n$ with an exact sequence $\Lambda^m \to \Lambda^n \to \mathfrak{M} \to 0$, where it may be $m=+\infty$. (Note that we can always choose to make m finite, since Λ is Noetherian.) Let $A^{(i)}$, $i=0, 1, \cdots, n-1$, be the g.c.d. of the minors of P of the order $n-i$. For $i \geq n$ we define $A^{(i)}=1$. It is well-known that the polynomials $A^{(i)}=A^{(i)}(t_1, \cdots, t_\mu)$, $i=0, 1, 2, \cdots$, are the invariants of the Λ-module $\mathfrak{M}$ up to units of Λ. Let $\mathrm{Tor}_\Lambda(\mathfrak{M})$ be the Λ-torsion part of $\mathfrak{M}$.

*) The notation $\doteq$ means "equal up to $\pm t_1^{a_1}t_2^{a_2}\cdots t_\mu^{a_\mu}$ for all integers $a_1, a_2 \cdots a_\mu$".

Lemma 1.1. *Let $A^{(d)}$ be the first non-zero polynomial of $\mathfrak{M}$. $A^{(d)}$ is the 0–th polynomial of* $\mathrm{Tor}_\Lambda(\mathfrak{M})$ *and* $d=\dim_{Q(\Lambda)}\mathfrak{M}\otimes_\Lambda Q(\Lambda)$, *where* $Q(\Lambda)$ *is the quotient field of* Λ.

For a proof, see R.C. Blanchfield [1], Lemma 4.10.

Now consider a *finitely generated* group $G=(x_1, \cdots, x_n | r_1, \cdots, r_m)$ (possible $m=+\infty$) with an epimorphism $\gamma\colon G\to\langle t_1, \cdots, t_\mu\rangle$. Let $K=\mathrm{Ker}\,\gamma$ and K' be the commutator subgroup of K. K/K' admits a canonical Λ-module structure. Fox's free calculus [3] produces a Jacobian matrix $\left(\gamma\left(\frac{\partial r_i}{\partial x_j}\right)\right)_{1\leq i\leq m,\, 1\leq j\leq n}$ evaluated at γ that is a presentation matrix of a certain Λ-module $\mathfrak{M}$ with an exact sequence $0\to K/K'\to\mathfrak{M}\to\mathcal{E}(\Lambda)\to 0$, where $\mathcal{E}(\Lambda)$ is the augmentation ideal, that is, the kernel of the augmentation $\mathcal{E}\colon\Lambda\to Z$. (See R.H. Crowell [2].) Since n is finite, $\mathfrak{M}$ and K/K' are finitely generated over Λ.

Lemma 1.2. *Let* $d=\dim_{Q(\Lambda)}(K/K')\otimes_\Lambda Q(\Lambda)$. *The* 0*-th polynomial of* $\mathrm{Tor}_\Lambda(K/K')$ *is the g.c.d. of the minors of the Jacobian matrix* $\left(\gamma\left(\frac{\partial r_i}{\partial x_j}\right)\right)$ *of the order* $n-d-1$. *Any minor of* $\left(\gamma\left(\frac{\partial r_i}{\partial x_j}\right)\right)$ *of an order greater than* $n-d-1$ is 0.

Proof. Since $\mathcal{E}(\Lambda)$ is torsion-free, we have $\mathrm{Tor}_\Lambda(K/K')=\mathrm{Tor}_\Lambda(\mathfrak{M})$. Using $\dim_{Q(\Lambda)}\mathfrak{M}\otimes_\Lambda Q(\Lambda)=d+1$, from Lemma 1.1 we obtain the desired results.

DEFINITION 1.3. The 0-th polynomial of $\mathrm{Tor}_\Lambda(K/K')$, denoted by $A_\gamma=A_\gamma(t_1, \cdots, t_\mu)$ is called the *Alexander polynomial* of G with $\gamma\colon G\to\langle t_1, \cdots, t_\mu\rangle$.

If $G(L)$ is a μ-link group (that is, the fundametal group of the exterior of a link L with μ components) and the epimorphism $\gamma\colon G(L)\to\langle t_1, \cdots, t_\mu\rangle$ is specified by the meridian curves of $L\subset S^3$, then A_γ is simply denoted by A and called the *Alexander polynomial* of the link L.

