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ON RINGS WITH SELF-INJECTIVE DIMENSION $\leqq 1$

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Let R be a ring with an identity and, for a left R-module $_RM$, $pd(M)$ and $id(M)$ denote the projective and injective dimension of $_RM$, respectively. A (left and right) noether ring R is called *n-Gorenstein* if $id(_RR)\leqq n$ and $id(R_R)\leqq n$ for $n\geqq 0$, and *Gorenstein* means n-Gorenstein for some n. This is slightly different from the well known definition in the commutative case unless a ring is local (see Bass [5]) and, as a generalization to the non-commutative case, there is another one by Auslander [1]. However, when we want to consider many interesting properties about a quasi-Frobenius ring and an hereditary ring in more general situation, we cannot conclude yet which definition is best. So, in this paper, we follow the above definition of a Gorenstein ring and try to generalize some interesting properties for a quasi-Frobenius ring. On the other hand, for a 1-Gorenstein ring, a few papers have appeared, for instance, Jans [12], Bass [4] and recently Sumioka [18], Sato [17] and, for a Gorenstein ring with squarezero radical, Zaks [19].

As the typical examples of 1-Gorenstein rings which are neither hereditary nor quasi-Forbenius, we have

1) Gorenstein orders, especially the group ring $\boldsymbol{Z}[G]$ where $\boldsymbol{Z}$ the ring of rational integers, G a finite group. (See Drozd-Kiricenko-Roiter [7], Roggenkamp [16] and Eilenberg-Nakayama [8].)

2) Triangular matrix rings over non-semisimple quasi-Frobenius rings. (See Sumioka [18] and Zaks [19].)

In §1, we shall show that for a 1-Gorenstein ring R, $E(_RR)\oplus E(_RR)/R$ is an injective cogenerator (Theorem 1) and as this corollary, an artin 1-Gorenstein ring which is $QF-1$ must be quasi-Frobenius (Corollary 3). This should compare with that for a quasi-Frobenius ring R, $_RR$ itself is an injective cogenerator. Next, as a generalization of "projectivity=injectivity" for modules over a quasi-Frobenius ring, we obtain that over a certain n-Gorenstein ring, finiteness of the projective dimension, projective dimension$\leqq n$, finiteness of the injective dimension and injective dimension$\leqq n$ for modules are all equivalent (Theorem 5).

In §2, first we attend to Nakayama's theorem [15] that a ring R is uniserial if and only if any homomorphic image of R is quasi-Frobenius, and replace "quasi-Frobenius" with "1-Gorenstein." Then we have three classes of rings, i.e. a uniserial ring, an hereditary ring with square-zero radical and a quasi-Frobenius ring with square-zero radical (Theorem 10). Moreover, as an application, we can classify a semiprimary ring whose proper homomorphic images are artin 1-Gorenstein (Theorem 12) and generalize [11, Theorem 1]. Also, in prime noether case, it will be shown that a restricted Gorenstein ring in the sense of Zaks [20] is equivalent to a restricted uniserial ring under certain hypothesis which always holds for commutative rings (Proposition 11).

Finally, Kaplansky's book [13] is suitable for looking at the recent development of commutative Gorenstein rings. In the present study about non-commutative Gorenstein rings, we should generalize the results described in [13] to the non-commutative case in appropriate form.

Notations. For a ring R and an R-module M, we denote
$n(R)$=the number of non-isomorphic simple left R-modules,
Rad R=the radical of R,
$\mathrm{Soc}({}_R R)$=the left socle of R,
$E(M)$=the injective hull of ${}_R M$,
$|M|$=the composition length of ${}_R M$.
A noether (artin) ring stands for left and right noetherian (artinian) and an ideal means twosided. Further, we say a non-zero ideal *twosided simple* if it contains no non-trivial ideal.

1. An injective cogenerator over a Gorenstein ring

In this section, first we consider which module is an injective cogenerator over a 1-Gorenstein ring, and next show the equivalence of the finiteness of projective dimension and injective dimension for modules over an n-Gorenstein ring which has a cogenerator with projective dimension$\leqq n$. These are well known for quasi-Frobenius rings, i.e. $n=0$.

Theorem 1. *Let R be a 1-Gorenstein ring, then $E({}_R R)\oplus E({}_R R)/R$ is an injective cogenerator.*

Proof. It is enough to show that any simple left R-module is monomorphic to $E({}_R R)\oplus E({}_R R)/R$. Otherwise, and suppose a simple left module S is not monomorphic to it, then

$$\mathrm{Hom}_R(S, R) = 0 = \mathrm{Ext}^1_R(S, R)\,.$$

Now represent S as

$$0 \to {}_R M \xrightarrow{i} {}_R R \to {}_R S \to 0$$

where M is a maximal left ideal and i is an inclusion map. If we denote $X^*=$

$\mathrm{Hom}_R(X, R)$ for an R-module X, we obtain an exact sequence:

$$S_R^* \to R_R^* \xrightarrow{i^*} M_R^* \to \mathrm{Ext}_R^1(S, R)$$

and so, by the assumption,

$$i^*\colon R_R^* \to M_R^* \text{ with } i^*(r^*) = (m \to mr) \qquad \text{for} \quad r \in R,\ m \in M$$

is an isomorphism. Hence

$$i^{**}\colon {}_RM^{**} \to {}_RR^{**} \simeq {}_RR \text{ with } i^{**}(f) = fi^*(1) \qquad \text{for} \quad f \in M^{**}$$

is an isomorphism, too. On the other hand, by Jans [12],

$$\sigma\colon {}_RM \to {}_RM^{**} \text{ with } \sigma(m) = (f \to f(m)) \qquad \text{for} \quad m \in M, f \in M^*$$

is also an isomorphism and therefore so is

$$i^{**}\sigma\colon {}_RM \to {}_RR\,.$$

However $i^{**}\sigma$ is an inclusion which contradicts $M \neq R$.

