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ON THE LENGTHS OF THE SECOND FUNDAMENTAL FORMS OF R-SPACES

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Introduction

The aim of this paper is to study the lengths of the second fundamental forms of a certain class of homogeneous submanifolds, called R-spaces, minimally imbedded into a unit sphere S. Among these submanifolds, we find Veronese surfaces and generalized Clifford surfaces. These have been characterized as minimal submanifolds with second fundamental form of minimal positive constant length by Chern-Do Carmo-Kobayashi [2]. Also Simons [9] discusses the lengths of the second fundamental forms of submanifolds in S.

Our main results are as follows. Let $||A||^2$ be the square of the length of the second fundamental form of an R-space N minimally imbedded into S. Then if N is regular (See section 2), $||A||^2$ is a certain multiple of dim N. If N is symmetric (See section 4), then $||A||^2$ is a rational number. These results are independent of the choice of an invariant Riemannian metric on N.

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1. Preliminaries

1.1. Let $(\mathfrak{g}, \sigma)$ be an orthogonal symmetric Lie algebra of compact type. Put $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$, where $\mathfrak{k}$ (resp. $\mathfrak{p}$) is the eigenspace of σ corresponding to the eigenvalue 1 (resp. -1). Let Aut$(\mathfrak{g})$ be the group of automorphisms of $\mathfrak{g}$. Identifying the Lie algebra of Aut$(\mathfrak{g})$ with $\mathfrak{g}$, let K be the connected Lie subgroup of Aut$(\mathfrak{g})$ corresponding to the Lie subalgebra $\mathfrak{k}$ of $\mathfrak{g}$. Then K leaves the subspace $\mathfrak{p}$ invariant. Let $(\ , \ )$ be an inner product on $\mathfrak{g}$ invariant under Aut$(\mathfrak{g})$. Then K acts as an isometry group on the Euclidean space $\mathfrak{p}$ with the inner product $(\ , \ )$, the restriction of the inner product $(\ , \ )$ on $\mathfrak{g}$ to $\mathfrak{p}$. Let S be the unit sphere of $\mathfrak{p}$, and H an element of S. Let N be the orbit of K through H. Denoting by L the stabilizer of H in K, the space N may be identified with the quotient space K/L, which is called an *R-space*.

1.2. Let $\mathfrak{a}$ be a maximal abelian subspace in $\mathfrak{p}$. We shall identify $\mathfrak{a}$ with the dual space $\mathfrak{a}^*$ of $\mathfrak{a}$ by the map ι: $\mathfrak{a}\rightarrow\mathfrak{a}^*$, $\iota(X)(Y)=(Y, X)$ for X, $Y\in\mathfrak{a}$. For $\lambda\in\mathfrak{a}$, we define the subspace $\mathfrak{k}_\lambda$ and $\mathfrak{p}_\lambda$ of $\mathfrak{g}$ as follows:

$$\mathfrak{k}_\lambda = \{X\in\mathfrak{k};\ ad(H)^2X = -(\lambda, H)^2X, \text{ for any } H\in\mathfrak{a}\}\ ,$$
$$\mathfrak{p}_\lambda = \{X\in\mathfrak{p};\ ad(H)^2X = -(\lambda, H)^2X, \text{ for any } H\in\mathfrak{a}\}\ .$$

Then $\mathfrak{k}_{-\lambda}=\mathfrak{k}_\lambda$, $\mathfrak{p}_{-\lambda}=\mathfrak{p}_\lambda$ and $\mathfrak{p}_0=\mathfrak{a}$. If we put

$$\mathfrak{r} = \{\lambda\in\mathfrak{a};\ \lambda\neq 0,\ \mathfrak{p}_\lambda\neq\{0\}\}\ ,$$