Lemma 1.4. *The Alexander polynomial of G/K' with the induced epimorphism $\gamma'\colon G/K'\to\langle t_1, \cdots, t_\mu\rangle$ is the Alexander polynomial of G with $\gamma\colon G\to\langle t_1, \cdots, t_\mu\rangle$.*

Proof. It follows from $\mathrm{Ker}\,\gamma'=K/K'$.

DEFINITION 1.5. Let $\beta^\gamma(G)=\dim_{Q(\Lambda)}(K/K')\otimes_\Lambda Q(\Lambda)$. For a link group $G=G(L^\mu)$ with the specified epimorphism $\gamma\colon G(L^\mu)\to\langle t_1, \cdots, t_\mu\rangle$, $\beta^\gamma(G(L))$ is simply denoted by $\beta(L)$.

The classical Alexander polynomial $A^0_\gamma(t_1, \cdots, t_\mu)$ is defined as the 0-th polynomial of the Λ-module K/K'. [In fact, it should be noted that $A^0_\gamma(t_1, \cdots, t_\mu)$ is the g.c.d. of the minors of the Jacobian matrix $\left(\gamma\left(\frac{\partial r_i}{\partial x_j}\right)\right)$ of the order $n-1$

by Lemma 1.2, provided $G=(x_1, \cdots, x_n | r_1, \cdots, r_m)$.]

The following is immediately clear from the definitions and Lemma 1.2:

Lemma 1.6.

$$A_\gamma^0(t_1, \cdots, t_\mu) = \begin{cases} A_\gamma(t_1, \cdots, t_\mu) & \text{if } \beta^\gamma(G)=0 \\ 0 & \text{if } \beta^\gamma(G)\neq 0 \, . \end{cases}$$

2. Proof of Theorem A

Now consider a finite connected complex X with an epimorphism $\gamma\colon \pi_1(X)\to\langle t_1, \cdots, t_\mu\rangle$. For a subcomplex X_0 of X (possible $X_0=\phi$), let $p\colon (\tilde{X}, \tilde{X}_0)\to(X, X_0)$ be the free abelian covering of (X, X_0) associated with the epimorphism γ. The integral homology group $H_*(\tilde{X}, \tilde{X}_0)$ admits a finitely generated Λ-module structure. Denote $\mathrm{Tor}_\Lambda H_*(\tilde{X}, \tilde{X}_0)$ by $T_*(\tilde{X}, \tilde{X}_0)$ and $\dim_{Q(\Lambda)}H_*(\tilde{X}, \tilde{X}_0)\otimes_\Lambda Q(\Lambda)$ by $\beta_*^\gamma(X, X_0)$. Clearly, the 0-th polynomial of $T_1(\tilde{X})$ is the Alexander polynomial of $\pi_1(X)$ with γ and $\beta_1^\gamma(X)$ is equal to $\beta^\gamma(\pi_1(X))$.

Lemma 2.1. *For some i, if $H_i(X, X_0)=0$, then $\beta_i^\gamma(X, X_0)=0$, i.e., $T_i(\tilde{X}, \tilde{X}_0)=H_i(\tilde{X}, \tilde{X}_0)$ and the 0-th polynomial A of $H_i(\tilde{X}, \tilde{X}_0)$ satisfies $|A(1, \cdots, 1)|=1$.*

