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DEGENERATE ELLIPTIC SYSTEMS OF PSEUDO-DIFFERENTIAL EQUATIONS AND NON-COERCIVE BOUNDARY VALUE PROBLEMS

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0. Introduction

In the present paper we shall study a class of degenerate elliptic systems of pseudo-differential equations, and apply the results obtained there to non-coercive boundary value problems of fourth order.

One of typical examples of non-coercive problems is the oblique derivative problem: Let Ω be a bounded open set in $\boldsymbol{R}^n$ with a smooth boundary Γ and consider the problem

$$(0.1) \qquad \begin{cases} A(x, D_x)u = f \ \text{ in } \ \Omega\,, \\ \dfrac{\partial u}{\partial \nu} = g \ \text{ on } \ \Gamma\,, \end{cases}$$

where $A(x, D_x)$ is an elliptic differential operator of second order on $\bar{\Omega}$ and ν is a non-vanishing real vector field tangent to Γ on its submanifold Γ_0. The behavior of ν near Γ_0 has a crucial effect on this problem (for details, see [5], [14], etc.). We shall consider in §4 a similar problem for an elliptic operator $L(x, D_x)$ of fourth order on $\bar{\Omega}$:

$$(0.2) \qquad \begin{cases} L(x, D_x)u = f \ \text{ in } \ \Omega\,, \\ \dfrac{\partial^2 u}{\partial \nu_1 \partial n} = g_1 \ \text{ on } \ \Gamma\,, \\ \dfrac{\partial u}{\partial \nu_2} = g_2 \ \text{ on } \ \Gamma\,, \end{cases}$$

where ν_1, ν_2 are vector fields of the same type as in (0.1). We study this problem by a usual method. Namely, let $\mathcal{P}$ be the Poisson operator of the Dirichlet problem

$$\begin{cases} L(x, D_x)u = f \ \text{ in } \ \Omega\,, \\ D_n u = h_1 \ \text{ on } \ \Gamma\,, \\ u = h_2 \ \text{ on } \ \Gamma\,, \end{cases}$$

where D_n denotes the normal derivative $-i\frac{\partial}{\partial n}$ on Γ. Then the mapping T: $h={}^t(h_1, h_2) \mapsto {}^t\left(\frac{\partial^2}{\partial \nu_1 \partial n}\mathscr{P}h|_\Gamma, \frac{\partial}{\partial \nu_2}\mathscr{P}h|_\Gamma\right)$ is a pseudo-differential operator on Γ, whose principal symbol is the Lopatinski matrix (, which is described in Chapter VI of [10]); the problem (0.2) can be reduced to investigation of the system of equations $Th=g$. The problem of such a system is not characterized even in the subelliptic case (, which means that the estimate of the type (0.3) below holds), while scalar subelliptic operators are done completely by Egorov [3], [4].

We shall give in §2 a sufficient condition for the subellipticity. Let $A(x, D_x)$ be an $m\times m$-matrix of pseudo-differential operators on an open ball $U(\subset \boldsymbol{R}^n)$ and $A(x, \xi)$ be homogeneous of order one in $\xi(|\xi|\geqq 1)$. On some assumptions (cf. [A-I]~[A-IV]) we derive the subelliptic estimate

$$(0.3) \qquad ||u||_{s+\varepsilon_0,U'} \leqq C(||Au||_{s,U'}+||u||_{s,U'}), \quad u\in C_0^\infty(U')\ (\varepsilon_0>0)\,,$$

where U' is an open ball $(\bar{U}'\subset U)$, $||\cdot||_{s,U'}$ is the norm of the Sobolev space $H_s(U')$ and $C_0^\infty(U')$ is the set of C^∞-functions in U' with compact support. In §3 we consider the system of equations $Au=f$ on a compact manifold such that its symbol represented by local coordinates satisfies locally the same assumptions as in §2. Constructing the almost right inverse (i.e. right regularizer) in the same way as in [1], [6], [16], etc., we show that the equation $Au=f$ is of Fredholm type (i.e., the kernel and cokernel are finite-dimensional in the Sobolev space). Finally, in §4 we study the solvability of (0.2) by using the reduction stated earlier. If the vector fields ν_1, ν_2 satisfy several assumptions (cf. (4.2)~(4.5)), (0.2) is of Fredholm type and the estimate

$$||u||_{s+3+\varepsilon_0,\Omega} \leqq C\left\{||Lu||_{s,\Omega}+\left\|\frac{\partial^2 u}{\partial \nu_1 \partial n}\right\|_{s+3/2,\Gamma}+\left\|\frac{\partial u}{\partial \nu_2}\right\|_{s+5/2,\Gamma} +||u||_{s+3,\Omega}\right\}, \quad u\in H_{s+4}(\Omega)\ (\varepsilon_0>0)$$

is obtained for $s\geqq 0$.

Èskin in [6] investigated the degenerate elliptic system $Au=f$ when det A is of principal type. We note that in our class det A may have multi-characteristics.

The main results of this paper are stated in our previous note [15] without proofs.

1. Notations and properties of pseudo-differential operators

We denote by $S^m_{\rho,\delta}(W)$ $(W\subset \boldsymbol{R}^n,\ m\in \boldsymbol{R},\ 0\leqq\delta\leqq\rho\leqq 1,\ \delta<1)$ the set of functions $p(x, \xi)\in C^\infty(W\times \boldsymbol{R}^n)$ satisfying for all multi-indices α, β

$$|D_x^\beta \partial_\xi^\alpha p(x, \xi)| \leqq C_{\alpha\beta}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}, \quad x\in W,\ \xi\in \boldsymbol{R}^n\,,$$

where $D_x^\beta=\left(-i\frac{\partial}{\partial x}\right)^\beta$, $\partial_\xi^\alpha=\left(\frac{\partial}{\partial\xi}\right)^\alpha$ and $\langle\xi\rangle=(|\xi|^2+1)^{1/2}$. For $p(x, \xi)\in S_{\rho,\delta}^m(W)$ we define a pseudo-differential operator $p(x, D_x)$ by

$$p(x, D_x)u(x)=\int e^{ix\xi}p(x, \xi)\hat{u}(\xi)đ\xi, \quad u\in\mathcal{S},$$

where $đ\xi=(2\pi)^{-n}d\xi$, $\mathcal{S}$ is the space of rapidly decreasing fuctions and $\hat{u}(\xi)$ is the Fourier transform $\int e^{-ix\xi}u(x)dx$. We denote by $S_{\rho,\delta}^m(W)$ the set of these operators $p(x, D_x)$, and call $p(x, \xi)$ the symbol of $p(x, D_x)$. It is well known that the estimate

$$\|p(x, D_x)u\|_s\leqq C|p|_l^{(m)}\|u\|_{s+m}, \quad u\in\mathcal{S}\ (s\in\boldsymbol{R})$$

holds for any $p(x, \xi)\in S_{\rho,\delta}^m(=S_{\rho,\delta}^m(\boldsymbol{R}^n))$, where

$$|p|_l^{(m)}=\max_{|\alpha|+|\beta|\leqq l}\ \inf_{x,\xi}|D_x^\beta\partial_\xi^\alpha p(x, \xi)\cdot\langle\xi\rangle^{-m+\rho|\alpha|-\delta|\beta|}|$$

and the constants C, l do not depend on $p(x, \xi)$. This is proved in Calderón-Vaillancourt [2], Kumano-go [9], etc. For $p(x, \xi)\in S_{\rho,\delta}^m$ and $q(x, \xi)\in S_{\rho,\delta}^{m'}$ we set

$$\sigma(p\circ q)(x, \xi)=\lim_{\varepsilon\to+0}\iint e^{-iy\eta}\chi(\varepsilon\eta, \varepsilon y)p(x, \xi+\eta)q(x+y, \xi)dyđ\eta,$$

where $\chi(\eta, y)\in\mathcal{S}(\boldsymbol{R}^{2n})$ and $\chi(0, 0)=1$. Then we have $\sigma(p\circ q)(x, \xi)\in S_{\rho,\delta}^{m+m'}$ and

$$\sigma(p\circ q)(x, D_x)u=p(x, D_x)\circ q(x, D_x)u(=p(x, D_x)(q(x, D_x)u)).$$

Furthermore the asymptotic expansion formula

$$\sigma(p\circ q)(x, \xi)-\sum_{|\alpha|<N}\frac{1}{\alpha!}\partial_\xi^\alpha p(x, \xi)D_x^\alpha q(x, \xi)\in S_{\rho,\delta}^{m+m'-(\rho-\delta)N} \tag{1.1}$$

is obtained for any integer $N(>0)$. These are explained in Kumano-go [8], [10]. As is considered in [12], we have

Proposition 1.1. *Let* $p(x, \xi)\in S_{0,0}^{m_1}$ *and* $q(x, \xi)\in S_{0,0}^{m_2}$. *If* $\partial_{\xi_j}p(x, \xi)\in S_{0,0}^{m_1-1}$ *for any* j, *then it follows that*

$$\sigma(p\circ q)(x, \xi)-p(x, \xi)q(x, \xi)\in S_{0,0}^{m_1+m_2-1}.$$

We can prove this proposition in the same way as in Chapter II of [10]. Replacing $\langle\xi\rangle$ in the above discussion with another basic weight function $\lambda(\xi)$ (i.e., $\lambda(\xi)\in C^\infty$, $1\leqq\lambda(\xi)\leqq A_0\langle\xi\rangle$ and $|\partial_\xi^\alpha\lambda(\xi)|\leqq A_\alpha\lambda(\xi)^{1-|\alpha|}$), we obtain the same results (cf. Chapter VII of [10]). We denote by $S_{\lambda,\rho,\delta}^m$ the set of symbols defined by $\lambda(\xi)$ $(\neq\langle\xi\rangle)$.