REMARK. In the theorem above, the assumption for R noetherian is necessary. For instance, let $R = \prod_\alpha K_\alpha$ be a direct product of infinitely many fields K_α, then R is self-injective but ${}_RR$ is not a cogenerator.

Next, we shall examine when only $E({}_RR)$ or $E({}_RR)/R$ is an injective cogenerator. A ring R is called a right S-ring if $E({}_RR)$ is a cogenerator and see Bass [3] or Morita [14] for details. In the latter case, we have the next result.

Corollary 2. *Let R be a 1-Gorenstein ring, then $E({}_RR)/R$ is a cogenerator if and only if $Soc({}_RR)=0$.*

Proof. "Only if": Suppose a simple left module S is monomorphic to ${}_RR$, then from the exact sequence

$$0 \to {}_RS \to {}_RR \to {}_RC \to 0\,,$$

we have an exact sequence

$$\mathrm{Ext}_R^1(R, R) \to \mathrm{Ext}_R^1(S, R) \to \mathrm{Ext}_R^2(C, R)\,.$$

Here, $\mathrm{Ext}_R^1(R, R)=0$ and $\mathrm{Ext}_R^2(C, R)=0$ since $id({}_RR)\leqq 1$, so $\mathrm{Ext}_R^1(S, R)=0$ which contradicts that $E({}_RR)/R$ is a cogenerator.

"If": Since $E_R(R) \oplus E({}_RR)/R$ is a cogenerator, for any simple left module ${}_RS$, S is either monomorphic to $E({}_RR)$ or $E({}_RR)/R$. However, from $\mathrm{Soc}({}_RR)=0$, ${}_RS$ must be monomorphic to $E({}_RR)/R$.

As an example of a ring R such that $E({}_RR)/R$ is a cogenerator, we obtain the following: Let R be an indecomposable semiprime 1-Gorenstein ring, then $E({}_RR)/R$ is a cogenerator unless R is artinian. More concretely, $R=\boldsymbol{Z}[G]$ is an example satisfying above assumption. Therefore Theorem 1 and Corollary 2 generalize Sato [17, Corollaries 3.3, 3.4 and Proposition 3.5].

As a second corollary of Theorem 1, we obtain a result about $QF-1$ rings. We recall a ring R is left $QF-1$ if every faithful R-module has the double centralizer property.

Corollary 3. *Let R be an artin 1-Gorenstein ring. If R is its own maximal left quotient ring, R is quasi-Frobenius. Hence an artin 1-Gorenstein ring which is left $QF-1$ is quasi-Frobenius.*

Proof. Since R is its own maximal left quotient ring, $E({}_RR)/R$ is monomorphic to a direct product of copies of $E({}_RR)$ and so $E({}_RR)$ is a cogenerator and, for any simple left module ${}_RS$, we have an exact sequence:

$$0 \to {}_RS \to {}_RR \to {}_RC \to 0\,,$$

which induces $\mathrm{Ext}^1_R(S, R)=0$ similarly to the proof of Corollary 2. Therefore ${}_RR$ is injective, i.e. R is quasi-Frobenius.

If R is left $QF-1$, $E({}_RR)$ has the double centralizer property and hence R is its own maximal left quotient by Lambek's result.

REMARK. Now, we have a further investigation about $QF-1$ rings, that is, we consider hereditary $QF-1$ rings. We have the following: "*A left non-singular left $QF-1$ ring is semisimple (artinian).*" In fact, if R is left non-singular, its maximal left quotient ring Q is semiprimitive. Furthermore, if R is left $QF-1$, $Q\simeq R$ by Lambek's result and hence R is semisimple by Camillo [6, Proposition 5].

As a consequence, *for a ring R the following are equivalent:*

(1) *R is left hereditary left $QF-1$,*

(2) *R is right hereditary right $QF-1$,*

(3) *R is semisimple (artinian).*

To investigate the latter problem in the beginning of this section, we require the next lemma.

Lemma 4. *For an exact sequence of modules over a ring R:*

$$0 \to {}_RA \to {}_RB \to {}_RC \to 0\,,$$

(1) *$id(A)$, $id(B)\leqq n$ implies $id(C)\leqq n$;*

(2) *$pd(B)$, $pd(C)\leqq n$ implies $pd(A)\leqq n$.*

Proof. (1) For any R-module ${}_RX$, we have

$$\mathrm{Ext}_R^{n+1}(X, B) \to \mathrm{Ext}_R^{n+1}(X, C) \to \mathrm{Ext}_R^{n+2}(X, A) \text{ (exact).}$$

Now, $\mathrm{Ext}_R^{n+1}(X, B)=\mathrm{Ext}_R^{n+2}(X, A)$ by the assumption, so $\mathrm{Ext}_R^{n+1}(X, C)=0$, i.e. $id(C)\leqq n$.