Proof. Let $\Delta_1^q, \cdots, \Delta_{s_q}^q$ be the q-simplexes of X forming a basis for the q-chain complex $C_q(X, X_0)$. Let $\tilde{\Delta}_1^q, \cdots, \tilde{\Delta}_{s_q}^q$ be the q-simplexes of $\tilde{X}$ such that, for each j, $\tilde{\Delta}_j^q$ corresponds to Δ_j^q under the projection p. $\{\tilde{\Delta}_1^q, \cdots, \tilde{\Delta}_{s_q}^q\}$ forms a Λ-basis for the q-chain complex $C_q(\tilde{X}, \tilde{X}_0)$. With these bases, the boundary homomorphism $\partial\colon C_q(\tilde{X}, \tilde{X}_0)\to C_{q-1}(\tilde{X}, \tilde{X}_0)$ represents a matrix (α_{jk}^q) with α_{jk}^q in Λ. Let $\tilde{r}_q$ be the rank of this matrix. The boundary homomorphism $\partial\colon C_q(X, X_0)\to C_{q-1}(X, X_0)$ is represented by the integral matrix $(\alpha_{jk}^q(1, \cdots, 1))$ whose rank r_q satisfies $r_q\leq\tilde{r}_q$. Since $H_i(X, X_0)=0$, the sequence $C_{i+1}(X, X_0)\xrightarrow{\partial} C_i(X, X_0)\xrightarrow{\partial} C_{i-1}(X, X_0)$ is exact at $C_i(X, X_0)$. Hence $r_{i+1}=s_i-r_i$. Using $H_q(\tilde{X}, \tilde{X}_0)\otimes_\Lambda Q(\Lambda)=H_q(C_*(\tilde{X}, \tilde{X}_0)\otimes_\Lambda Q(\Lambda))$, $\beta_q^\gamma(X, X_0)$ is equal to $s_q-\tilde{r}_q-\tilde{r}_{q+1}$. In particular, $\beta_i^\gamma(X, X_0)=s_i-\tilde{r}_i-\tilde{r}_{i+1}\leq s_i-r_i-r_{i+1}=0$. That is, $\beta_i^\gamma(X, X_0)=0$, $\tilde{r}_i=r_i$ and $\tilde{r}_{i+1}=r_{i+1}$.

Consider the short exact sequence

$$0 \to H_i(\tilde{X}, \tilde{X}_0) \to C_i(\tilde{X}, \tilde{X}_0)/\tilde{B}_i \to C_i(\tilde{X}, \tilde{X}_0)/\tilde{Z}_i \to 0 \, ,$$

where $\tilde{B}_i=\mathrm{Im}[\tilde{\partial}\colon C_{i+1}(\tilde{X}, \tilde{X}_0)\to C_i(\tilde{X}, \tilde{X}_0)]$ and $\tilde{Z}_i=\mathrm{Ker}[\tilde{\partial}\colon C_i(\tilde{X}, \tilde{X}_0)\to C_{i-1}(\tilde{X}, \tilde{X}_0)]$.

Note that the matrix (α_{jk}^{i+1}) is a presentation matrix of $C_i(\tilde{X}, \tilde{X}_0)/\tilde{B}_i$ and the $C_i(\tilde{X}, \tilde{X}_0)/\tilde{Z}_i$ is Λ-torsion-free of rank r_i. By lemma 1.1 the 0-th polynomial A of $H_i(\tilde{X}, \tilde{X}_0)$ is the r_i-th polynomial of $C_i(\tilde{X}, \tilde{X}_0)/\tilde{B}_i$. Now let $Z_i=\mathrm{Ker}[\partial\colon C_i(X, X_0)\to C_{i-1}(X, X_0)]=\mathrm{Im}[\partial\colon C_{i+1}(X, X_0)\to C_i(X, X_0)]$. Since

$C_i(X, X_0)/Z_i$ is free of rank r_i and $(\alpha_{jk}^{i+1}(1, \cdots, 1))$ is a presentation matrix of $C_i(X, X_0)/Z_i$, it follows that the g.c.d. of the minors of $(\alpha_{jk}^{i+1}(1, \cdots, 1))$ of the order $s_i - r_i$ is ± 1. This implies $|A(1, \cdots, 1)| = 1$. This completes the proof.

REMARK 2.2. For a finitely generated Λ-module $\mathfrak{M}$, $\mathfrak{M} \otimes_\Lambda Z = 0$ if and only if the 0-th polynomial A of $\mathfrak{M}$ satiesfies $|A(1, \cdots, 1)| = 1$. [Note that if (α_{jk}) is a presentation matrix of $\mathfrak{M}$, then $(\alpha_{jk}(1, \cdots, 1))$ is a presentation matrix of $\mathfrak{M} \otimes_\Lambda Z$.]