In this paper we use pseudo-differential operators on a C^∞ compact manifold

M defined in Seeley [13]. Let Q be a mapping: $C^\infty(M)\to C^\infty(M)$. Then for local coordinates (Φ_1, U_1), (Φ_2, U_2) (Φ_i is defined on an open set U_i) we have in a natural way a mapping $Q_{\Phi_2\Phi_1}$: $C_0^\infty(U_{\Phi_1})\to C^\infty(U_{\Phi_2})$ $(U_{\Phi_i}=\Phi_i(U_i))$. We say that a mapping P: $C^\infty(M)\to C^\infty(M)$ is a pseudo-differential operator on M of order m when there is a set of local coordinates $\{(\Phi_i, U_i)\}_{i=1,\cdots,N}$ covering M such that (i) for any $\varphi\in C_0^\infty(U_i)$, $\psi\in C_0^\infty(U_j)$ $(i\neq j)$ satisfying supp $(\varphi)\cap$ supp $(\psi)=\phi$ $(\varphi P\psi)_{\Phi_i\Phi_j}$ belongs to $S_{1,0}^{-\infty}$, (ii) for any φ, $\psi\in C_0^\infty(U_i)$ $(\varphi P\psi)_{\Phi_i\Phi_i}$ belongs to $S_{1,0}^m$ and (iii) the symbol $p(x, \xi)$ of $(\varphi P\psi)_{\Phi_i\Phi_i}$ has a homogeneous asymptotic expansion, that is, there exist symbols $p_{m-j}(x, \xi)$ $(j=0, 1, \cdots)\in S_{1,0}^{m-j}$ homogeneous of order $m-j$ in $\xi(|\xi|\geqq 1)$ such that for any integer $N(\geqq 0)$

$$p(x, \xi)-\sum_{j=0}^{N} p_{m-j}(x, \xi)\in S_{1,0}^{m-N-1}.$$

We call $p(x, \xi)$ the local symbol of P on V when $\varphi(x)=\psi(x)=1$ on V. Using the principal part $p_m(x, \xi)$ of the local symbol, we can define a function P_m on the cotangent space $T^*(M)-\{0\}$, which is called the pricipal symbol (part) of P. Let $A=\begin{bmatrix} a_{11}\cdots a_{1m} \\ \cdots \\ a_{m1}\cdots a_{mm}\end{bmatrix}$ be a matix of pseudo-differential operators on M such that the order of a_{ij} is $-s_i+t_j$ (s_i, $t_j\in\boldsymbol{R}$ and $i, j=1, \cdots, m$). We say that A is elliptic if its principal symbol is non-singular. In other words, the principal part $a_0(x, \xi)$ of the local symbol on V always satisfies det $a_0(x, \xi)\neq 0$ for $x\in\Phi_i(V)$ and $|\xi|\geqq 1$.

2. The system of first order operators

We set for ε, $\rho>0$

$$U_{\varepsilon,\rho}=\{x=(t, y)\in\boldsymbol{R}^n;\ -\varepsilon<t<\varepsilon,\ |y|<\rho\}.$$

Let $A(x; \xi)=\begin{bmatrix} a_{11}(x, \xi)\cdots a_{1m}(x, \xi) \\ \cdots \\ a_{m1}(x, \xi)\cdots a_{mm}(x, \xi)\end{bmatrix}$ be a matrix of symbols belonging to $S_{1,0}^1(\bar{U}_{\varepsilon_1\varepsilon_1})$ and homogeneous of order one in ξ, that is,

$$a_{ij}(x, \mu\xi)=\mu a_{ij}(x, \xi),\quad \mu\geqq 1,\ |\xi|\geqq 1.$$

We assume that when $t\neq 0$ $A(t, y; \tau, \eta)$ is elliptic (i.e., det $A(t, y; \tau, \eta)\neq 0$ for $|\tau, \eta|=(\tau^2+|\eta|^2)^{1/2}\geqq 1$) and that when $t=0$ the ellipticity is degenerate in the following way: For $(t, y)\in\bar{U}_{\varepsilon_1,\varepsilon_1}$ and $|\tau, \eta|\geqq 1$

[A-I] det $A(t, y; \tau, \eta)\neq 0$ when $t\neq 0$ or $t=0$ & $\tau\neq 0$;

[A-II] $A(0, y; 0, \eta)=[0]$ (zero-matrix), $|\eta|\geqq 1$;

[A-III] det $\dfrac{\partial A}{\partial\tau}(0, y; 0, \eta)\neq 0$, $|\eta|\geqq 1$;

there exist positive integers $k_1, \cdots, k_l$ independent of $(t, y; \eta')\in \bar{U}_{\varepsilon,\varepsilon_1}\times S$ ($\varepsilon>0$ is small enough and $S=\{\eta' : |\eta'|=1\}$) such that the following decomposition of the matrix

$$\tilde{A}(t, y; \eta') = \frac{\partial A}{\partial \tau}(t, y; 0, \eta')^{-1}\cdot A(t, y; 0, \eta'), \quad (t, y; \eta')\in \bar{U}_{\varepsilon,\varepsilon_1}\times S$$

is possible:

[A-IV] $t^{-k_1}\tilde{A}(t, y; \eta')$ is smooth on $t=0$ and has eigen-values $\lambda_1^1(t, y; \eta'), \cdots, \lambda_{m_1}^1(t, y; \eta')$ whose imaginary parts do not vanish on $\bar{U}_{\varepsilon,\varepsilon_1}\times S$. Other eigen-values vanish as $t\to 0$. Define a projection P^1 by

$$P^1(t, y; \eta') = \frac{1}{2\pi i}\oint_{\Gamma_1}(\lambda - t^{-k_1}\tilde{A}(t, y; \eta'))^{-1}d\lambda\,,$$

where Γ_1 is a Jordan curve surrounding $\lambda_1^1, \cdots, \lambda_{m_1}^1$ and having other eigen-values outside. Next for the matrix $t^{-k_1-k_2}\tilde{A}(I-P^1)$ the same statements hold. We can continue these decompositions one after another, and finally we have

$$\sum_{i=1}^{l} \operatorname{rank} P^i = m\,,$$

where P^i is the projection for the eigen-values $\lambda_1^i, \cdots, \lambda_{m_i}^i$ ($m_i=\operatorname{rank} P^i$) of $t^{-k_1-\cdots-k_i}\tilde{A}(I-P^1)\cdots(I-P^{i-1})$ with non-vanishing imaginary parts.

On the above assumptions we obtain

Theorem 2.1. *If* $\operatorname{Im}\lambda_1^i(0, y; \eta'), \cdots, \operatorname{Im}\lambda_{m_i}^i(0, y; \eta')$ *are all positive for every i such that $k_1+\cdots+k_i$ is odd, then we have the subelliptic estimate*

$$\|u\|_{s+\varepsilon_0}\leqq C(\|Au\|_{s,U_{\varepsilon_1,\varepsilon_1}}+\|u\|_s), \quad u\in C_0^\infty(U_{\varepsilon_2,\varepsilon_2})\ (0<\varepsilon_2<\varepsilon_1)\,, \tag{2.1}$$

where $\varepsilon_0=\dfrac{1}{k_1+\cdots+k_l+1}$ *and* $s\in \boldsymbol{R}$.

It is obvious that the estimate (2.1) for any $s\in \boldsymbol{R}$ is derived from the one for $s=0$. In order to prove this theorem we state several lemmas.

We write

$$A(x; \tau, \eta) = A(x; 0, \eta)+\int_0^1 \frac{\partial A}{\partial \tau}(x; \theta\tau, \eta)d\theta\tau\,,$$

and set

$$A^{(1)}(x; \tau, \eta) = \int_0^1 \frac{\partial A}{\partial \tau}(x; \theta\tau, \eta)d\theta\,,$$

which belongs to $S_{0,0}^0(\bar{U}_{\varepsilon_1,\varepsilon_1})$.

Lemma 2.1. *If $\varepsilon>0$ is sufficiently small, we have*

(i) *for all multi-indices α, β* ($|\alpha|\geqq 1$)

$$|D^{\beta}_{(t,y)}\partial^{\alpha}_{\xi}A^{(1)}(t, y; \xi)| \leqq C_{\alpha\beta}\langle\xi\rangle^{-1}, \quad (t, y)\in \bar{U}_{\varepsilon,\varepsilon_1}, \ \xi\in \boldsymbol{R}^n,$$

(ii) $|\det A^{(1)}(t, y; \xi)| \geqq \delta(>0), \quad (t, y)\in \bar{U}_{\varepsilon,\varepsilon_1}, \ |\xi| \geqq R,$

where R is a sufficiently large constant. (*For a matrix* $A(x)=(a_{ij}(x))$ $|A(x)|$ *denotes* $\max\limits_{i,j} |a_{ij}(x)|$).

