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ON MULTIPLY TRANSITIVE GROUPS

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1. Introduction

The known 4-fold transitive groups are A_n $(n\geq 6)$, S_n $(n\geq 4)$, M_{11}, M_{12}, M_{23} and M_{24}. Let G be one of these and assume G is a $(4, \mu)$-group on Ω with $\mu\geq 4$. Here we say that G is a (k, μ)-group on Ω if G is k-transitive on Ω and μ is the maximal number of fixed points of involutions in G. Let t be an involution in G with $|F(t)|=\mu$, then $G^{F(t)}=G(F(t))/G_{F(t)}$ is also a 4-fold transitive group. Here we set $F(t)=\{i\in\Omega \mid i^t=i\}$ and denote by $G(F(t))$, $G_{F(t)}$, the global, pointwise stabilizer of $F(t)$ in G, respectively.

In this paper we shall prove the following

Theorem 1. *Let G be a 4-fold transitive group on Ω. Assume that there exists an involution t in G satisfying the following conditions.*

(i) *G is a $(4, \mu)$-group on Ω where $\mu=|F(t)|$.*

(ii) *$G^{F(t)}$ is a known 4-fold transitive group; A_n $(n\geq 6)$, S_n $(n\geq 4)$ or M_n $(n=11, 12, 23$ or $24)$.*

Then G is also one of the known 4-fold transitive groups.

This theorem is a generalization of the Theorem of T. Oyama of [10]: the case that $G^{F(t)}\simeq A_n$ $(n\geq 6)$, S_n $(n\geq 4)$ or M_{12} has been proved by T. Oyama and the case that $G^{F(t)}\simeq M_{11}$, M_{23} or M_{24} by the author.

To consider the case that $G^{F(t)}\simeq M_{23}$ or M_{24}, we shall prove the following theorem in §3 and §4.

Theorem 2. *Let G be a $(1, 23)$-group on Ω. If there exists an involution t such that $|F(t)|=23$ and $G^{F(t)}\simeq M_{23}$. Then we have*

(i) *If P is a Sylow 2-subgroup of $G_{F(t)}$, then P is cyclic of order 2 and $N_G(P)\cap g^{-1}Pg\leq P$ for any $g\in G$.*

(ii) *$|\Omega|=69$ and G is imprimitive on Ω.*

(iii) *$O(G)\neq 1$ and is an elementary abelian 3-group. If we denote by ψ the set of $O(G)$-orbits on Ω, then $|\psi|=23$ and $G^{\psi}\simeq M_{23}$.*

It follows from this theorem that there is no $(3, 24)$-group such that for an involution t fixing exactly twenty-four points $G^{F(t)}\simeq M_{24}$.

In the remainder of this section we introduce some notations: Let G be a permutation group on Ω. For $X\leq G$ and $\Delta\subseteq\Omega$, we define $F(X)=\{i\in\Omega\mid i^x=i$ for all $x\in X\}$, $X(\Delta)=\{x\in X\mid\Delta^x=\Delta\}$, $X_\Delta=\{x\in X\mid i^x=i$ for every $i\in\Delta\}$ and $X^\Delta=X(\Delta)/X_\Delta$. If p is a prime, we denote by $O^p(X)$, the subgroup of X generated by all p'-elements in X and by $O^{p'}(X)$, the subtroup of X generated by all p-elements in X. $I(X)$ is the set of involutions in X.

Other notations are standard (cf. [6], [13]).

2. Preliminaries

First we describe the various properties of M_{23}.

(i) M_{23} is a 4-fold transitive group on twenty-three points $\{1, 2\cdots, 23\}$ and a Sylow 2-subgroup of the stabilizer of four points in M_{23} is of order 2^4. It has a seven fixed points and acts regularly on the remaining points.

(ii) M_{23} is a (4, 7)-group and has a unique conjugate class of involutions.

(iii) M_{23} is a simple group and the outer automorphism group of it is trivial.

(iv) The centralizer of an involution $\tilde{w}$ in M_{23} is a split extention of an elementary abelian normal subgroup $\tilde{E}$ of order 2^4 by a group $\tilde{M}$ which is isomorphic to $GL(3, 2)$.

(v) The center of a Sylow 2-subgroup of M_{23} is cyclic of order 2.

Set $\tilde{C}=C(\tilde{w})$ and $F(\tilde{w})=\Delta=\{1, 2, 3, 4, 5, 6, 7\}$. Then we have

(vi) $\tilde{E}^\Delta\simeq 1$ and $\tilde{E}$ is regular on $\{8, 9, \cdots, 23\}$.

(vii) $\tilde{M}$ is doubly transitive on Δ.

(viii) $M_{23}{}^\Delta\simeq A_7$ and $M_{23}(\Delta)=N(\tilde{E})$.

(ix) $O(\tilde{C})=1$, $O^2(\tilde{C})=\tilde{C}$ and $O^{7'}(\tilde{C})=\tilde{C}$

We now prove the following lemmas.

Lemma 1. *Let P be a 2-group and ϕ an automorphism of P of order 2. If $|C_P(\phi)|\leq 2^a$, then $|\Omega_1(P/P')|\leq 4^a$.*

Proof. Set $|\Omega_1(P/P')|=2^r$ and $Q/P'=\Omega_1(P/P')\cap C(\phi)$. Then $|Q/P'|\geq 2^{1/2r}$ (cf. (2.7) of [8]). Since $[\phi, Q]\leq P'$, $(\langle\phi\rangle Q)'\leq P'$, whence $|\langle\phi\rangle Q:(\langle\phi\rangle Q)'|\geq 2^{1/2r+1}$. On the other hand $|C_{\langle\phi\rangle Q}(\phi)|=|\langle\phi\rangle C_Q(\phi)|\leq 2^{a+1}$ and so $|\langle\phi\rangle Q:(\langle\phi\rangle Q)'|\leq 2^{a+1}$ (cf. (2.8) of [8]). Thus $r\leq 2a$.