(2) is dual to (1)

Theorem 5. *Let R be an artin n-Gorenstein ring and suppose there exists a cogenerator ${}_RW$ with $pd(W)\leqq n$. Then the following are equivalent for a left R-module ${}_RM$:*

$$(1)\ pd(M)<\infty,\quad (2)\ pd(M)\leqq n,\quad (3)\ id(M)<\infty,\quad (4)\ id(M)\leqq n.$$

Proof. (1)$\to$(2): Say $pd(M)=m<\infty$, there is a left module ${}_RX$ such that $\mathrm{Ext}_R^m(M, X)\neq 0$. Represent X as

$$0 \to {}_RK \to {}_RF \to {}_RX \to 0 \text{ (exact), } {}_RF \text{ free}$$

then this induces

$$\mathrm{Ext}_R^m(M, F) \to \mathrm{Ext}_R^m(M, X) \to \mathrm{Ext}_R^{m+1}(M, K) \text{ (exact)}.$$

Hence, $\mathrm{Ext}_R^{m+1}(M, R)=0$ implies $\mathrm{Ext}_R^m(M, F)\neq 0$, from which we have $id(F)\geqq m$. Now, $id(F)=id(R)\leqq n$ and hence $pd(M)=m\leqq n$.

(2)$\to$(3): Let

$$0 \to P_n \xrightarrow{f_n} P_{n-1} \to \cdots \to P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$$

be a projective resolution of M and $K_i=\mathrm{Ker}(f_i)$ $0\leqq i\leqq n-1$, $K_{-1}=M$, then first in an exact sequence:

$$0 \to P_n \to P_{n-1} \to K_{n-2} \to 0,$$

$id(P_n)$, $id(P_{n-1})\leqq id({}_RR)\leqq n$ implies $id(K_{n-2})\leqq n$ by Lemma 4 (1). For general i, in an exact sequence:

$$0 \to K_i \to P_i \to K_{i-1} \to 0,$$

if $id(K_i)\leqq n$, then $id(K_{i-1})\leqq n$ again by Lemma 4 (1). Therefore by the induction, $id(M)=id(K_{-1})\leqq n$.

(3)$\to$(4): Say $id(M)=m<\infty$, then there is a left module ${}_RX$ such that $\mathrm{Ext}_R^m(X, M)\neq 0$. Let

$$0 \to {}_RX \to {}_RE \to {}_RC \to 0 \text{ with } {}_RE \text{ injective}$$

be an injective presentation of X, then we have $\mathrm{Ext}_R^m(E, M)\neq 0$ from an exact sequence;

$$\mathrm{Ext}_R^m(M, E) \to \mathrm{Ext}_R^m(X, M) \to \mathrm{Ext}_R^{m+1}(C, M)$$

and so $pd(E)\geqq m$. On the one hand, as E is isomorphic to a direct summand of a direct product $\prod W$ of copies of ${}_RW$, $pd(E)\leqq pd(\prod W)=pd(W)\leqq n$ whence $id(M)=m\leqq n$.

(4)$\rightarrow$(1): Let

$$0\rightarrow M\xrightarrow{f_0}E_0\xrightarrow{f_1}E_1\rightarrow\cdots\rightarrow E_{n-1}\xrightarrow{f_n}E_n\rightarrow 0$$

be an injective resolution of ${}_RM$ and $C_i=\mathrm{Cok}(f_i)$ $0\leqq i\leqq n-1$, $C_{-1}=M$, then an exact sequence:

$$0\rightarrow C_{n-2}\rightarrow E_{n-1}\rightarrow E_n\rightarrow 0$$

and $pd(E_{n-1})$, $pd(E_n)\leqq pd(W)\leqq n$ imply $pd(C_{n-2})\leqq n$ by Lemma 4 (2). By the same discussion as the proof (2)$\rightarrow$(3), we obtain $pd(M)\leqq n$.

As a corollary of Theorems 1 and 5 we have the following where we recall a ring R is left $QF-3$ if $E({}_RR)$ is projective.

Corollary 6. *Let R be a 1-Gorenstein ring which is left $QF-3$, then the following are equivalent for a left R-module M:*

(1) $pd(M)<\infty$, (2) $pd(M)\leqq 1$, (3) $id(M)<\infty$, (4) $id(M)\leqq 1$.

Proof. By Theorem 1, ${}_RW=E({}_RR)\oplus E({}_RR)/R$ is a cogenerator with $pd(W)\leqq 1$ because

$$0\rightarrow {}_RR\xrightarrow{j}E({}_RR)\oplus E({}_RR)\rightarrow {}_RW\rightarrow 0$$

with $j(x)=(0, x)$ for $x\in R$ is a projective resolution of ${}_RW$. Further, it is well known a noetherian left $QF-3$ ring is artinian, so we may apply Theorem 5 in case $n=1$.

REMARK. (1) For any $n>0$, there exists a non-quasi-Frobenius ring satisfying the hypothesis in Theorem 5. For instance, let R be a serial (=generalized uniserial) ring with admissible sequence: 1, 2, $\cdots$, 2 (2 are n times), then $id({}_RR)=id(R_R)=gl.dim\,R=n$ and ${}_RW=\coprod_{i=0}^{n}E_i$ is an injective cogenerator with $pd(W)=n$ where $0\rightarrow{}_RR\rightarrow E_0\rightarrow E_1\rightarrow\cdots\rightarrow E_n\rightarrow 0$ is the minimal injective resolution of ${}_RR$. (See [10] for details of serial rings.)