$\mathfrak{r}$ is a root system in $\mathfrak{a}$ (Satake [7]). The root system $\mathfrak{r}$ is called the *restricted root system* of $(\mathfrak{g}, \sigma)$. We choose a linear order in $\mathfrak{a}$ and fix it once for all. We denote by $\mathfrak{r}^+$ the set of positive roots in $\mathfrak{r}$ with respect to this linear order in $\mathfrak{a}$. Then we have the following orthogonal decomposition of $\mathfrak{k}$ and $\mathfrak{p}$ with respect to the inner product (,) (cf. Helgason [3]):

$$\mathfrak{k} = \mathfrak{k}_0+\sum_{\lambda\in\mathfrak{r}^+}\mathfrak{k}_\lambda,\quad \mathfrak{p} = \mathfrak{a}+\sum_{\lambda\in\mathfrak{r}^+}\mathfrak{p}_\lambda\ . \tag{1.1}$$

1.3. Let (M, h) and (M', g) be Riemannian manifolds, and f: $M\rightarrow M'$ an isometric immersion. Let $T_x(M)$ be the tangent space of M at a point $x\in M$, and $T_x^\perp(M)$ the orthogonal complement of $T_x(M)$ in $T_{f(x)}(M')$. Let A: $T_x^\perp(M)\times T_x(M)\rightarrow T_x(M)$ be the Weingarten form at $x\in M$. Let $\{e_1, \cdots, e_n\}$ (resp. $\{f_1, \cdots, f_m\}$) be an orthonormal basis of $T_x(M)$ (resp. $T_x^\perp(M)$). Then the square of the length of the second fundamental form $||A||^2(x)$ at x is given by

$$||A||^2(x) = \sum_{p=1}^{n}\sum_{q=1}^{m} |A_{f_q}e_p|^2\ ,$$

where $|X|^2=g(X, X)$ for $X\in T_{f(x)}(M')$. Let $\rho(x)$ be the scalar curvature of M at x.

Lemma 1. *If the immersion f: $M\rightarrow M'$ is minimal and M' is a space form with the sectional curvature c, then we have*

$$\rho(x) = n(n-1)c-||A||^2(x)\ , \tag{1.2}$$

where $n=\dim M$.

Proof. If $c>0$, Simons [9] proves the formula. In the general case, we can prove the formula in the same way as in Simons [9].

2. Second fundamental forms of R-spaces

2.1. As in section 1, we assume that the point H is contained in the unit sphere S. Moreover we may assume that $H\in S\cap\mathfrak{a}$ and $(\lambda, H)\geqslant 0$ for any $\lambda\in\mathfrak{r}^+$, by virtue of the following lemma.

Lemma 2 (Helgason [3]). *For any $X\in\mathfrak{p}$, there exists an element $k\in K$ such* *that* $kX\in\mathfrak{a}$ *and* $(\lambda, kX)\geqq 0$ *for any* $\lambda\in\mathfrak{r}^+$.

We identify the tangent space $T_H(N)$ of N at H with a subspace of $\mathfrak{p}$ in a canonical manner. Then we have $T_H(N)=[\mathfrak{k}, H]$. Put

$$\mathfrak{r}_1^+ = \{\lambda\in\mathfrak{r}^+;\ (\lambda, H) = 0\},\ \mathfrak{r}_2^+ = \{\lambda\in\mathfrak{r}^+;\ (\lambda, H)>0\}\ .$$

The tangent space $T_H(N)$ and the orthogonal complement $T_H^\perp(N)$ in $T_H(S)$ are given by

$$T_H(N) = \sum_{\lambda\in\mathfrak{r}_2^+} \mathfrak{p}_\lambda\,, \tag{2.1}$$

$$T_H^\perp(N)= \mathfrak{a}_H+ \sum_{\lambda\in\mathfrak{r}_1^+} \mathfrak{p}_\lambda\,, \tag{2.2}$$

where $\mathfrak{a}_H=\{X\in\mathfrak{a};\ (X, H)=0\}$.
We shall call the submanifold N *regular*, if $\mathfrak{r}_2^+=\mathfrak{r}^+$.