Lemma 2. *Let (G,Ω) be a $(1, 23)$-group. Suppose there exists an involution t such that $|F(t)|=23$ and $G^{F(t)}\simeq M_{23}$. If P is a Sylow 2-subgroup of $G_{F(t)}$, then one of the following holds.*

(i) $C_G(P)^{F(P)}\simeq M_{23}$ and there is an involution u in $N_G(P)-P$ satisfying $u^G\cap P\neq\phi$.

(ii) $N_G(P)^{F(P)}\simeq M_{23}$ and $N_G(P)\cap g^{-1}Pg\leq P$ for every $g\in G$.

Proof. Since $G(F(t))=N_G(P)G_{F(t)}$, we have $N_G(P)^{F(P)}\simeq M_{23}$. Suppose that

$N_G(P) \cap g^{-1}Pg \nleq P$ for some g in G. Since $F(P) \neq F(g^{-1}Pg)$, there is an involution u in $g^{-1}Pg$ satisfying (i). As $|F(u^{F(P)})|=7$ (cf. (ii) of §2) and $|F(u)|=23$, $|((\Omega-F(P)) \cap F(u)|=16$ and so $|C_P(u)| \leq 16$ by the semi-regularity of P on $\Omega-F(P)$. By Lemma 1, $|\Omega_1(P/P')| \leq 2^8$. Since $|GL(n, 2)|$ is not divisible by the prime 23 when $1 \leq n \leq 8$, $O^{23'}(N_G(P))$ is a normal subgroup of $N_G(P)$ contained in $C_G(P)$ by Theorem 5.1.4 and 5.2.4 of [6]. Thus we obtain $C_G(P)^{F(P)} \simeq M_{23}$.

According as the lemma, the proof of Theorem 2 is divided into two cases.

3. Case (i)

In this section, we prove that the case (i) does not occur.

(3.1) The following hold.

(i) P is cyclic of order 2 and so we can choose P such that $P=\langle t\rangle$.

(ii) $N_G(P)=C_G(t)=\langle t\rangle \times O^2(C_G(t))$.

(iii) Set $O^2(C_G(t))=L(t)$. Then $L(t)/O(L(t)) \simeq M_{23}$, $O(L(t))^{F(t)}=1$, $t \notin \{g^2 \mid g \in G\}$ and $L(t)$ has a unique conjugate class of involutions.

(iv) Let s be an involution of $L(t)$, then $s \in \{g^2 \mid g \in G\}$, $I(C_G(t)) \subseteq t^G \cup s^G$, $t \not\sim s$ and s is a central involution.

Proof. Since P is a Sylow 2-subgroup of $N_G(P)_{F(P)}$, $Z(P)$ is a unique Sylow 2-subgroup of $C_G(P)_{F(P)}$ and so we have $C_G(P)_{F(P)}=Z(P) \times O(C_G(P))$. Set $\overline{C_G(P)}=C_G(P)/O(C_G(P))$. Considering the normal series of $C_G(P)$, $Z(\overline{C_G(P)})=\overline{Z(P)}$ and $\overline{C_G(P)}/\overline{Z(P)} \simeq M_{23}$. As the Schur multiplier of M_{23} is trivial ([7]), there exists a subgroup $\bar{L}$ of $\overline{C_G(P)}$ such that $\overline{C_G(P)}=\overline{Z(P)} \times \bar{L}$ and $\bar{L} \simeq M_{23}$. Let L be the inverse image of $\bar{L}$ in $C_G(P)$. Then $C_G(P)=Z(P)O(C_G(P))L$, hence $C_G(P)=Z(P) \times L$ because $O(C_G(P)) \leq L$. Since $L=O^2(C_G(P))$, $P \times L$ is a normal subgroup of $N_G(P)$ and so $O^{2'}(N_G(P)) \leq P \times L$. Hence if u is an involution satisfying (i) of Lemma 2 there are an element v in $I(P) \cup \{1\}$ and w in $I(L)$ with $u=vw$. Clearly $\tilde{C} \simeq C_{\bar{L}}(\bar{w})=\overline{C_L(w)} \simeq C_L(w)/O(C_L(w))$ where $O(C_L(w))=O(L) \cap C_L(w)$ (cf. (ix) of §2). We denote $O(C_L(w))=H$. Then $C_L(w)/H$ is isomorphic to $\tilde{C}$ and $C_L(w)/H=E/H \cdot M/H$ such that $E/H=E^{F(P)} \simeq E_{16}$, $E^{F(P) \cap F(w)}=1$, $C_L(w)^{F(P) \cap F(w)}=M^{F(P) \cap F(w)}=M/H \simeq GL(3, 2)$, E is a normal subgroup of $C_L(w)$ and $E^{F(P)} \cap M^{F(P)}=1$. By the fact that u is conjugate to some element of P, $G^{F(u)} \simeq M_{23}$ and it follows that either $y^{F(u)}=1$ or $y^{F(u)}$ is an involution for y in $I(E)$. If $y^{F(u)}=1$, then $F(y) \supseteq F(u)$. If $y^{F(u)}$ is an involution, $|F(y^{F(u)})|=7$ and so $F(y) \cap F(u)=F(u) \cap F(P)$ because $F(u) \cap F(P) \subseteq F(y) \cap F(u)$ and $|F(u^{F(P)})|=|F(w^{F(P)})|=7$.