More generally, an n-Gorenstein ring R with $dom.dim\,R\geqq n$ has an injective cogeneartor ${}_RW=\coprod_{i=0}^{n}E_i$ with $pd(W)\leqq n$ where $0\rightarrow{}_RR\rightarrow\{E_i;\ 0\leqq i\leqq n\}$ is the minimal injective resolution.

(2) We may construct an n-Gorenstein ring R_n with $gl{\cdot}dim\,R_n=\infty$ for any $n\geqq 0$ in the following way. Let R_0 be a non-semisimple quasi-Frobenius ring, and for any $n>0$, R_n the triangular matrix ring over R_{n-1}, i.e. $R_n=\begin{pmatrix}R_{n-1} & 0\\ R_{n-1} & R_{n-1}\end{pmatrix}$.

2. Rings whose homomorphic images are Gorenstein

In [19, §2], Zaks showed that, for a semiprimary ring R with square-zero radical, $id({}_RR)\leqq 1$ if and only if R is a direct product of a quasi-Frobenius ring and an hereditary ring, and hence $id({}_RR)\leqq 1$ is equivalent to $id(R_R)\leqq 1$. Such a decomposition theorem no longer holds unless the square of its radical is zero. For example, let Q be a local quasi-Frobenius ring with $(\mathrm{Rad}\,Q)^2=0$ and R the triangular matrix ring over Q, then R is artin 1-Gorenstein and indecomposable but is neither quasi-Frobenius nor hereditary.

Now, for a serial ring, we have a decomposition theorem as above.

Proposition 7. *Let R be a serial ring, then the following are equivalent:*
(1) $id(R_R)\leqq 1$,
(2) $id({}_RR)\leqq 1$,
(3) *R is a direct product of a quasi-Frobenius ring and a hereditary ring.*

Proof. Without loss of generality, we may assume that R is self-basic (twosided) indecomposable, and decompose ${}_RR$ as $R=Re_1\oplus\cdots\oplus Re_n$ such that $\{e_1, \cdots, e_n\}$ is the Kupisch series. If R is not quasi-Frobenius, Re_i is non-injective for some i $(1\leqq i\leqq n)$ and then, from $|Re_{j+1}|\leqq|Re_j|+1$ for $1\leqq j<n$, we obtain that if $i<n$, $|Re_{i+1}|=|Re_i|+1$, Re_i is monomorphic to Re_{i+1} and $E(Re_i)\simeq Re_j$ for some j $(i<j\leqq n)$ by [10, 1.1]. Now, let the number i be the smallest one with Re_i non-injective and Re_{i+1} injective. Here, we may suppose $i<n$ because, in case of $Ne_1=0$, Re_1 is monomorphic to Re_2 and if $Ne_1\neq 0$, by permuting $\{1, \cdots, n\}$, it is possible for Re_1 to be non-injective and Re_2 injective. Therefore we have

$$E(Re_i)\simeq Re_{i+1} \quad\text{and}\quad |Re_i|+1=|Re_{i+1}|.$$

So, saying $N=\mathrm{Rad}\,R$,

$$E(Re_i)/Re_i\simeq Re_{i+1}/Ne_{i+1}$$

is simple injective and from that Re_{i+1} is epimorphic to Ne_{i+2} if $i+1<n$,

$$Re_{i+1}/Ne_{i+1}\simeq Ne_{i+2}/N^2e_{i+2}\subsetneqq Re_{i+2}/N^2e_{i+2}$$

induces $Ne_{i+2}=0$ since Re_{i+2}/N^2e_{i+2} is indecomposable. This contradicts $|Re_j|\geqq 2$ for $j=2, \cdots, n$ and so $i+1=n$ and $|Re_{i+1}|=|Re_i|+1$ for $1\leqq i\leqq n$. Hence $Re_2\simeq Ne_{i+1}$ for $1\leqq i\leqq n-1$, i.e. Ne_i $(i=2, \cdots, n)$ are projective and R is hereditary.

Applying this proposition we classify the rings all of which homomorphic images are artin 1-Gorenstein. Before proceeding, we need two lemmas.

Lemma 8 (Bass [3]). *For a right perfect, right S-ring R, id $({}_RR)$ is finite if and only if ${}_RR$ is injective.*

Proof. Say, $id(R)=n<\infty$, then there exists a simple left module ${}_RS$ with $\mathrm{Ext}_R^n(S, R)\neq 0$. Now, since R is a right S-ring, we have an exact sequence:

$$0 \to {}_RS \to {}_RR \to {}_RC \to \to 0$$

which induces

$$\mathrm{Ext}_R^n(R, R) \to \mathrm{Ext}_R^n(S, R) \to \mathrm{Ext}_R^{n+1}(C, R) \text{ (exact)}.$$

Here, $\mathrm{Ext}_R^{n+1}(C, R)=0$ from $id({}_RR)=n$, so $\mathrm{Ext}_R^n(R, R)\neq 0$ and $n=0$, i.e. ${}_RR$ is injective.