Corollary 2.3. *Let $H_1(X)$ have a free abelian group of rank μ'. Then $\beta_1^\gamma(X) \leq \mu' - 1$ for any epimorphism $\gamma: \pi_1(X) \to \langle t_1, \cdots, t_\mu \rangle$ with $\mu \leq \mu'$. In particular, $\beta(L) \leq \mu - 1$ for a link L with μ components.*

Proof. Let X_0 be a connected graph in X with the inclusion isomorphism $H_1(X_0) \approx H_1(X)$. We have $H_1(X, X_0) = 0$. By Lemma 2.1 $H_1(\tilde{X}, \tilde{X}_0)$ is a torsion Λ-module. Since $H_1(\tilde{X}_0) \to H_1(\tilde{X}) \to H_1(\tilde{X}, \tilde{X}_0)$ is exact and $H_1(\tilde{X}_0)$ is a Λ-module of rank $\mu' - 1$, it follows that $\beta_1^\gamma(X) \leq \mu' - 1$, which completes the proof.

Consider a short exact sequence $0 \to T' \to T \to T'' \to 0$ of finitely generated torsion Λ-modules T', T and T''. Let us denote the 0-th polynomials of T', T and T'' by A', A and A'', respectively.

Lemma 2.4. $$A \doteq A'A''.$$

Proof. The proof will depend on the fact that Λ is a unique factorization domain. For a prime element p of Λ, let $A' = p^{\lambda'} q'$, $A = p^\lambda q$ and $A'' = p^{\lambda''} q''$, where q', q and q'' are elements in Λ prime to p, and λ', λ and λ'' are non-negative integers. Denote by Λ_p the local ring of Λ at the element p. Note that Λ_p is a principal ideal domain. By using the presentation matrices of T', T and T'', it follows that the ideal orders (i.e., the generators of the order ideals) of $T' \otimes_\Lambda \Lambda_{p'}$ $T \otimes_\Lambda \Lambda_p$ and $T'' \otimes_\Lambda \Lambda_p$ are $p^{\lambda'}$, p^λ and $p^{\lambda''}$, respectively. Since the sequence $0 \to T' \otimes_\Lambda \Lambda_p \to T \otimes_\Lambda \Lambda_p \to T'' \otimes_\Lambda \Lambda_p \to 0$ is exact, $p^\lambda = p^{\lambda'} p^{\lambda''}$. Hence $A \doteq A'A''$. This proves Lemma 2.4.

Let X be a finite connected complex with an epimorphism $\gamma: \pi_1(X) \to \langle t_1, \cdots, t_\mu \rangle$ and A_γ be the Alexander polynomial of $\pi_1(X)$ with γ. Using the unique factorization domain Λ, one can decompose A_γ into two factors u_γ and v_γ uniquely up to units of Λ such that $|v_\gamma(1, \cdots, 1)| = 1$ (yet u_γ does not contain any non-unit factor f of Λ with $|f(1, \cdots, 1)| = 1$).

Theorem 2.5. *Let X_i, $i = 0, 1$, be finite, connected complexes with rank $H_1(X_i; Z) \neq 0$. If there exists a finite connected complex Y which contains X_i and such that $H_j(Y, X_i) = 0$, $j = 1, 2$, then $\beta_1^{\gamma_0}(X_0) = \beta_1^{\gamma_1}(X_1)$ and $u_{\gamma_0} \doteq u_{\gamma_0}$ for all com-*

patible epimorphisms[**] $\gamma_i\colon \pi_1(X_i)\to\langle t_1, \cdots, t_\mu\rangle$.

Proof. Let $(\tilde{Y}, \tilde{X}_0, \tilde{X}_1)$ be the free abelian cover of (Y, X_0, X_1) associated with an epimorphism $\gamma\colon\pi_1(Y)\to\langle t_1, \cdots, t_\mu\rangle$. Consider the following exact sequence of the pair $(\tilde{Y}, \tilde{X}_i)$:

$$\to H_2(\tilde{Y}, \tilde{X}_i) \xrightarrow{\partial} H_1(\tilde{X}_i) \xrightarrow{i_*} H_1(\tilde{Y}) \xrightarrow{j_*} H_1(\tilde{Y}, \tilde{X}_i) \to \cdots .$$