We argue $F(y) \cap F(u)=F(u) \cap F(P)$ for any y in $I(E)$. Suppose $F(y) \supseteq F(u)$. Since $|F(y)| \leq 23$, $F(y)=F(u)$ and hence $\langle y, u\rangle$ is contained in a Sylow 2-subgroup of $G_{F(u)}$ and so $y^G \cap P \neq \phi$. Since $G^{F(y)} \simeq M_{23}$, $[P, y]=1$, $F(P) \cap F(y)=F(P) \cap F(u)$ and P is semi-regular on $\Omega-F(P)$, we have $P \simeq P^{F(y)}$ and P is an elementary abelian 2-group of order at most 16. Hence any element which is conjugate to some element of $P-\{1\}$ is not a square of any element in G. But the element y in L is a square of some element in L because $L/O(L)\simeq M_{23}$ and (ii) of §2, which is a contradiction. This shows that $F(u)\cap F(y)=F(u)\cap F(P)$ for any y in $I(E)$.

Set $\Delta=F(u)-F(P)=F(u)-F(y)$. Since $|F(u)-F(P)|=|F(u)-(F(u)\cap F(P)|=16$ and a Sylow 2-subgroup T of E is isomorphic to E_{16}, T acts regularly on Δ.

We argue $|P|=2$. Suppose $|P|\geq 4$. Then $|C_P(v)|\geq 4$. Since $C_P(v)$ is semi-regular on Δ and $[C_P(v), C_L(w)]=1$, we have $O^{7'}(C_L(w))^{\Delta}=1$. As $E\triangleright O(C_L(w))$, $O(C_L(w))^{\Delta}=1$ and so by (ix) of §2, $C_L(w)^{\Delta}=1$, a contradiction. Thus (i), (ii) and (iii) are proved.

Let s be an involution of $L(t)$. Since t is not a square of any element of G, t is not conjugate to s and u is of the form tw where w is an element in $I(L(t))$. On the other hand w is conjugate to s in $L(t)$ by (iii) and so u is conjugate to ts. Hence t is conjugate to ts. The four-group $\langle t, s\rangle$ is the center of a Sylow 2-subgroup of $C_G(t)$ by (v) of §2. Hence to complete the proof of (iv), we may assume t is not a central involution. Since $\langle t, s\rangle$ contains a central involution and $t\sim ts$, s must be a central involution. Thus (iv) is proved.

(3.2) Let notations be as in (3.1). Then

(i) If $t_1\in t^G$, $u_1\in I(G)$ and $[t_1, u_1]=1$, then $t_1=u_1$ or $|F(t_1)\cap F(u_1)|=7$.

(ii) There exist an involution s in $L(t)$ and a four-group $\{u_i|0\leq i\leq 3\}$ of $L(t)$ satisfying the following.

$u_0=1$. $[s, u_i]=1$, $F(tu_i)\cap F(u_j)=F(t)\cap F(\langle u_1, u_2\rangle)$ if $0\leq i, j\leq 3$ and $j\neq 0$. Set $F(t)\cap F(\langle u_1, u_2\rangle)=\Delta$. Then $|\Delta|=7$ and $|F(s)\cap\Delta|=3$.

Proof. By (ii) and (iii) of (3.1), (i) is obvious.

Let w, E and M be as in the proof of (3.1) and s an involution in M. Let T be a Sylow 2-subgroup of E normalized by s. Since T is isomorphic to E_{16}, there is a subgroup $\{1, u_1, u_2, u_3\}$ of T centralized by s (cf. Lemma 1). By (vi) of §2, $|F(T)\cap F(t)|=7$ and T is regular on $F(t)-F(T)$ and so $|F(t)\cap F(\langle u_1, u_2\rangle)|=|\Delta|=7$. Since $F(tu_i)\cap F(u_j)$ contains Δ, $F(tu_i)\cap F(u_j)=\Delta$ follows from (i). By (viii) of §2, $|F(s)\cap(F(t)\cap F(T))|=3$, hence $|F(s)\cap\Delta|=3$.

(3.3) Let s, $\{u_0, u_1, u_2, u_3\}$ be as in (ii) of (3.2). For $t_1\in t^G$ and $s_1\in I(L(t_1))$, we set $L(t_1)\cap C(s_1)=L(t_1, s_1)$. Then we have

(i) Set $\Gamma_i=F(tu_i)\cap F(s)$ and $N_i=L(tu_i, s)$ $(0\leq i\leq 3)$, then $|\Gamma_i|=7$, $F(s)\supseteq\bigcup_{i=0}^{3}\Gamma_i$, $\Gamma_k\cap\Gamma_l=\bigcap_{i=0}^{3}\Gamma_i$ $(k\neq l)$, $|\bigcap_{i=0}^{3}\Gamma_i|=3$ and $N_i/O(N_i)=N_i^{F(tu_i)}\simeq\tilde{C}$.

(ii) There exist subgroups E_i, M_i of N_i for each $i\in\{0, 1, 2, 3\}$ such that $N_i/O(N_i)=E_i/O(N_i)\cdot M_i/O(N_i)\triangleright E_i/O(N_i)$, $E_i/O(N_i)\simeq E_{16}$, $M_i/O(N_i)\simeq GL(3, 2)$, $E_i^{\Gamma_i}=1$, $N_i^{\Gamma_i}=M_i^{\Gamma_i}\simeq GL(3, 2)$ and $M_i^{\Gamma_i}$ is doubly transtive.

Proof. By the choice of s and u_i $(0\leq i\leq 3)$, (i) is clear. Since tu_i is conjugate to t for each i, we can define E_i and M_i in exactly the same way as E and M mentioned in the proof of (3.1). From this, (ii) immediately follows.

(3.4) Let notations be as in (3.1), (3.2) and (3.3). Then

(i) There is a $C_G(s)$-orbit Λ on $F(s)$ with $F(s)\supseteq\Lambda\supseteq\bigcup_{i=0}^{3}\Gamma_i$.

(ii) $|\Lambda|=19$, 21 or 23 and $|F(s)|=19$, 21 or 23.

(iii) If $k\in\Lambda$, then $C_G(s)_k$ has an orbit on $\Lambda-\{k\}$ of length at least 18.