Lemma 9. *Let I be a (twosided) ideal in any ring R and R/I^n a left hereditary ring for some $n>1$. Then $I^n=I^{n+1}$. Hence, if we assume ${}_RN=Rad\,R$ is finitely generated (or nilpotent) and R/N^n is left hereditary for $n>1$, then $N^n=0$ and so R itself left hereditary.*

Proof. Since I^{n-1}/I^n is an ideal in R/I^n, it is R/I^n-projective and the exact sequence of R/I^n-modules:

$$0 \to I^n/I^{n+1} \to I^{n-1}/I^{n+1} \to I^{n-1}/I^n \to 0$$

splits, i.e.

$$I^{n-1}/I^{n+1} \simeq I^{n-1}/I^n \oplus I^n/I^{n+1}$$

as R/I^n-modules. However, $I\cdot(I^{n-1}/I^n\oplus I^n/I^{n+1})=0$, so $I\cdot(I^{n-1}/I^{n+1})=0$, i.e. $I^n=I^{n+1}$.

Theorem 10. *For an indecomposable semiprimary ring R, the following are equivalent:*

(1) *For any homomorphic image T of R, $id({}_TT)\leqq 1$,*
(2) *For any homomorphic image T of R, $id(T_T)\leqq 1$,*
(3) *R is one of the following;*
(i) *R is uniserial,*
(ii) *R is hereditary with $(Rad\,R)^2=0$,*
(iii) *R is quasi-Frobenius with $(Rad\,R)^2=0$ and $n(R)=2$.*

Proof. (3) is left-right symmetry, so we prove only the equivalence of (1) and (3).

(1)$\to$(3): Say, $N=\mathrm{Rad}\,R$, since R/N^2 is also indecomposable, R/N^2 is either hereditary or quasi-Frobenius by Zaks [19]. In case of hereditary, $N^2=0$ by Lemma 9 and hence R is of type (ii). In another case, R/N^2 is a serial ring, so R is artinian and serial, too whence R is either hereditary or quasi-Frobenius by Proposition 7. If R is hereditary, $gl.dim\,R/N^2<\infty$ by Eilenberg-Nagao-Nakayama [9, Theorem 8] and hence by Bass [4, Proposition 4.3], $gl.dim\,R/N^2=id({}_{R/N^2}R/N^2)\leqq 1$, i.e. R/N^2 is hereditary, so $N^2=0$ and R is hereditary again by Lemma 9.

Thus, let R be serial quasi-Frobenius and $n(R)=n(R/N^2)=n$. Further, $\bar{R}=R/N^2$ also satisfies (1) and since (1) is Morita-invariant, we may assume $\bar{R}$ is self-basic and decompose $\bar{R}$ as $\bar{R}=\bar{R}e_1\oplus\cdots\oplus\bar{R}e_n$ with $\{e_1, \cdots, e_n\}$ Kupisch series. If $n>2$, $Je_1=e_nJe_1$ ($J=\text{Rad }\bar{R}$) is an ideal of $\bar{R}$ and the ring:

$$T=\bar{R}/Je_1=T\bar{e}_1\oplus\cdots\oplus T\bar{e}_n \quad \text{where} \quad \bar{e}_i=e_i+Je_1\in T$$

satisfies $id({}_TT)\leqq 1$. Hence, from $Je_2\simeq Re_1/Je_1$,

$$E(T\bar{e}_1)/T\bar{e}_1\simeq T\bar{e}_2/\bar{J}\bar{e}_2 \qquad (\bar{J}=\text{Rad }T)$$

is T-injective. However, $\bar{e}_2\bar{J}\bar{e}_3\neq 0$, i.e. $T\bar{e}_2/\bar{J}\bar{e}_2\simeq\bar{J}\bar{e}_3\subsetneqq T\bar{e}_3$ which contradicts the indecomposability of $T\bar{e}_3$, so $n\leqq 2$. Then, since R is uniserial if $n=1$, let $n=2$, i.e. we may represent $R=Re_1\oplus Re_2$ with $\{e_1, e_2\}$ Kupisch series because R is self-basic, too. Furthermore, if $N^2\neq 0$, then N^2e_1 and $N^2e_2\neq 0$ as R is quasi-Frobenius and the homomorphic image $T=R/(N^3e_1\oplus N^2e_2)=T\bar{e}_1\oplus T\bar{e}_2$ where $\bar{e}_i=e_i+(N^3e_1\oplus N^2e_2)\in T$ satisfies $id({}_TT)\leqq 1$. Now, from $E(T\bar{e}_2)\simeq T\bar{e}_1$,

$$E(T\bar{e}_2)/T\bar{e}_2\simeq T\bar{e}_1/J\bar{e}_1 \qquad (J=\text{Rad }T)$$

is T-injective. However,

$$J^2\bar{e}_1\simeq N^2e_1/N^3e_1\simeq Re_1/Ne_1\simeq T\bar{e}_1/J\bar{e}_1$$

is T-injective which contradicts that ${}_TT\bar{e}_1$ is indecomposable. Hence $N^2=0$.

(3)$\rightarrow$(1): In any case of (i)—(iii), R may be assumed self-basic. It is well known that a uniserial ring is characterized as a ring all of which homomorphic images are quasi-Frobenius.