By Lemma 2.1, $H_j(\tilde{Y}, \tilde{X}_i)$, $j=1, 2$, is a torsion Λ-module. This implies that the following induced sequence

$$T_2(\tilde{Y}, \tilde{X}_i) \xrightarrow{\partial'} T_1(\tilde{X}_i) \xrightarrow{i_*} T_1(\tilde{Y}) \xrightarrow{j_*} T_1(\tilde{Y}, \tilde{X}_i)$$

is exact and that $\beta_1^{\gamma_i}(X_i)=\beta_1^{\gamma}(Y)$, where $\gamma_i\colon \pi_1(X_i)\to\langle t_1, \cdots, t_\mu\rangle$ are the epimorphisms induced from γ. Again by Lemma 2.1, we have $T_2(\tilde{Y}, \tilde{X})_i\otimes_\Lambda Z= T_1(\tilde{Y}, \tilde{X}_i)\otimes_\Lambda Z=0$ (cf. Remark 2.2). From this and Lemma 2.4, it follows that $u_{\gamma_i}\doteq u_\gamma$, where u_γ is the factor of the 0-th polynomial of $T_1(\tilde{Y})$, not containing any non-unit factor f with $|f(1, \cdots, 1)|=1$. Thus, $\beta_1^{\gamma_0}(X_0)=\beta_1^{\gamma_1}(X_1)$ and $u_{\gamma_0}\doteq u_{\gamma_1}$. This completes the proof.

Proof of Theorem A. Consider the union of piecewise-linearly embedded annuli $F_1\cup\cdots\cup F_\mu\subset S^3\times[0,1]$ that reveals the *PL* cobordism of two links $L_0\subset S^3$ and $L_1\subset S^3$ (with μ components). Take a regular neighborhood N of $F_1\cup\cdots\cup F_\mu$ in $S^3\times[0, 1]$ meeting the boundary $S^3\times 0\cup S^3\times 1$ regularly. Let $Y=S^3\times[0, 1]-\mathring{N}$ and $X_i=Y\cap S^3\times i$, $i=0, 1$. By applying Theorem 2.5 to the triple (Y, X_0, X_1), we obtain Theorem A. This completes the proof.

3. Proof of Theorem B

Consider a finite, connected and oriented 4-manifold W with an epimorphsim $\gamma\colon\pi_1(W)\to\langle t_1, \cdots, t_\mu\rangle$, and let $\tilde{W}$ be the free abelian cover of W associated with γ. Suppose $\partial\tilde{W}$ is connected.

For any element f in Λ, let us define $\bar{f}(t_1, \cdots, t_\mu)=f(t_1^{-1}, \cdots, t_\mu^{-1})$.

The following theorem is basic to the proof of Theorem B.

Theorem 3.1. *Assume the sequence* $T_2(\tilde{W}, \partial\tilde{W})\xrightarrow{\partial'} T_1(\partial\tilde{W})\xrightarrow{i_*} T_1(\tilde{W})$ *is exact at* $T_1(\partial\tilde{W})$, *where* ∂' *is defined by the boundary homomorphism* $\partial\colon H_2(\tilde{W}, \partial\tilde{W})\to H_1(\partial\tilde{W})$. *Then the* 0-th *polynomial* A *of* $T_1(\partial\tilde{W})$ *is of a form* $F\bar{F}$: $A\doteq F\bar{F}$ *for an element* F *in* Λ.

REMARK 3.2. For the special case that $\beta_*^{\gamma}(W)=0$, the torsions $\Delta(\tilde{W})$,

[**] $\gamma_i\colon \pi_1(X_i)\to\langle t_1, \cdots, t_\mu\rangle$ are *compatible epimorphisms*, if γ_i are the restrictions of a common epimorphism $\pi_1(Y)\to\langle t_1, \cdots, t^\mu\rangle$ to $\pi_1(X_i)$, respectively.

$\Delta(\partial\tilde{W})\in Q(\Lambda)-\{0\}/\langle t_1, \cdots, t_\mu\rangle$ may be defined and the conclusion of Theorem 3.1 also follows from the duality theorem for torsions due to J.W. Milnor [7], i.e., $\Delta(\partial\tilde{W})=\Delta(\tilde{W})\cdot\bar{\Delta}(\tilde{W})$.

Before proving Theorem 3.1, we shall prove Theorem B.