(iv) If $|\Lambda|=19$, then $C_G(s)^\Lambda\simeq A_{19}$ or S_{19}.

Proof. Since $N_i\leq C_G(s)$ and $N_i^{\Gamma_i}$ is doubly transitive for i with $0\leq i\leq 3$, (i) follows immediately from (i) of (3.3). By assumption, $|F(s)|\leq 23$ and obviously $|\bigcup_{i=0}^{3}\Gamma_i|=19$, hence $19\leq|\Lambda|\leq 23$. On the other hand $\Lambda\supseteq\Gamma_0=F(\langle t, s\rangle)$, so $|\Lambda|$ is odd. Thus (ii) holds. To prove (iii), we may assume $k\in\bigcap_{i=0}^{3}\Gamma_i$. Since $(N_i)_k\leq C_G(s)_k$ and $(N_i)_k$ is transitive on $\Gamma_i-\{k\}$, we have (iii).

Now suppose $|\Delta|=19$. Then $C_G(s)^\Lambda$ is primitive and $N_i^{\Gamma_i}\simeq GL(3, 2)$. Hence $C_G(s)^\Lambda$ posseses an element of order 7. By Theorem 13.10 of [13], $C_G(s)^\Lambda\geq A_{19}$ holds and (3.4) is proved.

(3.5) Let notations be as in (3.1)—(3.4). There exists a Sylow 2-subgroup Q of $G_{F(s)}$ such that $s\in Z(Q)$ and $t\in N_G(Q)$. Let Γ be the $G^{F(s)}$-orbit containing Λ. Then

(i) $F(Q)=F(s)$, $G^{F(s)}=N_G(Q)^{F(s)}$ and $|\Gamma|=19$, 21 or 23.

(ii) If $k\in\Gamma$, then $N_G(Q)_k$ has an orbit on $\Gamma-\{k\}$ of elngth at least 18.

(iii) If $|\Gamma|=19$, then $N_G(Q)^\Gamma\simeq A_{19}$ or S_{19}.

(iv) If $|\Gamma|=21$, then $N_G(Q)^\Gamma\simeq A_{21}$ or S_{21}.

(v) If $|\Gamma|=23$, then $N_G(Q)^\Gamma\simeq A_{23}$ or S_{23}.

Proof. Let T be a Sylow 2-subgroup of $C_G(s)$ containing t. As s is a central involution by (iv) of (3.1) and $C_G(s)\leq G(F(s))$, T is a Sylow 2-subgroup of $G(F(s))$. Set $Q=T\cap G_{F(s)}$. Then Q satisfies the condition of (3.5). Now we prove (i)—(v). (i), (ii) and (iii) follow immediately from (3.4).

To prove (iv), first we argue that $N_G(Q)^\Gamma$ is primitive. If $|\Lambda|=19$, $C_G(s)^\Gamma$ posseses an element of order 19 by (iv) of (3.4), hence $N_G(Q)^\Gamma$ is primitive. Therefore we may assume $|\Lambda|=|\Gamma|=21$ and we argue that $C_G(s)^\Lambda$ is primitive. Suppose $C_G(s)^\Lambda$ is imprimitive. Let B_1 be a nontrivial block of $C_G(s)^\Lambda$, then by (iii) of (3.4) we have $|B_1|=3$. Let $\Pi=\{B_1, B_2 \cdots, B_7\}$ be a complete system of blocks. Since N_i is transitive on Π and $[N_i, tu_i]=1$, tu_i fixes all blocks in Π. Hence $F(tu_i)\cap B_l\neq\phi$ for every l with $1\leq l\leq 7$. On the other hand $|F(tu_i)\cap\Lambda|=7$, hence $|F(tu_i)\cap B_l|=1$. From this $(tu_i\, tu_j)^\Lambda=(u_iu_j)^\Lambda=1$ for any $i, j\in\{0, 1, 2, 3\}$. If $F(Q)\neq\Lambda$, then $|F(Q)-\Lambda|=2$ and so $(tu_itu_j)^{\Lambda_1}=(u_iu_j)^{\Lambda_1}=1$ where $\Lambda_1=F(Q)-\Lambda$. Hence $F(\langle u_1, u_2\rangle)=F(Q)=F(s)$, whihc is contrary to (ii) of (3.2). Thus $N_G(Q)^\Gamma$ is primitive.

Next we shall show that we many assume $E_0^{F(Q)}=1$. Since $M_0^{\Gamma_0}\simeq GL(3, 2)$ and $M_0\leq G(F(Q))$, $M_0^{F(Q)}$ posesses an element of order 7. We may assume this element has no fixed point on Γ, for otherwise we obtain $N_G(Q)^{\Gamma}\geq A_{21}$ by Theorem 13.10 of [13]. Hence an arbitrary M_0-orbit on Γ has length 7 or 14 and so $O(M_0)^{\Gamma}=1$ holds because $M_0/O(M_0)=M_0^{\Gamma_0}\simeq GL(3, 2)$. Hence $O(M_0)^{F(Q)}=1$. Set $\Gamma-F(t)=\Delta_0$. Then $\Delta_0=\Gamma-\Gamma_0$ and $|\Delta_0|=14$. Since the element of $M_0^{F(Q)}$ of order 7 as above and the element t have no fixed point on Δ_0, $\langle t\rangle\times N_0$ is transitive on Δ_0. It follows from $N_0\triangleright E_0$ that the orbits of $\langle t\rangle\times E_0$ on Δ_0 form a complete system of blocks of $\langle t\rangle\times N_0$. We denote this $\prod=\{B_1, \cdots, B_r\}$. Since $O(M_0)=O(N_0)$, $O(M_0)^{F(Q)}=1$ and $E_0/O(N_0)\simeq E_{16}$, we have $\langle t\rangle\times E_0$ is a 2-group on Δ_0. Hence $|B_1|=2$ and $r=7$. By (i) of (3.2), $F(s)\cap F(tv)=F(s)\cap F(t)$ holds for every $v\in I(E_0)$ and so $\Delta_0\cap F(tv)=\phi$. Hence $v^{B_k}=t^{B_k}tv^{B_k}=1$ for each B_k with $1\leq k\leq 7$, which implies $E_0^{\Gamma}=1$. If $F(Q)\neq\Gamma$, then $|F(Q)-\Gamma|=2$. Since $(F(Q)-\Gamma)\cap F(tv)=(F(s)\cap F(tv))-\Gamma=\phi$ for every $v\in I(E_0)$, we get $v^{F(Q)-\Gamma}=t^{F(Q)-\Gamma}\, tv^{F(Q)-\Gamma}=1$. Thus $E_0^{F(Q)}=1$.