Proof. (i) From the definition of $A^{(1)}$, it follows that

$$|D^{\beta}_{(t,y)}\partial^{\alpha}_{\xi}A^{(1)}(t, y; \xi)| \leqq \int_0^1 |D^{\beta}_{(t,y)}\partial^{\alpha}_{\xi}\left\{\frac{\partial A}{\partial\tau}(t, y; \theta\tau, \eta)\right\}| d\theta$$
$$\leqq C_1\langle\eta\rangle^{-|\alpha|}.$$

Hence, if $\frac{\langle\eta\rangle}{\langle\tau, \eta\rangle} \geqq \delta_1(>0)$ we have the estimate (i). Let $\frac{\langle\eta\rangle}{\langle\tau, \eta\rangle} \leqq \delta_1$, then $\frac{\langle\tau, \eta\rangle}{|\tau|} \leqq C$ $(|\tau| \geqq 1)$ when δ_1 is small. Therefore, we get

$$|D^{\beta}_{(t,y)}\partial^{\alpha}_{\xi}A^{(1)}(t, y; \xi)| = \left|D^{\beta}_{(t,y)}\partial^{\alpha}_{(\tau,\eta)}\left[\frac{1}{\tau}\{A(t, y; \tau, \eta)-A(t, y; 0, \eta)\}\right]\right|$$
$$\leqq C_2\frac{1}{|\tau|}+C_3\frac{\langle\tau, \eta\rangle}{|\tau|^{|\alpha|+1}} \leqq C_4\langle\tau, \eta\rangle^{-1} \quad (|\tau| \geqq 1).$$

So we obtain the inequality (i) for any $(\tau, \eta)\in \boldsymbol{R}^n$.

(ii) By Taylor's expansion

$$A^{(1)}(x; \tau, \eta) = A^{(1)}(x; 0, \eta)+\int_0^1 \frac{\partial A^{(1)}}{\partial\tau}(x; \theta\tau, \eta)d\theta\tau$$

and the estimate (i) we have

$$|\det A^{(1)}(t, y; \tau, \eta)| \geqq |\det \frac{\partial A}{\partial\tau}(t, y; 0, \eta)| - C_5\sum_{j=1}^{m}\left(\frac{|\tau|}{|\eta|}\right)^j, \ (|\eta| \geqq 1),$$

where C_5 does not depend on τ, η. Hence, from [A-III] the estimate (ii) holds when $\frac{|\tau|}{|\eta|} \leqq \rho$ and $|t| < \mathcal{E}$ ($\mathcal{E}$, ρ are small enough). When $\frac{|\tau|}{|\eta|} \geqq \rho$, writing

$$A^{(1)}(t, y; \tau, \eta) = \frac{1}{\tau}\{A(t, y; \tau, \eta)-A(t, y; 0, \eta)\}$$
$$= \pm A\left(t, y; \pm 1, \frac{\eta}{|\tau|}\right)-A(t, y; 0, \eta)\frac{1}{\tau}, \ (|\tau| \geqq 1)$$

(where $\pm$ means the sign of τ), we have

$$|\det A^{(1)}(t, y; \tau, \eta)| \geqq |\det A\left(t, y; \pm 1, \frac{\eta}{|\tau|}\right)|$$
$$-C_6\sum_{j=1}^{m}\{|A(t, y; 0, \eta)|\,|\eta|^{-1}\rho^{-1}\}^j,$$

where C_6 is independent of τ, η. From [A-II] it follows that

$$|A(t, y;\, 0, \eta)| \leqq C_7 |t|\, |\eta| \leqq C_7 |\eta| \varepsilon \qquad (|\eta| \geqq 1)\,.$$

Therefore, by [A-I] we obtain the estimate (ii) if ε is sufficiently small for ρ. The proof is complete.

Now we set

$$B^{(1)}(t, y;\, \xi) = A^{(1)}(t, y;\, \xi)^{-1}\theta(\xi)\,,$$

where $(t, y) \in U_{\varepsilon, \varepsilon_1}$, $\theta(\xi)$ $(\in C^\infty(\boldsymbol{R}^n))$ is equal to 0 for $|\xi| \leqq R$ and to 1 for $|\xi| \geqq R+1$ (R is the constant in (ii) of Lemma 2.1). In view of Lemma 2.1 and Proposition 1.1 it is seen that $B^{(1)}(x;\xi)$ satisfies the same inequality as in (i) of Lemma 2.1 and that $B^{(1)}(x; D_x)$ is a local inverse of $A^{(1)}(x; D_x)$ modulo $S^{-1}_{0,0}(\bar{U}_{\varepsilon,\varepsilon_1})$, namely, for all $\varphi(x)$, $\psi(x) \in C_0^\infty(U_{\varepsilon,\varepsilon_1})$ such that $\psi(x)=1$ on a neighborhood of $\operatorname{supp}(\varphi)$, we have

$$\begin{aligned} &\varphi B^{(1)}(x;\, D_x) \circ \psi A^{(1)}(x;\, D_x) \equiv \varphi \quad \bmod S^{-1}_{0,0}\,, \\ &\varphi A^{(1)}(x;\, D_x) \circ \psi B^{(1)}(x;\, D_x) \equiv \varphi \quad \bmod S^{-1}_{0,0}\,. \end{aligned}$$

This implies

$$(2.2) \quad \varphi A(x;\, D_x) \equiv \varphi A^{(1)}(x;\, D_x) \circ \psi [D_t + B^{(1)}(t, y;\, D_t, D_y) A(t, y;\, 0, D_y)] \bmod S^0_{0,0}.$$

Therefore, noting that $A(x;\, D_x)$ is elliptic when $t \neq 0$, we have only to examine the operator

$$D_t + B^{(1)}(t, y;\, D_t, D_y) A(t, y;\, 0, D_y)$$

near $t=0$. Furthermore, this is approximated by $D_t + L(t, y;\, D_y)$, where $L(t, y;\, \eta) \in S^1_{0,0}(\bar{U}_{\varepsilon,\varepsilon_1})$ has the form

$$L(t, y;\, \eta) = \frac{\partial A}{\partial \tau}(t, y;\, 0, \eta)^{-1} \cdot A(t, y;\, 0, \eta), \quad (t, y) \in U_{\varepsilon,\varepsilon_1},\ |\eta| \geqq 1\,.$$

More precisely, we have

Lemma 2.2. *Let $\varphi(t, y) \in C_0^\infty(U_{\varepsilon,\varepsilon_1})$. Then we have*

$$\begin{aligned} &\varphi(t, y) B^{(1)}(t, y;\, \tau, \eta) A(t, y;\, 0, \eta) - \varphi(t, y) L(t, y;\, \eta) \\ &= t\varphi(t, y) Q(t, y;\, \tau, \eta)\tau + R(t, y;\, \tau, \eta)\,, \end{aligned}$$

where $Q(x;\, \xi)$, $R(x;\, \xi) \in S^0_{0,0}$ and $Q(x;\, \xi)$ does not depend on φ.

Proof. By Taylor's expansion of $B^{(1)}(x;\, \tau, \eta)$ in τ and $B^{(1)}(x;\, 0, \eta) = \dfrac{\partial A}{\partial \tau}(t, y;\, 0, \eta)^{-1}$ $(|\eta| \leqq R+1)$, we have

$$\varphi(t, y)B^{(1)}(t, y;\ \tau, \eta)A(t, y;\ 0, \eta)-\varphi(t, y)L(t, y;\ \eta)$$
$$\equiv\varphi\int_0^1\frac{\partial B^{(1)}}{\partial\tau}(t, y;\ \theta\tau, \eta)d\theta\tau A(t, y;\ 0, \eta) \quad \operatorname{mod} S^0_{0,0}\,.$$

[A-II] yields that

$$A(t, y;\ 0, \eta) = t\int_0^1\frac{\partial A}{\partial t}(\theta t, y;\ 0, \eta)d\theta, \qquad |\eta|\geqq 1\,.$$

Therefore we obtain the lemma.

Proof of Theorem 2.1. Let $||u(t, y)||_{s,s'}=||\langle D_t\rangle^s u||_0+||\langle D_y\rangle^{s'}u||_0\,(s\geqq 0, s'\geqq 0)$. It suffices to prove the following lemma:

Lemma 2.3. *Let the assumptions in Theorem* 2.1 *be satisfied. Then we have for sufficiently small* $\varepsilon(>0)$

$$||\varphi u||_{1,\varepsilon_0}\leqq C_1||(D_t+\psi L(t, y;\ D_y))\,(\varphi u)||_0+C_2||\varphi u||_0, \quad u\in C_0^\infty(\boldsymbol{R}^n)\,,$$

where $\varepsilon_0=\dfrac{1}{k_1+\cdots+k_l+1}$, $\varphi, \psi\in C_0^\infty(U_{\varepsilon,\varepsilon_1})$, $\psi(x)=1$ *on a neighborhood of* supp (φ) *and the constant* C_1 *is independent of* φ, ψ *and* ε.