Let R be of type (ii). For any ideal I contained in N, since ${}_RI$ is a direct summand of ${}_RN$, R/I is also hereditary by Eilenberg-Nagao-Nakayama [9, Proposition 9]. If I is not contained in N, I contains a primitive idempotent e_1 with $I=Re_1\oplus(I\cap R(1-e_1))$ and further, if $I\cap R(1-e_1)\nsubseteq N$, choose a primitive idempotent e_2 orthogonal to e_1 in $I\cap R(1-e_1)$. By repeating this method, we have

$$I=Re_1\oplus\cdots\oplus Re_n\oplus I'$$

where $e_i^2=e_i$ is primitive and $I'=I\cap R(1-\sum_{i=1}^n e_i)\subseteq N$. Then, let $e=1-(e_1+\cdots+e_n)$, from I', $eR(1-e)\subseteq N$,

$$I'R=I'eRe+I'eR(1-e)\subseteq I\cap Re=I'\,,$$

i.e. I' is an ideal. Hence $T'=R/I'$ is an hereditary ring with

$${}_{T'}\text{Rad }T'=N/I'\simeq{}_{T'}Ne\oplus{}_{T'}N(1-e)/I'$$

and so $N(1-e)/I'$ is T'-projective. On the other hand,

$$T = R/I \simeq R(1-e)/I'$$

implies Rad $T=N(1-e)/I'$ and, as T' is epimorphic to T, $N(1-e)/I'$ is T-projective, i.e. T is hereditary.

Let R be of type (iii) and $R=Re_1\oplus Re_2$ where $\{e_1, e_2\}$ Kupisch series. For any ideal I contained in N, I is a direct summand and, as $N=Ne_1\oplus Ne_2$ with Ne_i simple, ${}_RI$ is isomorphic to Ne_1 or Ne_2 provided $I\neq 0$, N. If $I\simeq Ne_1$,

$${}_RI \simeq {}_RNe_1 = e_2Ne_1 \simeq e_2I$$

implies $I=e_2I$ and so, saying $N=I\oplus K$,

$$e_2I\oplus e_2K = e_2(I\oplus K) = e_2N = e_2Ne_1\,.$$

Hence

$$I = e_2I = e_2Ne_1 = e_2N = Ne_1$$

and

$${}_TT = R/I \simeq {}_TRe_1/Ne_1\oplus {}_TRe_2$$

which induces ${}_TRe_1/Ne_1$ projective. Now, let $J=$Rad T,

$${}_TRe_1/Ne_1 \simeq {}_TNe_2 \simeq {}_TJ(e_2+I) = J\,,$$

so ${}_TJ$ is projective and T is hereditary. In case of $I\simeq Ne_2$, we have the same discussion. Next, let $e_1\in I$, then

$$2 = |Re_1| \leqq |{}_RI| \leqq |{}_RR| = 4\,.$$

However, $|{}_RI|=2$ implies $I=Re_1$ and $Ne_2\subseteqq Re_1R\subseteqq Re_1$ which is a contradiction. Therefore, we may take $|{}_RI|=3$ and then $|{}_RR/I|=1$, i.e. R/I is a division ring. This completes the proof.

Finally, we investigate a ring whose proper homomorphic images are artin 1-Gorenstein, and here consider in two cases of a prime noether ring and a semiprimary ring.

For a prime noether case, we have a generalization of Zaks [20, Theorem 3]. Here an ideal I is said to have the *Artin-Rees property* if for every left ideal L, there is an n with $I^n\cap L\subseteqq IL$.

Proposition 11. *Let R be a prime noether ring and assume every maximal ideal in R has the Artin-Rees property. Then any proper homomorphic image of R is artin Gorenstein if and only if R is restricted uniserial.*

Proof. "Only if": For any maximal ideal M in R, $M=0$ implies R a simple ring, so we may suppose $M\neq 0$. Then R/M^2 is primary Gorenstein and hence quasi-Frobenius (in this case, uniserial) by Lemma 8. Thus let

$n>2$, $T=R/M^n$ and $J=\mathrm{Rad}\ T$, then $T/J^2 \simeq R/M^2$ is uniserial which implies $T=R/M^n$ $(n>2)$ uniserial.

Next, for any nonzero ideal I in R, there exist maximal ideals $M_1, \cdots, M_n$ in R with $M_1, \cdots, M_n \supseteq I$. Since $M_1, \cdots, M_n$ have the Artin-Rees property, there are integers $k_1, \cdots, k_n$ such that

$$M_1^{k_1} \cap \cdots \cap M_n^{k_n} \subseteq M_1 \cdots M_n \subseteq I .$$

Hence, we may suppose all $M_1, \cdots, M_n$ are distinct and, by the Chinese Remainder Theorem,

$$R/(M_1^{k_1} \cap \cdots \cap M_n^{k_n}) \simeq R/M_1^{k_1} \oplus \cdots \oplus R/M_n^{k_n}$$

is uniserial. On the other hand, $R/(M_1^{k_1} \cap \cdots \cap M_n^{k_n})$ is epimorphic to R/I, so R/I is uniserial too.

Now, we state the last theorem which is of a semiprimary case.