2.2. Let Δ be the fundamental root system of $\mathfrak{r}$ with respect to the order in $\mathfrak{a}$. Put

$$\Delta_1 = \{\lambda\in\Delta;\ \lambda\in\mathfrak{r}_1^+\}\ .$$

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$. Let $\tilde{\mathfrak{g}}$ be the complexification of $\mathfrak{g}$, and $\tilde{\mathfrak{h}}$ the subspace of $\tilde{\mathfrak{g}}$ spanned by $\mathfrak{h}$. The inner product (,) on $\mathfrak{g}$ can be extended uniquely to a complex symmetric bilinear form, denoted also by (,) on $\tilde{\mathfrak{g}}$. Let $\tilde{\mathfrak{r}}$ be the root system of $\tilde{\mathfrak{g}}$ relative to $\tilde{\mathfrak{h}}$. An element $\alpha\in\tilde{\mathfrak{h}}$ belongs to $\tilde{\mathfrak{r}}$, if $\alpha\neq 0$ and there exists a non-zero vector $X\in\tilde{\mathfrak{g}}$ such that $[H, X]=(\alpha, H)X$ for any $H\in\tilde{\mathfrak{h}}$. Let $\mathfrak{h}_0$ be the real part of $\tilde{\mathfrak{h}}$, i.e. the real subspace of $\tilde{\mathfrak{h}}$ spanned by $\tilde{\mathfrak{r}}$. Note that then $\mathfrak{h}_0=\sqrt{-1}\ \mathfrak{h}$. We denote by the same letter σ the conjugation of $\tilde{\mathfrak{g}}$ with respect to $\mathfrak{k}+\sqrt{-1}\ \mathfrak{p}$. We choose a σ-order in $\mathfrak{h}_0$ in the sense of Satake [7] which has the following property. Let $\tilde{\Delta}$ be the fundamental system with respect to this order in $\mathfrak{h}_0$, and denote by p the projecion of $\mathfrak{h}_0$ onto $\sqrt{-1}\ \mathfrak{a}$. Then $\sqrt{-1}\ \Delta=p(\tilde{\Delta})-\{0\}$. We denote the Satake diagram of $\tilde{\Delta}$ also by $\tilde{\Delta}$. Put $\tilde{\Delta}_1=p^{-1}(\sqrt{-1}\ \Delta_1)$. It is known (Takeuchi [11]) that isomorphic pairs $(\tilde{\Delta}, \tilde{\Delta}_1)$ of Satake diagrams gives rise to isomorphic pairs (K, L): We say that the pair $(\tilde{\Delta}, \tilde{\Delta}_1)$ is isomorphic to the pair $(\tilde{\Delta}', \tilde{\Delta}_1')$ if there exists an isomorphism φ of $\tilde{\Delta}$ onto $\tilde{\Delta}'$ such that φ maps $\tilde{\Delta}_1$ onto $\tilde{\Delta}_1'$, and that the pair (K, L) is isomorphic to the pair (K', L') if there exists an isomorphism f of K onto K' such that f maps L onto L'.

2.3. Let Δ_1 be a subsystem of Δ. Put

$$A(\Delta_1) = \left\{H\in\mathfrak{a}\cap S;\ \begin{matrix}(\lambda, H)\geqq 0,\ \text{for any}\ \lambda\in\mathfrak{r}^+,\\ \{\lambda\in\Delta;\ (\lambda, H) = 0\} = \Delta_1\end{matrix}\right\}.$$

Then there exists an element $H\in A(\Delta_1)$ such that the orbit of K through H is minimal in S. This follows easily from Hsiang-Lawson [4] (Corollary 1.8). If $(\mathfrak{g}, \sigma)$ is irreducible and the pair (K, L) is symmetric, then for the subsystem Δ_1 of Δ obtained from $N=K/L$ as in 2.2 the set $A(\Delta_1)$ consists of only one element (cf. Takeuchi [11]). Therefore in this case the submanifold N is minimal.

2.4. Let $A\colon T_H^{\perp}(N)\times T_H(N)\to T_H(N)$ be the Weingarten form of the submanifold N of S at H. The following proposition is due to Takagi-Takahashi [10].