We denote $L(t)^{F(t)}=\overline{L(t)}$. Since $\overline{L(t)}=L(t)/O(L(t))$ and $O(E_0)=E_0\cap O(L(t))$, we have $(\overline{L(t)}\cap N(\overline{E}_0))^{\Gamma_0}\simeq A_7$ by (viii) of §2. Hence $(L(t)\cap N(E_0O(L(t))))^{\Gamma_0}\simeq A_7$ and so if T is a Sylow 2-subgroup of E_0, we have $N_{L(t)}(T)^{\Gamma_0}\simeq A_7$. We note that $F(T)=F(Q)$ because $E_0^{F(Q)}=1$ and $L(t)$ has a unique conjugate class of involutions. So we have $N_{L(t)}(T)\leq G(F(Q))\cap G(\Gamma_0)$. Let y_0 be a 5-element of $N_{L(t)}(T)$ such that the order of $y_0^{\Gamma_0}$ is 5. Since $y_0\in G(F(Q))\cap G(\Gamma_0)$, we get $y_0\in G(\Gamma)\cap G(\Gamma_0)$. Therefore $|F(y_0^{\Gamma})|\geq 6$. As $N_G(Q)^{\Gamma}$ is primitive, it follows from Theorem 13.10 of [13] atht $N_G(Q)^{\Gamma}\geq A_{21}$. Thus (iv) is proved.

Finally we prove (v). If $|\Gamma|=23$, $F(Q)=\Gamma$. Since $G^{\Gamma}\geq N_i^{\Gamma}$ and N_i^{Γ} involves the group isomorphic to $GL(3, 2)$, G^{Γ} is not solvable. Hence by the result of [11], we have $G^{\Gamma}\simeq M_{23}$, A_{23} or S_{23}. If $G^{\Gamma}=N_G(Q)^{F(Q)}\simeq M_{23}$, we can apply (iii) of (3.1) to s and obtain $s\notin\{g^2|g\in G\}$, which is contrary to (iv) of (3.1). (Here we note that $I(L(t))\subseteq s^G$ and hence (i) of Lemma 2 occurs with respect to s.)

(3.6) Let notations be as in (3.5). We set $N=C_G(Q)$ if $F(Q)=\Gamma$ and $N=C_G(Q)_{\psi}$ where $\psi=F(Q)-\Gamma$ if $F(Q)\neq\Gamma$. Then $N^{\Gamma}\geq A_{|\Gamma|}$.

Proof. Since $|\Gamma\cap F(t)|=7$, by (i) of (3.2) $C_Q(t)$ acts semi-regularly on $F(t)-\Gamma$ and so $|C_Q(t)|\leq 16$. Hence $|\Omega_1(Q/Q')|\leq 2^8$ by Lemma 1. Since $GL(n, 2)$ is a $19'$-group when $1\leq n\leq 8$, $O^{19'}(N_G(Q))$ is a normal subgroup of $N_G(Q)$ contained in $C_G(Q)$ by Theorem 5.1.4 and 5.2.4 of [6]. Hence $C_G(Q)^{\Gamma}\geq A_{|\Gamma|}$ by (iii), (iv) and (v) of (3.5), so that $N^{\Gamma}\geq A_{|\Gamma|}$ because $|\psi|\leq 4$.

(3.7) We have now a contradiction in the following way.

Let notations be as in (3.1)—(3.6). Set $H=\langle t\rangle N$. We denote $H^{\Gamma}=\bar{H}$. Since $|F(\bar{t})|=7$ and by (3.6) $\bar{N}\geq A_{|\Gamma|}$, there exists in N an element v such that the order of $\bar{v}$ is 5, $[\bar{t}, \bar{v}]=1$ and $\bar{v}^{F(t)}\neq 1$. We may assume v is a 5-element.

Cleary v normalizes $\langle t\rangle N_\Gamma$. Since $Z(Q)$ is a unique Sylow 2-subgroup of N_Γ, $\langle t\rangle Z(Q)$ is a Sylow 2-subgroup of $\langle t\rangle N_\Gamma$. By the Frattini argument there is a 5-element w in N such that $\bar{v}=\bar{w}$ and w normalizes $\langle t\rangle Z(Q)$. It follows from $Z(Q)\leq Z(N)$ that w stabilizes a normal series $\langle t\rangle Z(Q)\triangleright Z(Q)\triangleright 1$. By Theorem 5.3.2 of [6], w centralizes $\langle t\rangle Z(Q)$ and hence $w\in L(t, s)$. Since $F(t)\cap F(s)=F(t)\cap\Gamma$, $w^{F(t)\cap F(s)}=v^{F(t)\cap\Gamma}\neq 1$. Hence $L(t, s)^{F(t)\cap F(s)}\simeq GL(3, 2)$ has a nontrivial 5-element, a contradiction.

4. Case (ii)

In this section we shall prove that if the case (ii) of Lemma 2 holds, then (G, Ω) is an imprimitive group of degree 69 and has properties listed in the conclusion of Theorem 2. From now on we assume the involution t is contained in P because P is an arbitrary Sylow 2-subgroup of $G_{F(t)}$.