In fact, combining Lemma 2.2 and this lemma, we obtain

$$||\varphi u||_{1,\varepsilon_0}\leqq C_1||(D_t+\psi B^{(1)}(t, y;\ D_t, D_y)A(t, y;\ 0, D_y))\,(\varphi u)||_0$$
$$+C_2\varepsilon||D_t(\varphi u)||_0+C_3||\varphi u||_0, \quad u\in C_0^\infty(\boldsymbol{R}^n)\,.$$

Here φ, ψ are the functions stated in Lemma 2.2 and Lemma 2.3. Since the above constant C_2 does not depend on ε, if $\varepsilon>0$ is small enough it follows (from (2.2)) that

$$||\varphi u||_{1,\varepsilon_0}\leqq C_4(||\psi A(x, D_x)\varphi u||_0+||\varphi u||_0)\,,$$

which proves Theorem 2.1.

Now let us derive Lemma 2.3. [A-IV] yields

Lemma 2.4. *There exist a finite open covering* $\{V_\alpha\}$ *on* $S(=\{\eta':\ |\eta'|=1\})$ *and a set of functions* $N_\alpha(x;\ \eta')\in C^\infty(\bar U_{\varepsilon,\varepsilon_1}\times V_\alpha)$ *such that for any* $(x, \eta')\in\bar U_{\varepsilon,\varepsilon_1}\times V_\alpha$

(i) $\det N_\alpha(x;\ \eta')\neq 0$

(ii) $N_\alpha(x;\ \eta')\tilde A(x;\ \eta')$

$$=\begin{pmatrix} t^{k_1}\tilde A_1(x;\ \eta') & & 0 \\ & t^{k_1+k_2}\tilde A_2(x;\ \eta') & \\ 0 & \ddots & \\ & & t^{k_1+\cdots+k_l}\tilde A_l(x;\ \eta')\end{pmatrix} N_\alpha(x;\ \eta')\,,$$

where $\tilde A_l(x;\ \eta')$ *is an* $m_l\times m_l$*-matrix with the eigen-values* $\lambda_1^i(x;\ \eta'), \cdots, \lambda_{m_i}^i(x;\ \eta')$

stated in [A-IV].

Proof. Let us recall that

$$P^i(t, y; \eta') = \frac{1}{2\pi i}\oint_{\Gamma_i}(\lambda - A_i(t, y; \eta'))^{-1}d\lambda\,. \tag{2.3}$$

Here $A_i = t^{-k_i - \cdots - k_1}\tilde{A}(I-P^0)\cdots(I-P^{i-1})$ $(P^0=[0])$ and Γ_i is a Jordan curve surrounding $\lambda_1^i, \cdots, \lambda_{m_i}^i$ and having other eigen-values outside. Obviously $P^i(t, y; \eta')$ is infinitely differentiable on $\bar{U}_{\varepsilon,\varepsilon_1}\times S$. From the definition (2.3) it is easily seen that for all $i, j=1, \cdots, l$

$$P^i\tilde{A} = \tilde{A}P^i, \quad P^iA_j = A_jP^i, \quad P^iP^j = P^jP^i\,.$$

Set

$$Q^i(t, y; \eta') = (I-P^0)\cdots(I-P^{i-1})P^i \quad (i = 1, \cdots, l)\,.$$

Then we have

$$\begin{aligned} &\operatorname{rank} Q^i = \operatorname{rank} P^i = m_i, \quad Q^iQ^j = [0] \quad \text{if } i \neq j\,,\\ &\tilde{A}Q^i = t^{k_1+\cdots+k_i}A_iQ^i\,. \end{aligned}$$

Choose generalized eigen (row) vectors $\psi_1^i(t, y; \eta'), \cdots, \psi_{m_i}^i(t, y; \eta')$ linearly independent such that $\psi_j^iQ^i = \psi_j^i$. These vectors can be taken smoothly on $\bar{U}_{\varepsilon,\varepsilon_1}\times V$ (V is an open set in S). Put

$$N(t, y; \eta') = \begin{pmatrix} \psi_1^1\ (t, y; \eta') \\ \vdots \\ \psi_{m_1}^1(t, y; \eta') \\ \vdots \\ \psi_1^l\ (t, y; \eta') \\ \vdots \\ \psi_{m_l}^l(t, y; \eta') \end{pmatrix}.$$

Then we see easily that $N(t, y; \eta')$ satisfies (i) and (ii) of the lemma. The proof is complete.

Proposition 2.1. *Let* $A'(\eta')\in C^\infty(S)$ $(S=\{\eta'\colon |\eta'|=1\})$ *be an* $m'\times m'$-*matrix whose eigen-values all have non-vanishing imaginary parts on* S, *and let* k *be a constant positive integer. Set*

$$L'(t; \eta) = t^kA'\Big(\frac{\eta}{|\eta|}\Big)|\eta|\theta(\eta)\,,$$

where $\theta(\eta)$ $(\in C^\infty)=1$ *for* $|\eta|\geqq 1$ *and* $\theta(\eta)=0$ *for* $|\eta|\leqq\frac{1}{2}$. *Then, we have*

(i) *If either '*k *is even' or '*k *is odd and every imaginary part of the eigen-value is positive', the following estimate holds*:

$$C^{-1}(\|D_t v\|_0+\|t^k\langle D_y\rangle v\|_0)\leqq\|(D_t+L'(t;\,D_y))v\|_0+\|t^k v\|_0$$
$$\leqq C(\|D_t v\|_0+\|t^k\langle D_y\rangle v\|_0),\quad v(t,\,y)\in\mathcal{S}\,.$$

(ii) *If k is even, there exists a continuous operator R' from $L^2(\boldsymbol{R}^n)$ to $H_1^{(k)}(\boldsymbol{R}^n)$ $=\{u\in L^2;\ D_t u\in L^2,\ t^k\langle D_y\rangle u\in L^2\}$ such that $S'=(D_t+L')R'-I$ is continuous from $L^2(\boldsymbol{R}^n)$ to $H_{0,\infty}(\boldsymbol{R}^n)=\{u\in L^2;\ \langle D_y\rangle^N u\in L^2$ for any $N>0\}$.*

We can prove this proposition in the same way as in [14]. A sketch of the proof is given in Appendix. In this section we do not need (ii) of the proposition, which is used in the next section.

Proof of Lemma 2.3. Let $\theta(\eta)\in C^\infty(\boldsymbol{R}^{n-1})$ be equal to 1 for $|\eta|\geqq 1$ and to 0 for $|\eta|\leqq\dfrac{1}{2}$. For any η we define $N_\alpha(x;\,\eta)$ by $N_\alpha\Bigl(x;\,\dfrac{\eta}{|\eta|}\Bigr)\theta(\eta)$. Then $N_\alpha(x;\,\eta)$ belongs to $S^0_{\langle\eta\rangle,1,0}(\bar{U}_{\varepsilon,\varepsilon_1})$. Let $\{\varphi'_\alpha(\eta')\}$ be a partition of unity on S subject to the covering $\{V_\alpha\}$ stated in Lemma 2.4, and define $\varphi_\alpha(\eta)=\varphi'_\alpha\Bigl(\dfrac{\eta}{|\eta|}\Bigr)$ $\cdot\theta(\eta)$ for any η. By (i) of Lemma 2.4 and the asymptotic expansion formula (1.1) we have for $p(t,\,y;\,\eta)\in S^1_{\langle\eta\rangle,1,0}$ and $\varphi\in C_0^\infty(U_{\varepsilon,\varepsilon_1})$

$$\|\varphi p(t,\,y;\,D_y)u\|_0\leqq C_1\sum_\alpha\|\varphi N_\alpha(t,\,y;\,D_y)\varphi_\alpha(D_y)\circ p(t,\,y;\,D_y)u\|_0$$
$$+C_2\|u\|_0\,. \tag{2.4}$$

Similarly, from (ii) of Lemma 2.4 it follows that

$$\varphi N_\alpha(x;\,D_y)\varphi_\alpha(D_y)\circ\psi L(x;\,D_y)\equiv\varphi D(x;\,D_y)\circ\psi N_\alpha(x;\,D_y)\varphi_\alpha(D_y)$$
$$\mod S^0_{0,0}(\boldsymbol{R}^n_x)\,. \tag{2.5}$$

Here $D(x;\,\eta)\in S^1_{\langle\eta\rangle,1,0}(\bar{U}_{\varepsilon,\varepsilon_1})$ is of the form

$$D(x;\,\eta)=\begin{pmatrix} t^{k_1}\tilde{A}_1\Bigl(x;\,\dfrac{\eta}{|\eta|}\Bigr)|\eta| & & 0 \\ & t^{k_1+k_2}\tilde{A}_2\Bigl(x;\,\dfrac{\eta}{|\eta|}\Bigr)|\eta| & \\ & & \ddots \\ 0 & & t^{k_1+\cdots+k_l}\tilde{A}_l\Bigl(x;\,\dfrac{\eta}{|\eta|}\Bigr)|\eta| \end{pmatrix},\quad |\eta|\geqq 1\,.$$