Proposition 3. *For $X_\lambda\in\mathfrak{p}_\lambda$, $\lambda\in\mathfrak{r}_2^+$, the Weingarten form A is given by*

$$A_{Z_0}X_\lambda = -\frac{(\lambda, Z_0)}{(\lambda, H)}X_\lambda, \quad \text{if } Z_0\in\mathfrak{a}_H ,$$

$$A_{Z_\mu}X_\lambda = -\frac{1}{(\lambda, H)^2}[[H, X_\lambda], Z_\mu], \quad \text{if } Z_\mu\in\mathfrak{p}_\mu,\ \mu\in\mathfrak{r}_1^+ .$$

There exists an orthonormal basis $\{X_{\lambda\cdot 1}, \cdots, X_{\lambda\cdot m_\lambda}\}$ (resp. $\{Y_{\lambda\cdot 1}, \cdots, Y_{\lambda\cdot m_\lambda}\}$) of $\mathfrak{p}_\lambda$ (resp. $\mathfrak{k}_\lambda$) such that

$$\text{(2.3)}\qquad \begin{cases}[H, X_{\lambda\cdot p}] = -(\lambda, H)Y_{\lambda\cdot p} , \\ [H, Y_{\lambda\cdot p}] = (\lambda, H)X_{\lambda\cdot p} \qquad \text{for any } H\in\mathfrak{a} ,\end{cases}$$

where m_λ is the multiplicity of $\lambda\in\mathfrak{r}^+$, i.e. $m_\lambda=\dim\mathfrak{p}_\lambda$.

Proposition 4. *The square of the length of the second fundamental form $||A||^2$ at H is given by*

$$\text{(2.4)}\qquad ||A||^2 = -n+\sum_{\lambda\in\mathfrak{r}_2^+}\frac{1}{(\lambda, H)^2}\Big(m_\lambda|\lambda|^2 + \sum_{p=1}^{m_\lambda}\sum_{\mu\in\mathfrak{r}_1^+}\sum_{q=1}^{m_\mu}|X_{(\lambda\cdot p:\mu\cdot q)}|^2\Big) .$$

Here $n=\dim N$, $X_{(\lambda\cdot p:\mu\cdot q)}=[Y_{\lambda\cdot p}, X_{\mu\cdot q}]$ and $|X|^2=(X, X)$ for $X\in\mathfrak{g}$. In particular when N is regular, we have

$$\text{(2.5)}\qquad ||A||^2 = -n+\sum_{\lambda\in\mathfrak{r}^+} m_\lambda\frac{|\lambda|^2}{(\lambda, H)^2} .$$

Proof. Let $\{H, H_1, \cdots, H_l\}$ be an orthonormal basis of $\mathfrak{a}$. Applying Proposition 3 and (2.3), we have

$$\begin{aligned}||A||^2 &= \sum_{\lambda\in\mathfrak{r}_2^+}\sum_{p=1}^{m_\lambda}\Big(\sum_{k=1}^{l}|A_{H_k}X_{\lambda\cdot p}|^2 + \sum_{\mu\in\mathfrak{r}_1^+}\sum_{q=1}^{m_\mu}|A_{X_{\mu\cdot q}}X_{\lambda\cdot p}|^2\Big) \\ &= \sum_{\lambda\in\mathfrak{r}_2^+}\frac{1}{(\lambda, H)^2}\Big(m_\lambda\sum_{k=1}^{l}(\lambda, H_k)^2\end{aligned}$$

$$+\sum_{p=1}^{m_\lambda}\sum_{\mu\in\mathfrak{r}_1^+}\sum_{q=1}^{m_\mu}|[Y_{\lambda\cdot p}, X_{\mu\cdot q}]|^2)$$

$$=\sum_{\lambda\in\mathfrak{r}_2^+}\frac{1}{(\lambda, H)^2}(m_\lambda(|\lambda|^2-(\lambda, H)^2)$$

$$+\sum_{p=1}^{m_\lambda}\sum_{\mu\in\mathfrak{r}_1^+}\sum_{q=1}^{m_\mu}|X_{(\lambda\cdot p:\mu\cdot q)}|^2)$$

$$=-n+\sum_{\lambda\in\mathfrak{r}_2^+}\frac{1}{(\lambda, H)^2}(m_\lambda|\lambda|^2$$

$$+\sum_{p=1}^{m_\lambda}\sum_{\mu\in\mathfrak{r}_1^+}\sum_{q=1}^{m_\mu}|X_{(\lambda\cdot p:\mu\cdot q)}|^2),$$

which proves the first formula of the proposition. The second formula (2.5) is the immediate consequence of (2.4).