(4.1) $O(G)\neq 1$.

Proof. Let (G, Ω) be a minimal counterexample to (4.1).
Since $|G: N_G(P)|$ is odd, there is a Sylow 2-subgroup S of G such that $S\triangleright P$. Set $H=G(F(t))$. If $t\in H^g$ for some $g\in G$, then $t^{g^{-1}}\in H$ and $(t^{g^{-1}})^h\in S$ for some $h\in H$ because S is a Sylow 2-subgroup of H. Since $N_G(P)\cap P^{g^{-1}h}\leq P$, $F(t^{g^{-1}h})=F(P)=F(t)$, hence $g^{-1}h\in H$, which implies $g\in H$. Consequently $t\in H^g$ if and only if $g\in H$. If $t_1(\neq t)$ is an involution in $t^G\cap C(t)$, then as above $t_1\in H_{F(t)}$ and so $tt_1\in I(H_{F(t)})$. Hence $(tt_1)^g\in H$ if and only if $g\in H$.

Thus we can apply Theorem 3.3 of [1] to t, H and G. Set $\langle t^G\rangle=L$. Since $O(G)=O_2(G)=1$, the 2-rank of any nontrivial characteristic subgroup of L is at least 2 by the Theorem of Brauer-Suzuki ([3]) and Theorem 7.6.1 of [6]. Hence $H\cap L'$ is strongly embedded in L'. By the Theorem of Bender ([2]), L^∞ is a simple group isomorphic to $PSL(2, q)$, $Sz(q)$ or $PSU(3, q)$ for $q=2^n\geq 4$. Here L^∞ is the last term of the derived series of L. Set $L^\infty=N$. We note that N is a normal subgroup of G and $|N: N\cap H|\geq 5$.

Since $G^{F(t)}\triangleright N^{F(t)}$ and $G^{F(t)}\simeq M_{23}$, we have $N^{F(t)}\simeq M_{23}$ or 1. Suppose $N^{F(t)}\simeq M_{23}$. Since $N\not\simeq M_{23}$, we have $N\not\leq G(F(t))$. If $|N_{F(t)}|$ is odd, $G=\langle t\rangle N$ and $P=\langle t\rangle$ by the minimality of G. By the Glauberman's Z^*-theorem ([5]), $G\triangleright\langle t\rangle O(G)=\langle t\rangle$, a contradiction. If $|N_{F(t)}|$ is even, by the minimality of G, $G=N$. Since N has a unique conjugate class of involution, $I(N_G(P))\subseteq I(P)$ by the assumption (ii) of Lemma 2. Hence S/P is an elementary abelian 2-group (cf. section 3 of [2]), which is contrary to $N_G(P)^{F(P)}\simeq M_{23}$.

Now we suppose $N^{F(t)}=1$. Since $N\cap P\neq 1$ and $NC_G(N)=N\times C_G(N)$, the assumption (ii) of Lemma 2 forces $|C_G(N)_{F(t)}|$ is odd. Hence if $|C_G(N)|$ is even, $C_G(N)^{F(t)}\neq 1$ and so $C_G(N)^{F(t)}\simeq M_{23}$ because $M_{23}\simeq G^{F(t)}\triangleright C_G(N)^{F(t)}$. Obviously $C_G(N)\leq G(F((N\cap P)^g))=G(F(t^g))$ for any $g\in G$. Therefore $\{F(t)^g\mid g\in G\}$ forms a complete system of blocks of G on Ω and an involution of $C_G(N)$

has exactly seven fixed points on each block. But (G, Ω) is a $(1, 23)$-group and hence $|\{F(t)^g | g \in G\}| = 3$, which implies $|N: N \cap H| = 3$, a contradiction. Thus we have $C_G(N)=1$. From this G/N is isomorphic to a subgroup of outer automorphism group of N. Hence G/N is solvable ([12]) and so $G^{\infty}=N$. Thus $N^{F(t)} \geq (G^{F(t)})^{\infty} \simeq M_{23}$, a contradiction.

(4.2) P is cyclic or generalized quaternion.

Proof. Suppose that P contains a four-group Q. Then $O(G)=\langle C_{O(G)}(x) | 1 \neq x \in Q\rangle$ by Theorem 5.3.16 of [6] and $O(G) \leq G(F(P)) = G(F(t))$. Since $O(G)^{F(t)} \lhd G^{F(t)} \simeq M_{23}$, $O(G)^{F(t)}=1$. Hence $O(G) \leq G_{F(t)}$, so that $O(G)=1$, which is contrary to (4.1). Thus P is cyclic or generalized quaternion.

Let us note that the automorphism group of P is a $\{2, 3\}$-group. Hence $N_G(P)^{F(P)}=C_G(P)^{F(P)} \simeq M_{23}$. By the similar argument as in the first paragraph of the proof of (3.1), we have

(4.3) $C_G(P)^{F(P)} \simeq M_{23}$. $C_G(P) = Z(P) \times O^2(C_G(P))$. Set $L = O^2(C_G(P))$. Then $L^{F(P)}=L/O(L) \simeq M_{23}$.

By the Feit-Thompson theorem ([4]), $O(G)$ is solvable. Hence we have

(4.4) Let N be a minimal normal subgroup of G contained in $O(G)$. Then N is an elementary abelian p-group for some odd prime p.

(4.5) Set $K=\{x \in N | x^t=x^{-1}\}$. Then

(i) L normalizes K and $K \not\leq G(F(t))$.

(ii) Set $X=(\langle t\rangle \times L)K$ and $\Gamma=\alpha^X$ where $\alpha \in F(t)$. Then $\Gamma \supseteq F(t)$, $|\Gamma| > 23$ and $|\Gamma|$ is odd.