We get easily

$$\|\langle D_y\rangle^{1/(k+1)}v(t,\,y)\|_0\leqq C_3(\|D_t v\|_0+\|t^k\langle D_y\rangle v\|_0)$$

for $v(t,\,y)\in C_0^\infty(\boldsymbol{R}^n)$ whose support lies in $|t|\leqq 1$ (k is a positive integer). Combining this inequality and (2.4), we have

$$\|\langle D_y\rangle^{\varepsilon_0}\varphi u\|_0 \leqq C_4 \sum_{\alpha}\sum_{i} (\|D_t v_i^\alpha\|_0 + \|t^{k_1+\cdots+k_i}\langle D_y\rangle v_i^\alpha\|_0)\,,$$

where ${}^t(v_1^\alpha, \cdots, v_l^\alpha) = \psi N_\alpha(x; D_y)\varphi_\alpha(D_y)(\varphi u)$ ($\psi(x)=1$ in a neighborhood of supp (φ)). (2.4) and (2.5) yield that

$$\begin{aligned}\|D_t(\varphi u)\|_0 &\leqq \|(D_t+\psi L)(\varphi u)\|_0 + \|\psi L(\varphi u)\|_0 \\ &\leqq \|(D_t+\psi L)(\varphi u)\|_0 \\ &\qquad + C_6 \sum_{\alpha} \|\psi D(x; D_y)\circ\psi N_\alpha(x; D_y)\varphi_\alpha(D_y)(\varphi u)\|_0 \\ &\qquad\qquad + C_7\|\varphi u\|_0\,.\end{aligned}$$

Noting that $\|\psi D\circ\psi N_\alpha\varphi_\alpha(\varphi u)\|_0 \leqq C_8 \sum_i \|t^{k_1+\cdots+k_i}\langle D_y\rangle v_i^\alpha\|_0$, we have only to show

$$\|D_t v_i^\alpha\|_0 + \|t^{k_1+\cdots+k_i}\langle D_y\rangle v_i^\alpha\|_0 \leqq C_9(\|(D_t+\psi L)(\varphi u)\|_0 + \|\varphi u\|_0)\,.$$

This is guaranteed by (i) of Proposition 2.1. The proof is complete.

3. The system on a compact manifold

Let M be a compact C^∞ manifold and let $A = \begin{bmatrix} a_{11}\cdots a_{1m} \\ \cdots \\ a_{m1}\cdots a_{mm}\end{bmatrix}$ be a matrix of pseudo-differential operators on M such that the order of a_{ij} is $-s_i+t_j$ ($s_i, t_j \in \boldsymbol{R}$). We define for $s\in\boldsymbol{R}$, $s'=(s_1', \cdots, s_m')$ ($s_i'\in\boldsymbol{R}$)

$$\boldsymbol{H}_s^{(s')}(M) = \prod_{i=1}^m H_{s+s_i'}(M)\,,$$

$$\|u\|_s^{(s')} = \Big(\sum_{i=1}^m \|u_i\|_{s+s_i'}^2\Big)^{1/2} \quad (u = {}^t(u_1, \cdots, u_m))\,.$$

Then A is a continuous operator from $\boldsymbol{H}_s^{(t)}(M)$ to $\boldsymbol{H}_s^{(s)}(M)$.

In this section we consider the system of equations

$$Au = f \tag{3.1}$$

on the following assumptions. Let M ($n=\dim M\geqq 2$) be separated into two connected components by a C^∞ submanifold M_0. We assume that A is elliptic outside M_0 and degenerate on M_0 in the following way: Let $\{x^i=(x_0^i, x_1^i, \cdots, x_{n-1}^i)\}_{i=1,\cdots,N}$ be a set of local coordinates such that each x^i transforms an open set V_i to $U_{\varepsilon_1,\varepsilon_1}(=\{(x_0^i, x_1^i, \cdots, x_{n-1}^i) : |x_0^i| < \varepsilon_1,\ \{(x_1^i)^2+\cdots+(x_{n-1}^i)^2\}^{1/2} < \varepsilon_1\}$ ($\varepsilon_1>0$)) and that $\bigcup_{i=1}^N V_i$ covers M_0. Furthermore, let M_0 be expressed by the equation $x_0^i=0$ and the transition from x^i to x^j in the domain where both x^i and x^j are defined be given by the form

$$x_0^j = x_0^i, \quad x_k^j = \varphi_k^j(x_1^i, \cdots, x_{n-1}^i) \quad (k = 1, \cdots, n-1)\,.$$

We suppose that when A is represented near V_i by $x^i=(t, y_1, \cdots, y_{n-1})$ ($i=1, \cdots,$

N) the principal part $A_0(t, y; \tau, \eta)$ of the local symbol on V_i always satisfies the assumptions [A-I]~[A-IV] stated in §2, and that the constants $k_1, \cdots, k_l$ and $m_1, \cdots, m_l$ in [A-IV] are all independent of a choice of x^ι.

Our purpose is to show the following theorem:

Theorem 3.1. (i) *If* $\operatorname{Im} \lambda_1^\iota(0, y; \eta'), \cdots, \operatorname{Im} \lambda_{m_i}^\iota(0, y; \eta')$ *are all positive for every i such that* $k_1+\cdots+k_\iota$ *is odd, then we have the estimate*

$$\|u\|_{s+\varepsilon_0-1}^{(t)} \leqq C(\|Au\|_s^{(s)}+\|u\|_{s-1}^{(t)}), \quad u \in \boldsymbol{H}_s^{(t)}(M)$$

for $s \in \boldsymbol{R}\left(\varepsilon_0=\dfrac{1}{k_1+\cdots+k_l+1}\right)$.

(ii) *If every* $k_\iota(i=1, \cdots, l)$ *is even, A is of Fredholm type as a mapping from* $\boldsymbol{H}_{s+\varepsilon_0-1}^{(t)}(M)$ *to* $\boldsymbol{H}_s^{(s)}(M)$ *(i.e., the kernel and cokernel of* (3.1) *are finite-dimensional).*

Remark 3.1. Let $A=A_\mu$ have a parameter $\mu(>0)$ as a covariable and be elliptic including μ outside M_0. Furthermore, assume that the local symbol $A_0(t, y; \tau, \eta, \mu)$ inclusive of μ satisfies the same hypotheses as in the theorem. Then the equation $A_\mu u=f$ is uniquely solvable in the same spaces if μ is large enough.

Proof of Theorem 3.1. Since A is elliptic outside M_0, we have only to investigate the equation $Au=f$ locally near M_0. Let $A(t, y; \tau, \eta)$ be the local symbol of A (on V_ι) in the local coordinates $x^\iota=(t, y)=x$. Set for $s'=(s_1', \cdots, s_m')$ $(s_i' \in \boldsymbol{R})$

$$\Lambda^{s'}=\begin{bmatrix} \langle D_x\rangle^{s_1'} & & 0 \\ & \ddots & \\ 0 & & \langle D_x\rangle^{s_m'} \end{bmatrix}.$$

Then $\Lambda^{s'}$ is a topological isomorphism from $\boldsymbol{H}_s^{(s')}(\boldsymbol{R}^n)=\prod_{j=1}^m H_{s+s_j'}(\boldsymbol{R}^n)$ to $H_s(\boldsymbol{R}^n)=\prod_{\iota=1}^m H_s(\boldsymbol{R}^n)$. We examine

$$A'(x; D_x)=\Lambda^{1+s} \circ A(x; D_x) \circ \Lambda^{-t} \quad (1+s=(1+s_1, \cdots, 1+s_m))$$

instead of $A(x; D_x)$. Its principal part $A_0'(x; \xi)$ is of the form

$$A_0'(x; \xi)=A_0\left(x; \frac{\xi}{|\xi|}\right)|\xi|, \quad |\xi| \geqq 1,$$

which is homogenuous of order one in ξ $(|\xi| \geqq 1)$. $A_0'(x; \xi)$ satisfies all the assumptions for $A(x; \xi)$ stated in §2. Therefore, by Theorem 2.1 we obtain for $\varphi(t, y) \in C_0^\infty(U_{\varepsilon_1, \varepsilon_1})$

$$\|\varphi u\|_{s+\varepsilon_0} \leqq C_1(\|A_0'(x; D_x)(\varphi u)\|_s+\|\varphi u\|_s), \quad u \in H_{s+1}(\boldsymbol{R}^n),$$

which proves (i) of Theorem 3.1.