2.5. Let α: $T_H(N)\times T_H(N)\to T_H^\perp(N)$ be the second fundamental form at H. Then we have (cf. Kobayashi-Nomizu [5])

(2.6) $(\alpha(X, Y), Z)=(A_Z X, Y)$ for $X, Y\in T_H(N)$ and $Z\in T_H^\perp(N)$.

Proposition 5. *The submanifold N of S is minimal if and only if the following condition is satisfied*:

$$\sum_{\lambda\in\mathfrak{r}_2^+}\frac{m_\lambda}{(\lambda, H)}\lambda=nH. \tag{2.7}$$

Proof. By definition, N is minimal if and only if

$$\sum_{\lambda\in\mathfrak{r}_2^+}\sum_{p=1}^{m_\lambda}\alpha(X_{\lambda\cdot p}, X_{\lambda\cdot p})=0.$$

By (2.6) and Proposition 3, we have

$$\alpha(X_{\lambda\cdot p}, X_{\lambda\cdot p})=\sum_{k=1}^{l}(A_{H_k}X_{\lambda\cdot p}, X_{\lambda\cdot p})H_k$$

$$+\sum_{\mu\in\mathfrak{r}_1^+}\sum_{q=1}^{m_\mu}(A_{X_{\mu\cdot q}}X_{\lambda\cdot p}, X_{\lambda\cdot p})X_{\mu\cdot q}$$

$$=-\frac{1}{(\lambda, H)}\sum_{k=1}^{l}(\lambda, H_k)H_k$$

$$=H-\frac{1}{(\lambda, H)}\lambda.$$

Therefore we have

$$0=\sum_{\lambda\in\mathfrak{r}_2^+}\sum_{p=1}^{m_\lambda}\alpha(X_{\lambda\cdot p}, X_{\lambda\cdot p})=\sum_{\lambda\in\mathfrak{r}_2^+}m_\lambda\left(H-\frac{1}{(\lambda, H)}\lambda\right)$$

$$=nH-\sum_{\lambda\in\mathfrak{r}_2^+}\frac{m_\lambda}{(\lambda, H)}\lambda,$$

which proves the proposition.

2.6. Assume that the algebra $\mathfrak{g}$ decomposes into the direct sum $\mathfrak{g}=\mathfrak{g}_1+\mathfrak{g}_2$ of two ideals $\mathfrak{g}_1$ and $\mathfrak{g}_2$ invariant under σ. For $i=1, 2$, let $\mathfrak{g}_i=\mathfrak{k}_i+\mathfrak{p}_i$, where $\mathfrak{k}_i=\mathfrak{g}_i\cap\mathfrak{k}$ and $\mathfrak{p}_i=\mathfrak{g}_i\cap\mathfrak{p}$, and put $S_i=S\cap\mathfrak{p}_i$, $\mathfrak{a}_i=\mathfrak{a}\cap\mathfrak{p}_i$. Assume that an element $H_i\in\mathfrak{a}_i\cap S_i$ satisfies $(\lambda, H_i)\geqslant 0$ for any $\lambda\in\mathfrak{r}^+$. Let N_i be the orbit of K through H_i, and suppose that the submanifold N_i of S_i is minimal. Let $||A_i||^2$ be the square of the second fundamental form of the submanifold N_i of S_i. Then we have

Proposition 6. *Assume that the submanifold N is the orbit of K through $H=\sqrt{\frac{n_1}{n}}H_1+\sqrt{\frac{n_2}{n}}H_2$, where $n_i=\dim N_i$. Then N is a minimal submanifold of the unit sphere S and we have*

$$||A||^2=n\left(1+\frac{1}{n_1}||A_1||^2+\frac{1}{n_2}||A_2||^2\right). \tag{2.8}$$

Proof. Put $(\mathfrak{r}_i)_s^+=\mathfrak{r}_s^+\cap\mathfrak{p}_i$, $i, s=1, 2$. By (2.7) we have

$$\sum_{\lambda\in(\mathfrak{r}_i)_2^+}\frac{m_\lambda}{(\lambda, H_i)}\lambda=n_iH_i\,.$$