Proof of Theorem B. Let $F_1\cup\cdots\cup F_\mu\subset S^3\times[0, 1]$ be the cobordism annuli between the links $L_0\subset S^3$, $L_1\subset S^3$. Let N be the regular neighborhood of $F_1\cup\cdots\cup F_\mu$ in $S^3\times[0, 1]$ meeting the boundary $S^3\times 0\cup S^3\times 1$ regularly. Since each F_i is locally flat, it follows that N is homeomorphic to $F_1\times D^2\cup F_2\times D^2\cup\cdots\cup F_\mu\times D^2$, D^2 being a 2-cell. Let $W=S^3\times[0, 1]-\mathring{N}$ and $W\cap S^3\times i=X_i$, $i=0, 1$. Consider the specified epimorphisms $\gamma\colon \pi_1(W)\to\langle t_1, \cdots, t_\mu\rangle$ and $\gamma_i\colon \pi_1(X_i)\to\langle t_1, \cdots, t_\mu\rangle$, $i=0, 1$.

Now, consider the following diagram:

$$\begin{tikzcd} H_2(\tilde{W}) \arrow[r, "j_*"] \arrow[d] & H_2(\tilde{W}, \partial\tilde{W}) \arrow[r, "\partial"] & H_1(\partial\tilde{W}) \arrow[r, "i_*"] & H_1(\tilde{W}) \\ H_2(\tilde{W}, \tilde{X}_i) \arrow[ur] & & & \end{tikzcd}$$

Here, the row sequence is exact and the triangle is commutative.

By Lemma 2.1, $H_2(\tilde{W}, \tilde{X}_i)=T_2(\tilde{W}, \tilde{X}_i)$. Then, the above diagram implies that the sequence $T_2(\tilde{W}, \partial\tilde{W})\xrightarrow{\partial'} T_1(\partial\tilde{W})\xrightarrow{i_*} T_1(\tilde{W})$ is exact. Henc from Theroem 3.1, $A\doteq f\bar{f}$ for an element f in Λ, where A is the 0-th polynomial of $T_1(\partial\tilde{W})$. Notice that ∂W is obtained from X_0 and X_1 by pasting along the tori of the boundaries ∂X_0 and ∂X_1, and that the restriction epimorphism $\pi_1(\partial W)\to\langle t_1, \cdots, t_\mu\rangle$ of γ is determined by the epimorphisms $\gamma_0\colon \pi_1(X_0)\to\langle t_1, \cdots, t_\mu\rangle$ and $\gamma_1\colon \pi_1(X_1)\to\langle t_1, \cdots, t_\mu\rangle$.

Consider the following exact sequence (obtained from the Mayer-Vietories sequence),

$$\sum_{i=1}^{\mu}\Lambda/t_i-1 \to T_1(\tilde{X}_0)\oplus T_1(\tilde{X}_1) \to T_1(\partial\tilde{W}) \to \sum_{i=1}^{\mu}\Lambda/t_i-1$$

Let A_i, $i=0, 1$, be the Alexander polynomials of $\pi_1(X_i)$ with γ_i, and split $A_i=u_iv_i$, where $|v_i(1, \cdots, 1)|=1$ (yet u_i does not contain any non-unit factor f' with $|f'(1, \cdots, 1)|=1$). Also, split $f=f_uf_v$, where $|f_v(1, \cdots, 1)|=1$ (yet f_u does not contain any non-unit factor f'' with $|(f''(1, \cdots, 1)|=1)$.

From the sequence above, Lemma 2.4 and the reciprocity $A_i\doteq\bar{A}_i$ (see R.C. Blanchfield [1]), it follows that $v_0v_1=f_v\bar{f}_v$ and hence that there exist F_i in Λ, $i=0, 1$, with $|F_i(1, \cdots, 1)|=1$ and such that $v_0F_0\bar{F}_0\doteq v_1F_1\bar{F}_1$. Theorem 2.5 implies $u_0\doteq u_1$ and hence we have $A_0F_0\bar{F}_0=A_1F_1\bar{F}_1$. This completes the proof.