Now let us show (ii) of Theorem 3.1. From the estimate in (i) of the theorem the kernel of (3.1) is finite-dimensional. We shall show that the cokernel is also finite-dimensional by constructing the (right) regularizer R, that is, R is continuous from $\boldsymbol{H}_s^{(\boldsymbol{s})}(M)$ to $\boldsymbol{H}_{s+\varepsilon_0-1}^{(t)}(M)$ and $S=AR-I$ is a compact operator in $\boldsymbol{H}_s^{(\boldsymbol{s})}(M)$. Obviously it suffices to do so for $s=0$. Furthermore, as is easily seen, we can make such an operator R by the local analysis of A in the same way as in Agranovich [1], Višik-Grušin [17], etc. Therefore we have only to construct a local regularizer of $D_t+B_0^{(1)}(t, y; D_t, D_y)A_0'(t, y; 0, D_y)$, where $B_0^{(1)}(x; \tau, \eta)=A_0^{(1)}(x; \tau, \eta)^{-1}\left(=\left[\int_0^1\frac{\partial A_0'}{\partial\tau}(x; \theta\tau, \eta)d\theta\right]^{-1}\right)$ for large $|\tau, \eta|$ (cf. Lemma 2.1 and (2.2)). We obtain the required local regularizer:

Lemma 3.1. *Let* $\varphi(t, y)$, $\psi(t, y)\in C_0^\infty(U_{\varepsilon,\varepsilon})$ ($\varepsilon>0$ *is small enough*) *and* $\psi(t, y)=1$ *in a neighborhood of supp* (φ). *Then, there exists an operator* R_0 *continuous from* $L^2(\boldsymbol{R}^n)$ *to* $H_{1,\varepsilon_0}(\boldsymbol{R}^n)$ ($=\{u\in L^2; D_tu\in L^2, \langle D_y\rangle^{\varepsilon_0}u\in L^2\}$) *such that*

(i) *the estimate*

$$\|D_t(R_0f)\|_0+\|\langle D_y\rangle^{\varepsilon_0}(R_0f)\|_0\leqq C\|f\|_0 \tag{3.2}$$

holds for a constant C independent of φ, ε, *and*

(ii) $$\psi(D_t+B_0^{(1)}(t, y; D_t, D_y)A_0'(t, y; 0, D_y))\circ R_0\varphi f$$
$$=\varphi f+Qf+K_\varphi f,$$

where K_φ *is a continuous operator from* $L^2(\boldsymbol{R}^n)$ *to* $H_{\varepsilon_0}(\boldsymbol{R}^n)$ *and* Q *is a continuous one from* $L^2(\boldsymbol{R}^n)$ *to* $L^2(\boldsymbol{R}^n)$ *with a norm* $\leqq\varepsilon C'$ (C' *is independent of* ε, φ, ψ).

Proof. We denote by R_0^i the operator R' in Proposition 2.1 for $k=k_i$ and $A'(\eta')=\tilde{A}_i(0; \eta')$, and define the functions $N_\alpha(t, y; \eta)$, $\varphi_\alpha(\eta)$, $\psi_\alpha(\eta)$[1] in the same way as in the proof of Lemma 2.3. Set

$$R_0'=\sum_\alpha\psi N_\alpha^{-1}(t, y; D_y)\psi_\alpha(D_y)\circ\begin{bmatrix}\psi R_0^1 & & 0\\ & \ddots & \\ 0 & & \psi R_0^l\end{bmatrix}\circ\psi N_\alpha(t, y; D_y)\varphi_\alpha(D_y).$$

Then Proposition 2.1 yields the estimate for R_0' of the type (3.2). Moreover, using the asymptotic expansion formula (1.1), we have

$$\psi(D_t+L'(t, y; D_y))R_0'\varphi f=\varphi f+Q_0'\varphi f+\psi K_0'\varphi f,$$

where $L'(t, y; \eta)=\left[\frac{\partial A_0'}{\partial\tau}(t, y; 0, \eta)\right]^{-1}\cdot A_0'(t, y; 0, \eta)\theta(\eta)$, K_0' is continuous from $L^2(\boldsymbol{R}^n)$ to $H_{0,\varepsilon_0}(\boldsymbol{R}^n)$ and Q_0' is a continuous operator on L^2 with a norm $\leqq\varepsilon C_1$ (C_1 does not depend on ε). Let $\chi_N(\tau, \eta)\in C^\infty(\boldsymbol{R}^n)$ be equal to 1 if $|\tau|\geqq(N+1)$

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[1] $\psi_\alpha(\eta)$ is of the same type as $\varphi_\alpha(\eta)$ and equal to 1 on a neighborhood of supp (φ_α).

$\cdot|\eta|$ & $|\tau,\eta|\geqq 1$ and to 0 if $|\tau|\leqq N|\eta|$ or $|\tau,\eta|\leqq\frac{1}{2}$. When N is large enough, the symbol $R_0''(t, y; \tau, \eta)=\psi'(t, y)(\tau+L'(t, y; \eta))^{-1}\chi_N(\tau, \eta)$ belongs to $S^{-1}_{0,0}$ ($\psi'(t, y)\in C_0^\infty(U_{\varepsilon\ \varepsilon})$ and $\psi'(t, y)=1$ in a neighborhood of supp (ψ)) and it follows from Proposition 1.1 that

$$\psi(D_t+L'(t, y; D_y))\circ R_0''(t, y; D_t, D_y)\equiv\psi\chi_N(D_t, D_y) \quad \bmod S^{-1}_{0,0}\,.$$

On the other hand, $(1-\chi_N(D_t, D_y))$ is a continuous operator from $H_{0,\varepsilon_0}(\boldsymbol{R}^n)$ to $H_{\varepsilon_0,\varepsilon_0}(\boldsymbol{R}^n)$. Therefore, setting $\tilde{R}_0=R_0'-R_0''K_0'$, we have

$$\psi(D_t+L'(t, y; D_y))\tilde{R}_0\varphi f = \varphi f+Q_0'\varphi f+\tilde{K}f\,,$$

where $\tilde{K}$ is continuous from $L^2(\boldsymbol{R}^n)$ to $H_{\varepsilon_0,\varepsilon_0}(\boldsymbol{R}^n)$. Hence, by Lemma 2.2 we easily obtain (ii) of Lemma 3.1. The proof is complete.

4. An application to boundary value problems

Let Ω be a bounded open set in $\boldsymbol{R}^n (n\geqq 3)$ with a C^∞ boundary Γ, and consider the boundary value problem (mentioned in Introduction):

$$\text{(4.1)} \qquad \begin{cases} L(x, D_x)u=f & \text{in } \Omega\,, \\ \dfrac{\partial^2 u}{\partial\nu_1\partial n}=g_1 & \text{on } \Gamma\,, \\ \dfrac{\partial u}{\partial\nu_2}=g_2 & \text{on } \Gamma\,, \end{cases}$$

where $L(x, D_x)$ is an elliptic differential operator of fourth order on $\bar{\Omega}$ with C^∞ coefficients, n is an inner (unit) normal vector of Γ and ν_1, ν_2 are non-vanishing real vector fields on Γ. Let the following assumptions (4.2)~(4.5) be satisfied:

Γ is separated into two connected components by a C^∞ submanifold Γ_0, and ν_1, ν_2 are tangent to Γ on Γ_0 and transversal to Γ_0. We write

$$\nu_i=\nu_{it}+\nu_{in} \quad (i=1, 2)$$

where ν_{in} is the normal component $(=\langle\nu_i, n\rangle n)$ and ν_{it} is the tangential component to Γ;

(4.2) the directions of ν_{1t} and ν_{2t} coincide in a neighborhood of Γ_0;

(4.3) the sigh of $\langle\nu_i, n\rangle$ $(i=1, 2)$ does not change near Γ_0 and $\langle\nu_i(x), n(x)\rangle$ has a zero of finite (even) order κ_i along the curves on Γ defined by the vector field ν_{it} (κ_i is constant on Γ_0).

Furthermore, the following inequality holds near Γ_0:

$$\text{(4.4)} \qquad \left|\left\langle\frac{\nu_2}{|\nu_{2t}|}, n\right\rangle\right| < \left|\left\langle\frac{\nu_1}{|\nu_{1t}|}, n\right\rangle\right|.$$

(The assumption (4.3) implies that $v_i (i=1, 2)$ belongs to the third class stated in Egorov-Kondrat'ev [5] or the author [14].) Let $\omega_1^+(x; \zeta)$, $\omega_2^+(x; \zeta)$ denote the roots of the equation $L_0(x, \zeta+\omega n)=0$ in ω with the positive imaginary part where L_0 is the principal part of L and ζ is any vector $(\neq 0)$ parallel to Γ;

(4.5) $\omega_1^+(x; \zeta)$ and $\omega_2^+(x; \zeta)$ are always purely imaginary in a neighborhood of Γ_0.

On these assumptions we obtain

Theorem 4.1. *Let the problem* (4.1) *be coercive outside* Γ_0 (*i.e., the Shapiro-Lopatinski condition* (*see* [10], [11], *etc.*) *is satisfied*) *and* (4.2)~(4.5) *hold. Then we have for any* $s \geqq 0$

$$\text{(i)} \quad \|u\|_{s+3+\varepsilon_0,\Omega} \leqq C\{\|Lu\|_{s,\Omega}+\|B_1u\|_{s+3/2,\Gamma}+\|B_2u\|_{s+5/2,\Gamma}+\|u\|_{s+3,\Omega}\}, \qquad u\in H_{s+4}(\Omega),$$

where $\varepsilon_0=\min\left(\frac{1}{\kappa_1+1}, \frac{1}{\kappa_2+1}\right)$, $B_1u=\left.\frac{\partial^2 u}{\partial v_1 \partial n}\right|_\Gamma$ and $B_2u=\left.\frac{\partial u}{\partial v_2}\right|_\Gamma$.

(ii) *The operator* $u \mapsto (Lu, B_1u, B_2u)$ *is of Fredholm type from* $H_{s+3+\varepsilon_0}(\Omega)$ *to* $H_s(\Omega)\times H_{s+3/2}(\Gamma)\times H_{s+5/2}(\Gamma)$.