Hence

$$\begin{aligned}\sum_{\lambda\in\mathfrak{r}_2^+}\frac{m_\lambda}{(\lambda, H)}\lambda&=\sum_{\lambda\in(\mathfrak{r}_1)_2^+}\frac{m_\lambda}{(\lambda, H)}\lambda+\sum_{\lambda\in(\mathfrak{r}_2)_2^+}\frac{m_\lambda}{(\lambda, H)}\lambda\\&=\sqrt{nn_1}\,H_1+\sqrt{nn_2}\,H_2\\&=nH\,,\end{aligned}$$

which proves the minimality of N. By (2.4) we have

$$\begin{aligned}||A||^2&=-n+\sum_{\lambda\in(\mathfrak{r}_1)_2^+}\frac{1}{(\lambda, H)^2}\Big(m_\lambda|\lambda|^2\\&\qquad+\sum_{p=1}^{m_\lambda}\sum_{\mu\in(\mathfrak{r}_1)_1^+}\sum_{q=1}^{m_\mu}|X_{(\lambda\cdot p:\mu\cdot q)}|^2\Big)\\&\quad+\sum_{\lambda\in(\mathfrak{r}_2)_2^+}\frac{1}{(\lambda, H)^2}\Big(m_\lambda|\lambda|^2\\&\qquad+\sum_{p=1}^{m_\lambda}\sum_{\mu\in(\mathfrak{r}_2)_1^+}\sum_{q=1}^{m_\mu}|X_{(\lambda\cdot p:\mu\cdot q)}|^2\Big)\\&=-n+\frac{n}{n_1}(||A_1||^2+n_1)+\frac{n}{n_2}(||A_2||^2+n_2)\\&=n\left(1+\frac{1}{n_1}||A_1||^2+\frac{1}{n_2}||A_2||^2\right),\end{aligned}$$

which proves (2.8).

2.7. EXAMPLE. Let $(\mathfrak{g}, \sigma)$ be the orthogonal symmetric Lie algebra corresponding to a symmetric pair $(SU(3), SO(3))$. Then if N is not regular, the pair (K, L) is either $(SO(3), S(O(1)\times 0(2)))$ or $(SO(3), S(O(2)\times O(1)))$. In these cases the submanifolds N are minimal, and they are isometric. They are the so-called Veronese surfaces. Applying (2.4) and (2.7), we get

$$||A||^2 = \begin{cases} 6, & \text{if } N \text{ is regular and minimal,} \\ \dfrac{4}{3}, & \text{if } N \text{ is the Veronese surface.} \end{cases}$$

3. The case where the submanififold N is regular

3.1. In this section we assume that the submanifold N is regular. Put

$$\mathfrak{S} = \{\lambda \in \mathfrak{r};\ 2\lambda \notin \mathfrak{r}\} \quad \text{and} \quad \mathfrak{S}^+ = \{\lambda \in \mathfrak{S};\ \lambda \in \mathfrak{r}^+\}\ .$$

Then $\mathfrak{S}$ is a reduced root system. For $\lambda \in \mathfrak{S}^+$, put $k_\lambda = m_\lambda + m_{\lambda/2}$, where $m_{\lambda/2}=0$ unless $\frac{\lambda}{2} \in \mathfrak{r}$. Then by Proposition 4, we get

$$||A||^2 = -n + \sum_{\lambda \in \mathfrak{S}^+} k_\lambda \frac{|\lambda|^2}{(\lambda, H)^2}\,, \tag{3.1}$$

and the submanifold N is minimal if and only if

$$\sum_{\lambda \in \mathfrak{S}^+} \frac{k_\lambda}{(\lambda, H)} \lambda = nH\,, \tag{3.2}$$

by Proposition 5.