Theorem 12. *Let R be an indecomposable semiprimary ring and R_0 the basic subring of R with $N=Rad\ R_0$. Then any proper homomorphic image of R is* 1*-Gorenstein if and only if R is one of the following:*

(1) *R is uniserial;*

(2) *R is serial with admissible sequence* 3, 2;

(3) *R is hereditary with square-zero radical;*

(4) $n(R) \leqq 2$, $(Rad\ R)^2=0$ *and for any primitive idempotent e in R_0,* (*a*) $eNe=0$ *provided* $e \neq 1$, (*b*) *If Ne contains a nonzero ideal properly, it is a maximal left and right subideal in Ne and $N(1-e)$ is a simple left and right ideal of R_0;*

(5) $n(R)=2$, $(Rad\ R)^2=0$ *and R_0 has a primitive idempotent e such that* (*a*) *eNe is simple left and right ideal of R_0,* (*b*) *Either* $(1-e)Ne=0$ *or* $N(1-e)=0$, (*c*) *Each of* $(1-e)Ne$ *and* $N(1-e)$ *is twosided simple unless it is zero and* $N(1-e)=eN(1-e)$;

(6) *R is triangular with $n(R)=3$, $(Rad\ R)^2=0$ and Ne is twosided simple for a primitive idempotent e in R_0 provided $Ne \neq 0$.*

Proof. Throughout the proof, we may assume R self-basic and then $N=\mathrm{Rad}\ R$.

"Only if." If $N^3 \neq 0$, R/N^3 is uniserial by Theorem 10 and so is R by [15].

Let $N^3=0$ but $N^2 \neq 0$, then R/N^2 is quasi-Frobenius with $n(R/N^2)=2$ again by Theorem 10 and Lemma 9 and hence R is serial with $n(R)=2$. Thus, let $\{e_1, e_2\}$ be a Kupisch series, then $Ne_1 \neq 0$. For, $Ne_1=0$ implies $N^2=0$ (contradiction) because Re_1 is epimorphic to Ne_2. So $Ne_1 \neq 0$ and Re_2 is epimorphic to Ne_1. If both N^2e_1 and N^2e_2 are nonzero, R/N^2e_1 is neither hereditary since Ne_1/N^2e_1 is not projective nor quasi-Frobenius since R/N^2e_1 has non-constant admissible sequence 2, 3. Therefore

$$N^2e_1 \neq 0\,, \quad N^2e_2 = 0 \quad \text{or} \quad N^2e_1 = 0\,, \quad N^2e_2 \neq 0\,.$$

In either case, R has the admissible sequence 2, 3; i.e. R is of type (2).

In the following, we may assume $N^2=0$, $N\neq 0$ and R not hereditary because otherwise R is of type (3). Here, we remark that for a semiprimary ring R with square-zero radical N, R is hereditary if and only if any primitive idempotent e in R satisfies either $eN=0$ or $Ne=0$. Now, if $n(R)=1$, i.e. R is local and N contains a nonzero ideal $I\neq N$, R/I must be quasi-Frobenius. Hence ${}_RN/I$, N/I_R are simple and R is of type (4).

Therefore, now suppose $n(R)=2$, then there exists a primitive idempotent e with $eN\neq 0$, $Ne\neq 0$ and $1-e$ is primitive too. In case of $eNe\neq 0$, $I=(1-e)Ne\oplus N(1-e)\neq 0$ since R is indecomposable and $R/I\simeq eRe\oplus(1-e)R(1-e)$ as rings implies that eRe is quasi-Frobenius, so ${}_ReNe$, eNe_R are simple. Next, if both $(1-e)Ne$ and $N(1-e)$ were nonzero, $R/N(1-e)$ is indecomposable but neither hereditary nor quasi-Frobenius. Hence either $(1-e)Ne=0$ or $N(1-e)=0$ and each of them is twosided simple unless it is zero. Further, $N(1-e)=eN(1-e)$ because R is indecomposable. These show that R is of type (5) in case of $eNe\neq 0$. So we assume $eNe=0$, in which case $eN(1-e)\neq 0$ as e was chosen with $eN\neq 0$. Then $R/eN(1-e)$ must be hereditary and $(1-e)N(1-e)=0$. Here, if Ne contains properly a nonzero ideal I, R/I has to be quasi-Frobenius whence both ${}_RN(1-e)=eN(1-e)$ and ${}_RNe/I$ are simple. These also hold for a right side. On the one hand, if $N(1-e)$ contains properly a nonzero ideal I, by exchanging the idempotent e with $1-e$, the same argument as above holds. Hence R becomes of type (4).