REMARK. Let $\tilde{L}(x, D_x)$ be a strongly elliptic operator and replace L in (4.1) with $\tilde{L}+\mu^4$ (μ is a parameter $(\geqq 0)$). Then (4.1) is uniquely solvable for large μ, that is, for any $(f, g_1, g_2)\in H_s(\Omega)\times H_{s+3/2}(\Gamma)\times H_{s+5/2}(\Gamma)$ the solution $u\in H_{s+3+\varepsilon_0}(\Omega)$ is found uniquely in $H_3(\Omega)$ if μ is sufficiently large. This is obtained by means of Remark 3.1.

Proof of Theorem 4.1. Let $\mathcal{P}$ be the Poisson operator, that is, $\mathcal{P}$ is a continuous mapping from $\boldsymbol{H}^{(\boldsymbol{t})}_{s+5/2}(\Gamma)$ $(\boldsymbol{t}=(0, 1))$ to $H_{s+4}(\Omega)$ satisying $L(x, D_x)\mathcal{P}=0$ and $D\mathcal{P}=I+K_\Gamma$ (where $Du={}^t(D_nu|_\Gamma, u|_\Gamma)$ and K_Γ is continuous from $\boldsymbol{H}^{(\boldsymbol{t})}_{s+5/2}(\Gamma)$ to $\boldsymbol{H}^{(\boldsymbol{t})}_{s+7/2}(\Gamma)$). The construction of $\mathcal{P}$ is described in [10]. Set

$$T=B\mathcal{P} \quad (Bu={}^t(B_1u, B_2u)).$$

Then T is continuous from $\boldsymbol{H}^{(\boldsymbol{t})}_{s+5/2}(\Gamma)$ to $\boldsymbol{H}^{(\boldsymbol{s})}_{s+5/2}(\Gamma)$ $(\boldsymbol{s}=(-1, 0))$. If we have for T the estimate and the regularizer of the same type as for A in (3.1), we can obtain the theorem. In fact, combining the inequalities

$$\|u\|_{s+3+\varepsilon_0,\Omega} \leqq C_1(\|Lu\|_{s-\varepsilon_0,\Omega}+\|Du\|^{(\boldsymbol{t})}_{s+\varepsilon_0+3/2,\Gamma}+\|u\|_{s+3,\Omega})$$

and

$$\|h\|^{(\boldsymbol{t})}_{s+\varepsilon_0+3/2,\Gamma} \leqq C_2(\|Th\|^{(\boldsymbol{s})}_{s+5/2,\Gamma}+\|h\|^{(\boldsymbol{t})}_{s+3/2,\Gamma}),$$

we have

$$\|u\|_{s+3+\varepsilon_0,\Omega} \leqq C_3(\|Lu\|_{s,\Omega}+\|Bu\|^{(\boldsymbol{s})}_{s+5/2,\Gamma}+\|B(u-\mathcal{P}Du)\|^{(\boldsymbol{s})}_{s+5/2,\Gamma}+\|u\|_{s+3,\Omega}),$$

which yields the estimate (i) of the theorem. There exists an operator R_D continuous from $H_s(\Omega)$ to $H_{s+4}(\Omega)$ such that $LR_D=I+S_D$ and $DR_D=0$ where S_D is a continuous operator from $H_s(\Omega)$ to $H_{s+1}(\Omega)$. Using this R_D and the regularizer R of T, we set $\mathcal{R}(f, g)=\mathcal{P}R(g-BR_Df)+R_Df$ for $(f, g)\in H_s(\Omega)\times \boldsymbol{H}^{(s)}_{s+5/2}(\Gamma)$. Then $\mathcal{R}$ is the (right) regularizer for the problem (4.1) (i.e., $\mathcal{R}$ is continuous from $H_s(\Omega)\times \boldsymbol{H}^{(s)}_{s+5/2}(\Gamma)$ to $H_{s+3+\varepsilon_0}(\Omega)$, and $I-(L\mathcal{R}, B\mathcal{R})$ is a compact oprator on $H_s(\Omega)\times \boldsymbol{H}^{(s)}_{s+5/2}(\Gamma)$), which proves (ii) of the theorem. Therefore it suffices to examine T.

T is a matrix of pseudo-differential operators on Γ modulo a continuous operator from $\boldsymbol{H}^{(t)}_{s+5/2}(\Gamma)$ to $\boldsymbol{H}^{(s)}_{s+5/2+N'}(\Gamma)$ (N' is an arbitrary positive constant), whose (i, j)-element is of order $-s_i+t_j$ ($\boldsymbol{t}=(t_1, t_2)=(0, 1)$, $\boldsymbol{s}=(s_1, s_2)=(-1, 0)$). The principal symbol T_0 of T is expressed by the Lopatinski matrix of (4.1) on Γ. These are explained in Kumano-go [10].

Now, setting $M=\Gamma$ and $M_0=\Gamma_0$, we shall show that T satisfies all the assumptions for A in (3.1) by choosing appropriate local coordinates near Γ_0. Let $z(x)=\mathrm{dis}\,(x, \Gamma)$ $(x\in\Omega)$ (, which is a C^∞-function when z is small enough). From (4.2) and (4.3) we can take a set of local coordinates $\{(x^i, z)\}_{i=1,\cdots,N}$ covering Γ_0 such that (i) $\{x^i\}_{i=1,\cdots,N}$ is of the same type as $\{x^i\}$ stated in §3 and (ii) $\dfrac{\partial}{\partial \nu_j}$ $(j=1, 2)$ is transformed by $x^i=(t, y)$ $(i=1, \cdots, N)$ to

$$a_j(t, y)\frac{\partial}{\partial t}+t^{\kappa_j}b_j(t, y)\frac{\partial}{\partial z},$$

where a_j and b_j are not equal to 0 near the origin. Representing T locally by $x^i=(t, y)$ $(i=1, \cdots, N)$, its principal symbol $A_0(t, y; \tau, \eta)$ is of the following form (near $(t, y)=0$):

$$A_0(t, y; \tau, \eta)=\begin{bmatrix} -t^{\kappa_1}b_1(\omega_1^+ +\omega_2^+)-a_1\tau & t^{\kappa_1}b_1\omega_1^+\omega_2^+ \\ i\,t^{\kappa_2}b_2 & i\,a_2\tau \end{bmatrix}, \quad (|\tau, \eta|\geqq 1).$$

From this form and (4.5) it is seen that [A-I] and [A-II] (see §2) are satisfied. Since

$$\frac{\partial A_0}{\partial\tau}(t, y; 0, \eta)=\begin{bmatrix} -t^{\kappa_1}b_1\partial_\tau(\omega_1^+ +\omega_2^+)|_{\tau=0}-a_1 & t^{\kappa_1}b_1\partial_\tau(\omega_1^+\omega_2^+)|_{\tau=0} \\ 0 & ia_2 \end{bmatrix}, \quad (|\eta|\geqq 1),$$

[A-III] also holds. Finally let us check [A-IV]. We have for $\eta'\in S(=\{|\eta'|=1\})$

$$\begin{aligned} \tilde{A}_0(t, y; \eta') &= \frac{\partial A_0}{\partial\tau}(t, y; 0, \eta')^{-1}\cdot A_0(t, y; 0, \eta') \\ &= \{a_1a_2+t^{\kappa_1}a_2b_1\partial_\tau(\omega_1^+ +\omega_2^+)|_{\tau=0}\}^{-1} \\ &\quad\cdot\begin{bmatrix} t^{\kappa_1}\{a_2b_1(\omega_1^+ +\omega_2^+)+t^{\kappa_2}b_1b_2\partial_\tau(\omega_1^+\omega_2^+)\}|_{\tau=0} & -t^{\kappa_1}a_2b_1\omega_1^+\omega_2^+|_{\tau=0} \\ t^{\kappa_2}\{a_1b_2+t^{\kappa_1}b_1b_2\partial_\tau(\omega_1^+ +\omega_2^+)|_{\tau=0}\} & 0 \end{bmatrix}. \end{aligned}$$

(4.4) yields $\left|\dfrac{t^{\kappa_2}b_2}{a_2}\right|<\left|\dfrac{t^{\kappa_1}b_1}{a_1}\right|$ for small t, which implies that $\kappa_1<\kappa_2$ or $\kappa_1=\kappa_2$ &

$\left|\frac{a_1b_2}{a_2b_1}\right|<1$. In the latter case, by (4.5) we see that [A-IV] is satisfied ($l=1$ and $k_1=\kappa_1=\kappa_2$). Let us consider the former case. Set $k_1=\kappa_1$, $k_2=\kappa_2-\kappa_1$ ($l=2$). Then k_1 and k_2 are positive even integers. Since

$$\begin{aligned}&\det(\lambda-t^{-k_1}\tilde{A}_0(t,y;\eta'))\\&\quad=\lambda^2-(\alpha_2+t^{k_1+k_2}\beta_1)(a_1a_2+t^{k_1}\alpha_1)^{-1}\lambda+t^{k_2}\beta_2(a_1b_2+t^{k_1}\alpha_3)(a_1a_2+t^{k_1}\alpha_1)^{-2}\end{aligned}$$