Theorem 1. *If the submanifold N is regular and minimal, then*

$$||A||^2 = n\,(|\mathfrak{S}^+| - 1)\,. \tag{3.3}$$

Proof. By (3.1) it is sufficient to show that

$$\sum_{\lambda \in \mathfrak{S}^+} k_\lambda \frac{|\lambda|^2}{(\lambda, H)^2} = n \cdot |\mathfrak{S}^+|\ .$$

On the other hand, we have

$$\Big(\sum_{\lambda \in \mathfrak{S}^+} \frac{1}{(\lambda, H)} \lambda,\ nH\Big) = n \cdot |\mathfrak{S}^+|\ .$$

Therefore by (3.2) it is sufficient to prove

$$\sum_{\lambda \in \mathfrak{S}^+} k_\lambda \frac{|\lambda|^2}{(\lambda, H)^2} = \Big(\sum_{\lambda \in \mathfrak{S}^+} \frac{1}{(\lambda, H)} \lambda,\ \sum_{\mu \in \mathfrak{S}^+} \frac{k_\mu}{(\mu, H)} \mu\Big)\,. \tag{3.4}$$

To prove the formula, we prepare two lemmas. Let V be an h-dimensional real vector space. Let Φ be a reduced root system in V, and W the Weyl group of Φ. Let $(\ ,\ )$ be an inner product on V invariant under W. We choose a linear order in V. Let Φ^+ be the set of positive roots with respect to this order. For $\lambda\in\Phi^+$, put

$$\Phi_\lambda^+ = \{\xi\in\Phi^+;\ \xi = s\lambda \quad \text{for some } s\in W\}\ .$$

We can take a subset Λ of Φ^+ such that the union $\Phi^+=\bigcup_{\lambda\in\Lambda}\Phi_\lambda^+$ is disjoint. For $\lambda\in\Lambda$ and $H\in V$ such that $(\eta, H)\neq 0$ for any $\eta\in\Phi$, put

$$K(\lambda, H) = \sum_{\xi\in\Phi_\lambda^+}\frac{1}{(\xi, H)}\,\xi\ .$$

Lemma 7. $|K(\lambda, H)|^2 = \sum_{\xi\in\Phi_\lambda^+}\frac{|\xi|^2}{(\xi, H)^2}$.

Proof. Since

$$|K(\lambda, H)|^2 = \sum_{\xi\in\Phi_\lambda^+}\frac{|\xi|^2}{(\xi, H)^2}+2\sum_{\substack{\xi,\eta\in\Phi_\lambda^+\\ \xi<\eta}}\frac{(\xi, \eta)}{(\xi, H)(\eta, H)}\ ,$$

it suffices to prove

$$\sum_{\substack{\xi,\eta\in\Phi_\lambda^+\\ \xi<\eta}}\frac{(\xi, \eta)}{(\xi, H)(\eta, H)} = 0\ . \tag{3.5}$$

Assume that $\xi, \eta\in\Phi_\lambda^+$ and $\xi<\eta$. Then $|\xi|=|\eta|=|\lambda|$. If $(\xi, \eta)>0$ (resp. <0), we have $(\xi, \eta)=\frac{|\lambda|^2}{2}\left(\text{resp.}\ -\frac{|\lambda|^2}{2}\right)$ (cf. Serre [8]). Suppose $(\xi, \eta)<0$. Then $(\xi, \xi+\eta)=|\xi|^2+(\xi, \eta)=\frac{|\lambda|^2}{2}$, and similarly $(\eta, \xi+\eta)=\frac{|\lambda|^2}{2}$. It follows easily

$$\frac{(\xi, \eta)}{(\xi, H)(\eta, H)}+\frac{(\xi, \xi+\eta)}{(\xi, H)(\xi+\eta, H)}+\frac{(\eta, \xi+\eta)}{(\eta, H)(\xi+\eta, H)} = 0\ . \tag{3.6}$$

Put

$$A^+ = \{(\xi;\ \eta)\in\Phi_\lambda^+\times\Phi_\lambda^+;\ (\xi, \eta)>0,\ \xi<\eta\}\ ,$$
$$A^- = \{(\xi;\ \eta)\in\Phi_\lambda^+\times\Phi_\lambda^+;\ (\xi, \eta)<0,\ \xi<\eta\}\ .$$