By using a similar argument in the proof of Theorem B, from Theorems 2.5 and 3.1 we also obtain the following:

Corollary 3.3. *Let M be a closed, connected and orientable 3-manifold with* *an epimorphism* $\gamma\colon \pi_1(M)\rightarrow\langle t_1, \cdots, t_\mu\rangle$. *The integer* $\beta^\gamma=(M)$ *and the Alexander polynomial* A_γ(*modulo* $F\bar{F}$-*form for* $F\in\Lambda$ *with* $|F(1, \cdots, 1)|=1$) *are the invariants of the homology cobordism of* M.

NOTATION. For a Λ-module $\mathfrak{M}$ let
$D(M)=\{x\in M|$ There exist coprime elements $\alpha_1, \cdots, \alpha_s$ in $\Lambda(s>1)$ with $\alpha_i x=0$, $i=1, 2, \cdots, s\}$ and $\hat{\mathfrak{M}}=\mathfrak{M}/D(\mathfrak{M})$.

Proof of Theorem 3.1. According to R.C. Blanchfield [1], there exist the (linking) pairings $V'\colon T_1(\partial\tilde{W})\times T_1(\partial\tilde{W})\rightarrow Q(\Lambda)/\Lambda$ and $V\colon T_2(\tilde{W}, \partial\tilde{W})\times T_1(\tilde{W})\rightarrow Q(\Lambda)/\Lambda$ and the induced pairings $\hat{V}'\colon \hat{T}_1(\partial\tilde{W})\times\hat{T}_1(\partial\tilde{W})\rightarrow Q(\Lambda)/\Lambda$ and $\hat{V}\colon \hat{T}_2(\tilde{W}, \partial\tilde{W})\times\hat{T}_1(\tilde{W})\rightarrow Q(\Lambda)/\Lambda$ are primitive. By the assumption, the sequence $0\rightarrow\operatorname{Im}\partial'\overset{\subset}{\rightarrow}T_1(\partial\tilde{W})\overset{i_*}{\rightarrow}\operatorname{Im}i_*\rightarrow 0$ is exact. Note that $V'(\partial'(y), x)=V(y, i_*(x))$ for all $y\in T_2(\tilde{W}, \partial\tilde{W})$ and $x\in T_1(\partial\tilde{W})$. Suppose for all $y'=\partial'(y)\in\operatorname{Im}\partial'$, $V'(y', x)=0$. This is equivalent to $V(y, i_*(x))=0$ for all $y\in T_2(\tilde{W}, \partial\tilde{W})$, since $V'(y', x)=V'(\partial'(y), x)=V(y, i_*(x))$. Using the primitive pairing $\hat{V}$, we obtain that $V'(y', x)=0$ for all $y'\in\operatorname{Im}\partial'$ is equivalent to $i_*(x)\in D(T_1(\tilde{W}))$ and hence $i_*(x)\in D(\operatorname{Im}i_*)$, i.e., $x\in i_*^{-1}(D(\operatorname{Im}i_*))$. Thus, the primitive pairing $\hat{V}'$ induces the primitive pairing $\hat{V}''\colon \hat{\operatorname{Im}}\,\partial\times[T_1(\partial\tilde{W})/i_*^{-1}(D(\operatorname{Im}i_*))]\rightarrow Q(\Lambda)/\Lambda$.

Let f, A and g be the 0-th polynomials of $\operatorname{Im}\partial'$, $T_1(\partial\tilde{W})$ and $\operatorname{Im}i_*$, respectively. By Lemma 2.4 we have $A\doteq fg$. By a result of R.C. Blanchfield ([1, Theorem 4.7]), f and g are also the 0-th polynomials of $\hat{\operatorname{Im}}\,\partial'$ and $\hat{\operatorname{Im}}\,i_*\approx T_1(\partial\tilde{W})/i_*^{-1}(D(\operatorname{Im}i_*))$, respectively. The primitive pairing $\hat{V}''$ asserts the equality $\bar{f}\doteq g$. (See [1, Theorem 4.5].) Therefore, $A\doteq fg\doteq f\bar{f}$. This completes the proof.

FINAL REMARK. Theorem B was independently proved by Y. Nakagawa slightly earlier than the present author, whose proof is based on the Fox's free calculus [3]. (cf. A. Kawauchi and Y. Nakagawa [6].)

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