Proof. Since $N^{F(t)} \lhd G^{F(t)} \simeq M_{23}$, $N^{F(t)}=1$. Hence $N \not\leq G(F(t))$. By Lemma 2.1 of [2], $N=C_N(t)K$ and so $K \not\leq G(F(t))$. If $x \in K$ and $y \in L$, $x^y \in N$. It follows from (4.2) that $t \in Z(P)$. Hence $[L, t]=1$ and $(x^y)^t=x^{yt}=x^{ty}=(x^{-1})^y =(x^y)^{-1}$. So we have $x^y \in K$. Thus (i) holds.

Since $L^{F(t)} \simeq M_{23}$ and $K \not\leq G(F(t))$, $\Gamma \supseteq \alpha^L=F(t)$ and $\Gamma \neq F(t)$. Let T be a Sylow 2-subgroup of L. Then $F(T) \cap F(t) \neq \phi$ and $\langle t\rangle \times T$ is a Sylow 2-subgroup of X. Therefore $|\Gamma|$ is odd. Thus (ii) holds.

(4.6) Let $\Pi=\{\Delta_1, \Delta_2, \cdots, \Delta_r\}$ be the set of K-orbits on Γ. Then the following hold.

(i) $r=23$, $K^{\Pi}=t^{\Pi}=O(L)^{\Pi}=1$, $X^{\Pi}=L^{\Pi} \simeq M_{23}$ and $X_{\Pi}=\langle t\rangle O(L)K$.

(ii) If $y \in I(\langle t\rangle \times L)$ and $y \neq t$. Then $|F(y^{\Pi})|=7$ and for $\Delta_i, \Delta_j \in F(y^{\Pi})$, $|\Delta_i \cap F(y)| = |\Delta_j \cap F(y)|$.

(iii) For $y \in I(\langle t\rangle \times L) - \{t\}$ and $\Delta_i \in F(y^{\Pi})$ we set $|\Delta_i \cap F(y)| = m(y)$. Then $m(y)=1$ or 3 and $|F(y) \cap \Gamma| = 7 \times m(y)$.

Proof. If $r=1$, then K^{Γ} is regular, so that $|F(t^{\Gamma})| = |K^{\Gamma} \cap C(t^{\Gamma})|$. On the other hand $|F(t^{\Gamma})|=23$ and by the definition of K, $|K^{\Gamma} \cap C(t^{\Gamma})|=1$, a contradiction. Thus $r \neq 1$.

We consider the action of X on the set Π. Since $K^{\Pi}=1$, $[t, L]=1$ and X

is transitive on Π, we have $t^{\Pi}=1$ and L is transitive on Π. Hence for Δ_i, $\Delta_j \in \Pi$, there is an element $x \in L$ such that $(\Delta_i)^x=\Delta_j$. Then $|F(t) \cap \Delta_i| = |(F(t) \cap \Delta_i)^x| = |F(t) \cap \Delta_j|$, so that $|F(t)| = |\Delta_i \cap F(t)| \times r$ for any $\Delta_i \in \Pi$. Hence $|\Delta_i \cap F(t)|=1$ and $r=23$. Since $F(O(L)) \supseteq F(t)$, $O(L)^{\Pi}=1$ and $X_{\Pi}=\langle t\rangle O(L)K$. Thus (i) holds.

Let $y \in I(\langle t\rangle \times L)$ and $y \neq t$. Then $y^{\Pi} \neq 1$ and by (ii) of §2, $|F(y^{\Pi})|=7$. Since $X_{\Pi}=\langle t\rangle O(L)K$, $L^{\Pi} \cap C(y^{\Pi}) = (C_L(y))^{\Pi}$. By (vii) of §2, $L^{\Pi} \cap C(y^{\Pi})$ is transitive on $F(y^{\Pi})$. Therefore as above we obtain (ii).

Since $23 \geq |F(y) \cap \Gamma| = |F(y^{\Pi})| \times m(y) = 7 \times m(y)$, we have $m(y) \leq 3$. By (ii) of (4.5), $|\Gamma|$ is odd and so $m(y)$ is odd. Thus (iii) holds.

(4.7) Let $s \in I(L)$. Then the following hold.

(i) $m(s)=3$ and $|F(s) \cap \Gamma|=21$.

(ii) If $\Delta \in F(s^{\Pi})$, then $F(s) \supseteq \Delta$. Moreover $|\Delta|=3$ and N is an elementary abelian 3-group.

(iii) $F(s) \subseteq \Gamma$ and $|F(s)|=21$.

Proof. Suppose $m(s) \neq 3$. Then by (iii) of (4.6) $m(s)=1$. Since K^{Δ} is regular for any $\Delta \in \Pi$, if $\Delta \in F(s^{\Pi})$, s^{Δ} inverts K^{Δ}. Hence $(ts)^{\Delta}$ centralizes K^{Δ} and so $F(ts) \supseteq \Delta$ and $m(ts)=|\Delta|$. Since $|\Delta| \neq 1$, by (iii) of (4.6) we have $|\Delta|=m(ts)=3$. Therefore by (iii) of (4.6) $|F(ts) \cap \Gamma|=21$. Since $L/O(L) \simeq M_{23}$, s^{Γ} is an even permutation. Furthermore $|F(s) \cap \Gamma|=7$ because $m(s)=1$. On the other hand $|\Gamma| = |\Delta| \times 23 = 69$ and s^{Γ} is an odd permutation, a contradiction. Thus (i) holds.

Since $|F(s) \cap \Gamma|=21$ and s^{Γ} is an even permutation, t^{Γ} is an odd permutation because $|F(t) \cap \Gamma|=23$. Hence $(ts)^{\Gamma}$ is an odd permutation and so $m(ts)=1$ and $(ts)^{\Delta}$ inverts K^{Δ} for $\Delta \in F(s^{\Pi})=F((ts)^{\Pi})$. Therefore $s^{\Delta}=(t^{\Delta})(ts)^{\Delta}$ centralizes K^{Δ} and $F(s) \supseteq \Delta$, so that $m(s)=|\Delta|=3$. Hence K and N are elementary abelian 3-groups, so (ii) holds.