Finally, suppose $n(R)\geqq 3$. As ${}_RN$ is not projective, there are primitive idempotents e, f with $fNe\neq 0$ and $Nf\neq 0$. Now, assume $(1-e)Ne=0$, then eNe is a nonzero ideal, $n(R/eNe)=n(R)\geqq 3$ and R/eNe is indecomposable, so R/eNe must be hereditary by Theorem 10. Therefore there exists a primitive idempotent $e'\neq e$ with $eNe'\neq 0$ by an indecomposability of R and then $I=(1-e)Ne'+N(1-e-e')$ is a nonzero ideal since R is indecomposable and $n(R)\geqq 3$. If we put $\bar{R}=R/I$, $\bar{e}=e+I$ and $\bar{e}'=e'+I$, $\bar{R}\bar{e}\oplus\bar{R}\bar{e}'$ is a block of R and not any of the ring stated in Theorem 10 (contradiction). Thus $(1-e)Ne\neq 0$, i.e. $f\neq e$ and, by setting $e_1=e$, $e_2=f$, R is expressible as $R=Re_1\oplus Re_2\oplus\cdots\oplus Re_n$ where $n=n(R)\geqq 3$, e_i $(1\leqq i\leqq n)$ are primitive idempotents and either $e_2Ne_3\neq 0$ or $e_3Ne_2\neq 0$. If an ideal $I=(1-e_2)Ne_1+(1-e_1-e_3)Ne_2+(1-e_2)Ne_3+\sum_{j>3}Re_j$ is nonzero, then R/I must be hereditary by Theorem 10 as R/I is indecomposable and $n(R/I)=3$, and so we obtain that $Ne_1=e_2Ne_1+Ie_1$, $e_1Ne_2=0=e_3Ne_2$ and $Ne_3=e_2Ne_3+Ie_3\neq 0$. In this case $R/\sum_{j\geqq 3}Ne_j$ has to be quasi-Frobenius, which contradicts $e_1Ne_2=0$. Hence $I=0$ implies $n=3$, $Ne_1=e_2Ne_1\neq 0$, $Ne_2=e_1Ne_2+e_3Ne_2\neq 0$ and $Ne_3=e_2Ne_3$. Moreover, if $Ne_3\neq 0$, $e_1Ne_2=0=e_3Ne_2$ for R/Ne_1 or R/Ne_3 is indecomposable but neither hereditary nor quasi-Frobenius according to $e_1Ne_2 \neq 0$ or $e_3Ne_2 \neq 0$, but it contradicts $Ne_2 \neq 0$. Therefore $Ne_3 = 0$ and $e_3Ne_2 \neq 0$ induces $e_1Ne_2 = 0$ since gl. dim $R/e_1Ne_2 = 2$, i.e. R is of type (6).

"If." Case (1): By Nakayama [15], R is uniserial if and only if any homomorphic image of R is quasi-Frobenius.

Case (2): Let $R = Re_1 \oplus Re_2$ where e_1, e_2 are primitive idempotents and $|Re_1| = 3$, $|Re_2| = 2$. Then, for any nonzero proper ideal I in R,

$$0 \neq I \cap \mathrm{Soc}({}_RR) = I \cap (N^2e_1 \oplus Ne_2) = (I \cap N^2e_1) \oplus (I \cap Ne_2)$$

implies either $I \cap N^2e_1 \neq 0$ or $I \cap Ne_2 \neq 0$. In either case, we obtain $N^2e_1 \subseteq I$. Now, suppose $N^2e_1 = I$, then R/I is quasi-Frobenius with the admissible sequence 2,2. Next, if $N^2e_1 \neq I$, R/I is a proper homomorphic image of R/N^2e_1 and hence has the admissible sequence $\{1, 2\}$, $\{1, 1\}$ or $\{1\}$. In all cases, R/I is hereditary.

Case (3): Any homomorphic image of R is hereditary by [9, Proposition 9].

Case (4): For any nonzero ideal I of R, if $I \subseteq N$, $I = Ie \oplus I(1-e)$ with Ie, $I(1-e)$ ideals for a primitive idempotent e and $R/I \simeq Re/Ie \oplus R(1-e)/I(1-e)$ is either hereditary or quasi-Frobenius by the property (b). If $I \subseteq\!\!\!\!/\ N$, I contains a primitive idempotent e and so R/I is isomorphic to $(1-e)R(1-e)$ or 0.

Case (5): For any nonzero ideal I of R, if $I \subseteq N$, $I = eIe \oplus (1-e)Ie \oplus I(1-e)$ and these summands are all ideals. By the property (b), in case of $(1-e)Ne = 0$, $R/I \simeq Re/eIe \oplus R(1-e)/I(1-e)$ implies that R/I is hereditary or quasi-Frobenius according to $eIe \neq 0$ or $I(1-e) \neq 0$. In case of $N(1-e) = 0$, $R/I \simeq Re/I \oplus R(1-e)$ shows that R/I is quasi-Frobenius (resp. hereditary) provided $(1-e)Ie \neq 0$ (resp. $eIe \neq 0$). Next, if I is not contained in N, e or $1-e$ belongs to I and so $I = Re \oplus (I \cap R(1-e))$ or $I = (I \cap Re) \oplus R(1-e)$ respectively. In the former case, we may assume $I \cap R(1-e) \subseteq N$ and hence $R/I \simeq (1-e)R(1-e)/(1-e)N(1-e)$ is a division ring. Also, in the latter case, we have the same conclusion.

Case (6): R has a complete set e_1, e_2, e_3 of mutually orthogonal primitive idempotents satisfying $e_iNe_j = 0$ if $i \leqq j$. Hence, for any nonzero ideal I of R, if $I \subseteq N$, $I = Ie_1 \oplus Ie_2$ with Ie_1, Ie_2 ideals and $R/I \simeq Re_1/Ie_1 \oplus Re_2/Ie_2 \oplus Re_3$ is hereditary since $Ie_i = Ne_i$ or 0 $(i = 1, 2)$. If $I \subseteq\!\!\!\!/\ N$, some e_i for $i = 1, 2, 3$ is contained in I and we may show similarly that R/I is hereditary.

REMARK. In [20], Zaks showed that, for a commutative noether ring R, any (proper) homomorphic image of R is Gorenstein if and only if any (proper) homomorphic image of R is quasi-Frobenius. For a non-commutative case, however, we see it no longer holds by Theorems 10 and 12. In prime noether case (see Proposition 11), we don't know whether the hypothesis of the Artin-Ress property is superfluous or not.

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