(where $\alpha_1=a_2b_1\partial_\tau(\omega_1^++\omega_2^+)|_{\tau=0}$, $\alpha_2=a_2b_1(\omega_1^++\omega_2^+)|_{\tau=0}$, $\alpha_3=b_1b_2\partial_\tau(\omega_1^++\omega_2^+)|_{\tau=0}$, $\beta_1=b_1b_2\partial_\tau(\omega_1^+\omega_2^+)|_{\tau=0}$ and $\beta_2=a_2b_1\omega_1^+\omega_2^+|_{\tau=0}$), the eigen-values λ_1, λ_2 of $t^{-k_1}\tilde{A}_0(t,y;\eta')$ are of the forms

$$\begin{aligned}\lambda_1(t,y;\eta')&=(a_1a_2)^{-1}\alpha_2+O(t^{k_1}),\\\lambda_2(t,y;\eta')&=-t^{k_2}a_2^{-1}b_2\alpha_2^{-1}\beta_2+O(t^{k_2+1}),\end{aligned}$$

where $O(t^k)$ means that $t^{-k}O(t^k)$ is smooth (in t, y and η'). On the other hand, we have

$$(I-P^1)=\frac{1}{2\pi i}\oint_{|\lambda-\lambda_2|=\delta}(\lambda-t^{-k_1}\tilde{A}_0)d\lambda=(\lambda_2-\lambda_1)^{-1}\operatorname{cof}[\lambda_2-t^{-k_1}\tilde{A}_0].$$

Therefore,

$$\begin{aligned}t^{-k_1}\tilde{A}_0(I-P^1)&=\frac{\lambda_2}{\lambda_2-\lambda_1}\operatorname{cof}[\lambda_2-t^{-k_1}\tilde{A}_0]\\&=t^{k_2}\begin{bmatrix}0&-a_2^{-1}b_2\alpha_2^{-2}\beta_2^2\\0&-a_2^{-1}b_2\alpha_2^{-1}\beta_2\end{bmatrix}+O(t^{k_2+1}).\end{aligned}$$

Hence, by (4.5) we see that [A-IV] is satisfied. The proof is complete.

Appendix. Proof of Proposition 2.1

Proposition 2.1 is derived from the following lemma:

Lemma A. *Let A' be a constant $m'\times m'$-matrix whose eigen-values all have non-vanishing imaginary parts, and let k be a constant positive integer.*

(i) *If either 'k is even' or 'k is odd and every imaginary part of the eigen-value is positive', we have the estimate*

$$\begin{aligned}C^{-1}(\|D_tw(t)\|_{0,R^1}+\|t^kw\|_{0,R^1})&\leq\|(D_t+t^kA')w\|_{0,R^1}\\&\leq C(\|D_tw\|_{0,R^1}+\|t^kw\|_{0,R^1}),\quad w(t)\in\mathcal{S},\end{aligned}$$

where the constant C can be taken uniformly in η' when $A'=A'(\eta')$ ($A'(\eta')$ is stated in Proposition 2.1).

(ii) *If k is even, the operator*

$$(D_t+t^kA'):\ W_1^k(\boldsymbol{R}^1)\rightarrow L^2(\boldsymbol{R}^1)$$

is a topological isomorphism ($W_1^k(\boldsymbol{R}^1)=\{w(t)\in H_1(\boldsymbol{R}^1);\ t^k w(t)\in L^2(\boldsymbol{R}^1)\}$).

Transforming A' to Jordan's normal form, we can prove this lemma in the same way as in the proof of Theorem 2.1 in the author [14].

Proof of Proposition 2.1. The idea of the proof is referred to Višik-Grušin [16], Grušin [7]. By the change of the variable: $t=|\eta|^{-1/(k+1)}t'$, we have

$$\begin{aligned}
&||D_t w(t)||_{0,\boldsymbol{R}^1}^2 = |\eta|^{1/(k+1)}||D_{t'}w'(t')||_{0,\boldsymbol{R}^1}^2 \quad (w'(t')=w(|\eta|^{-1/(k+1)}t')),\\
&||t^k|\eta|w(t)||_{0,\boldsymbol{R}^1}^2 = |\eta|^{1/(k+1)}||t'^k w'(t')||_{0,\boldsymbol{R}^1}^2,\\
&||\{D_t+t^k A'\Big(\frac{\eta}{|\eta|}\Big)|\eta|\theta(\eta)\}w(t)||_{0,\boldsymbol{R}^1}^2 = |\eta|^{1/(k+1)}||\{D_{t'}+t'^k A'\Big(\frac{\eta}{|\eta|}\Big)\theta(\eta)\}w'(t')||_{0,\boldsymbol{R}^1}^2 .
\end{aligned}$$

Therefore, from (i) of Lemma A it follows that

$$\begin{aligned}
&C^{-1}\Big(\int_{|\eta|\geqq 1}||D_t v(t,\eta)||_{0,\boldsymbol{R}^1}^2 d\eta+\int_{|\eta|\geqq 1}||t^k\langle\eta\rangle v(t,\eta)||_{0,\boldsymbol{R}^1}^2 d\eta\Big)\\
&\leqq\int_{|\eta|\geqq 1}||(D_t+L'(t;\eta))v||_{0,\boldsymbol{R}^1}^2 d\eta\\
&\leqq C\Big(\int_{|\eta|\geqq 1}||D_t v||_{0,\boldsymbol{R}^1}^2 d\eta+\int_{|\eta|\geqq 1}||t^k\langle\eta\rangle v||_{0,\boldsymbol{R}^1}^2 d\eta\Big),\ v(t,\eta)\in\mathcal{S},
\end{aligned}$$

which proves (i) of Proposition 2.1.

Let $\psi(\eta)$ ($\in C^\infty$)$=1$ for $|\eta|\geqq 2$ and $\psi(\eta)=0$ for $|\eta|\leqq 1$. By (ii) of Lemma A the operator $D_t+t^k A'\Big(\frac{\eta}{|\eta|}\Big)|\eta|$ has an inverse Q_η for any $\eta(\neq 0)$. We define

$$R'f(t,y)=\mathscr{F}_{\eta\to y}^{-1}[Q_\eta\psi(\eta)\tilde{f}(t,\eta)],$$

where $\tilde{f}(t,\eta)=\int e^{-iy\eta}f(t,y)dy$. Then R' satisfies the requirement of (ii) in Proposition 2.1.

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References

[1] M.S. Agranovich: *Elliptic singular integro-differential operators*, Uspehi Mat. Nauk **20** no. 5 (1965), 3–120=Russian Math. Surveys **20** no. 5/6 (1965), 1–122.

[2] A.P. Calderón and R. Vaillancourt: *A class of bounded pseudo-differential operators*, Proc. Nat. Acad. Sci. U.S.A. **69** (1972), 1185–1187.

[3] Ju. V. Egorov: *Subelliptic operators*, Uspehi Mat. Nauk **30** no. 2 (1975), 57–114= Russian Math. Surveys **30** no. 2 (1975), 59–118.

[4] Ju. V. Egorov: *On subelliptic operators*, Uspehi Mat. Nauk **30** no. 3 (1975), 57–104 =Russian Math. Surveys **30** no. 3 (1975).

[5] Ju. V. Egorov and V.A. Kondrat'ev: *The oblique derivative problem*, Mat. Sb. **78** (120) (1969), 148–176=Math. USSR Sb. **7** (1969), 139–169.

[6] G.I. Èskin: *Degenerate elliptic pseudodifferential equations of principal type*, Mat. Sb. **82** (124) (1970), 585–628 = Math. USSR Sb. **11** (1970), 539–582.

[7] V.V. Grušin: *On a class of hypoelliptic operators*, Mat. Sb. **83** (125) (1970), 458–476 = Math. USSR Sb. **12** (1970), 456–473.

[8] H. Kumano-go: *Algebras of pseudo-differential operators*, J. Fac. Sci. Univ. Tokyo **17** (1970), 31–50.

[9] H. Kumano-go: *Pseudo-differential operators of multiple symbol and the Calderón-Vaillancourt theorem*, J. Math. Soc. Japan **27** (1975), 113–120.

[10] H. Kumano-go: Pseudo-differential operators, Iwanami Publisher, Tokyo, 1974 (in Japanese).

[11] J.L. Lions and E. Magenes: Non homogeneous boundary value problems and applications I, Springer Verlag, Berlin, 1972.

[12] M. Nagase: *On the Cauchy problem for parabolic pseudo-differential equations*, Osaka J. Math. **11** (1974), 239–264.

[13] R. Seeley: *Topics in pseudo-differential operators*, Pseudo-Diff. Operators (C.I.M.E., Stresa, 1968), Edizioni Cremonese, Rome (1969), 169–305.

[14] H. Soga: *Boundary value problems with oblique derivative*, Publ. Res. Inst. Math. Sci. Kyoto Univ. **10** (1975), 619–668.

[15] H. Soga: *Degenerate elliptic systems of pseudodifferential equations*, Proc. Japan Acad. **52** (1976), 417–419.

[16] M.I. Višik and V.V. Grušin: *On a class of higher order degenerate elliptic equations* , Mat. Sb. **79** (121) (1969) 3–36 = Math. USSR Sb. **8** (1969), 1–32.

[17] M.I. Višik and V.V. Grušin: *Elliptic pseudodifferential operators on a closed manifold which degenerate on a submanifold*, Dokl. Akad. Nauk SSSR **189** (1969), 16–19 = Soviet Math. Dokl. **10** (1969), 1316–1320.