Since $L^{F(t)}=L/O(L) \simeq M_{23}$, by (vi) of §2, there exists a four-group $\langle s_1, s_2\rangle$ of L such that $F(s_1) \cap F(t)=F(s_2) \cap F(t)$. Since L has a unique conjugate class of involutions (cf. (ii) of §2), $m(s_1)=m(s_2)=m(s_1s_2)=3$. Hence $F(s_1) \cap \Gamma=F(s_2) \cap \Gamma =F(s_1s_2) \cap \Gamma$ and $|F(s_1) \cap \Gamma|=21$. To prove (iii) it will suffice to show that $|F(s_1)|=21$. Assume $|F(s_1)| \neq 21$. Then $|F(s_1)|=23$ and $|F(s_1) \cap (\Omega-\Gamma)|=2$. Since $L/O(L) \simeq M_{23}$, we have $C_L(s_1)/O(C_L(s_1)) \simeq \tilde{C}$ by the property of M_{23}. $C_L(s_1)$ acts on $F(s_1) \cap (\Omega-\Gamma)$ and $O^2(\tilde{C})=\tilde{C}$ by (ix) of §2, hence $C_L(s_1)$ acts trivialy on $F(s_1) \cap (\Omega-\Gamma)$. Therefore $F(s_1)=F(s_2)=F(s_1s_2)$ and $|F(s_1)|=23$. By Theorem 5.3.16 of [6], $N=\langle C_N(s) \mid 1 \neq s \in \langle s_1, s_2\rangle\rangle$ and hence N acts on $F(s_1)$. From this $3 \mid |F(s_1)|$, a contradiction. Thus (iii) holds.

(4.8) The following hold.

(i) $O(G)$ is an elementary abelian 3-group.

(ii) G is imprimitive on Ω and the length of an $O(G)$-orbit is three. $|P|=2$.

(iii) $|\Omega|=69$. Let ψ be the set of $O(G)$-orbits on Ω. Then $|\psi|=23$ and $G^{\psi}\simeq M_{23}$.

Proof. Since $L^{F(t)}=L/O(L)\simeq M_{23}$, there exist two subgroups $\langle s_1, s_2\rangle$, $\langle s_3, s_4\rangle$ of L satisfying the following (cf. §2). $\langle s_1, s_2\rangle\simeq\langle s_3, s_4\rangle\simeq E_4$, $F(s_1)\cap F(t)=F(s_2)\cap F(t)=F(s_1s_2)\cap F(t)$, $F(s_3)\cap F(t)=F(s_4)\cap F(t)=F(s_3s_4)\cap F(t)$, $|(F(s_1)\cap F(t))\cap(F(s_3)\cap F(t))|=3$. By (ii) and (iii) of (4.7), we have $\Gamma\supseteq F(s_1)=F(s_2)=F(s_1s_2)$, $|F(s_1)|=21$, $\Gamma\supseteq F(s_3)=F(s_4)=F(s_3s_4)$, $|F(s_3)|=21$ and $|F(s_1)\cap F(s_3)|=9$.

On the other hand $O(G)=\langle C_{0(G)}(s)\,|\,1\neq s\in\langle s_1, s_2\rangle\rangle=\langle C_{0(G)}(s)\,|\,1\neq s\in\langle s_3, s_4\rangle\rangle$ by Theorem 5.3.16 of [6]. Hence $O(G)$ acts on $F(s_1)$ and $F(s_3)$, so that also on $F(s_1)\cap F(s_3)$. Therefore the length of an $O(G)$-orbit is three because it is a common divisor of 9 and 21. From this $O(G)$ is an elementary abelian 3-subgroup and by (4.2) P is cyclic of order 2. Thus (i) and (ii) hold.

Let ψ be the set of $O(G)$-orbits on Ω. Since $\psi\supseteq\Pi$, $\Pi=F(t^{\psi})$ and $X^{\Pi}\simeq M_{23}$, we have $G^{\Pi}\geq M_{23}$. If $G^{\Pi}\neq M_{23}$, then $G^{\Pi}\geq A_{23}$ by the result of [11]. But if S is as in (4.1), the order of S/P is equal to that of a Sylow 2-sulgroup of M_{23}, a contradiction. Hence $G^{\Pi}\simeq M_{23}$.

Now we suppose $\psi\neq\Pi$. Then $t^{\psi}\neq 1$ and G^{ψ} satisfies (ii) of Lemma 2. On the other hand $O(G^{\psi})=1$, which is contrary to (4.1), so (iii) holds.

5. Proof of Theorem 1

The proof of Theorem 1 is obtained in the following way: By the Theorem of Oyama and his lemma of [10], it will suffice to consider the case that $G^{F(t)}$ is isomorphic to M_{11}, M_{23} or M_{24}. Since G is 4-fold transitive on Ω, $G^{F(t)}\not\simeq M_{23}$ and M_{24} by Theorem 2. Hence we consider the case that $G^{F(t)}\simeq M_{11}$.

Suppose that $G^{F(t)}\simeq M_{11}$. Let P be a Sylow 2-subgroup of $G_{F(t)}$ and S a Sylow 2-subgroup of a stabilizer of four points of Ω in G such that $S\geq P$. Then $N_S(P)\leq G(F(P))$, hence $N_S(P)^{F(t)}=1$ by the structure of M_{11}, so $F(N_S(P))=F(t)$. Since P is a Sylow 2-subgroup of $G_{F(t)}$, $N_S(P)=P$, which forces $S=P$, hence $|F(S)|=11$. By the Theorem of [9], $G^{\Omega}=M_{11}$, a contradiction.

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