We define a mapping f of A^+ to A^- by

$$f(\xi;\ \eta) = \begin{cases}(\xi;\ \eta-\xi), & \text{if } \xi<\eta-\xi\ ,\\ (\eta-\xi;\ \xi), & \text{if } \eta-\xi<\xi\ .\end{cases}$$

Let S_ξ be the symmetry with respect to ξ. Then, if $(\xi, \eta)>0$, $(\xi, \eta)=\frac{|\lambda|^2}{2}$ and so $S_\xi(\eta)=\eta-\xi$. Therefore the above mapping is well-defined. If $(\xi, \eta)<0$, then $(\xi, \eta)=-\frac{|\lambda|^2}{2}$ and so $S_\xi(\eta)=\xi+\eta$. Therefore we have esaily

$$f^{-1}(\xi; \eta) = \{(\xi; \xi+\eta), (\eta; \xi+\eta)\} . \tag{3.7}$$

This, together with (3.6), implies (3.5). The proof of Lemma 7 is completed.

Lemma 8. $(K(\lambda, H), K(\mu, H))=0$ *for* $\lambda, \mu\in\Lambda, \lambda\neq\mu$.

Proof. We have

$$(K(\lambda, H), K(\mu, H)) = \sum_{\xi\in\Phi_\lambda^+} \sum_{\eta\in\Phi_\mu^+} \frac{(\xi, \eta)}{(\xi, H)(\eta, H)} .$$

If λ and μ are contained in the different irreducible components of Φ, the formula is trivially true, and so we may assume that the root system Φ is irreducible. Then if $\alpha, \beta\in\Phi$ are such that $|\alpha|=|\beta|$, there exists an element $s\in W$ such that $\beta=s\alpha$. Therefore we have $|\lambda|\neq|\mu|$. We may assume $|\lambda|<|\mu|$. Since the root system Φ is reduced, we have $|\mu|^2=2|\lambda|^2$ or $3|\lambda|^2$ (cf. Serre [8]).

In the case of $|\mu|^2=3|\lambda|^2$, Φ is of type G_2 and we may assume that Λ is a fundamental root system of Φ. Then we have $(\lambda, \mu)=-\frac{3}{2}|\lambda|^2$, $\Phi_\lambda^+=\{\lambda, \lambda+\mu, 2\lambda+\mu\}$ and $\Phi_\mu^+=\{\mu, 3\lambda+\mu, 3\lambda+2\mu\}$. In this case the proof is straightforward.

In the case of $|\mu|^2=2|\lambda|^2$, assume that $\xi\in\Phi_\lambda^+$ and $\eta\in\Phi_\mu^+$. If $(\xi, \eta)>0$ (resp. <0), then we have $(\xi, \eta)=|\lambda|^2$ (resp. $-|\lambda|^2$) (cf. Serre [8]). If $(\xi, \eta)<0$, it follows easily

$$\begin{aligned} &\frac{(\xi, \eta)}{(\xi, H)(\eta, H)}+\frac{(\xi+\eta, \eta)}{(\xi+\eta, H)(\eta, H)}+\frac{(\xi+\eta, 2\xi+\eta)}{(\xi+\eta, H)(2\xi+\eta, H)} \\ &\qquad +\frac{(\xi, 2\xi+\eta)}{(\xi, H)(2\xi+\eta, H)}=0 . \end{aligned} \tag{3.8}$$

Put

$$\begin{aligned} A^+ &= \{(\xi; \eta)\in\Phi_\lambda^+\times\Phi_\mu^+; (\xi, \eta)>0\} , \\ A^- &= \{(\xi; \eta)\in\Phi_\lambda^+\times\Phi_\mu^+; (\xi, \eta)<0\} . \end{aligned}$$

We define a mapping f of A^+ to A^- by

$$f(\xi; \eta) = \begin{cases} (\xi; \eta-2\xi), & \text{if } \eta-2\xi\in\Phi^+ , \\ (\xi-\eta; \eta), & \text{if } 2\xi-\eta\in\Phi^+ \text{ and } \xi-\eta\in\Phi^+ , \\ (\eta-\xi; 2\xi-\eta), & \text{if } 2\xi-\eta\in\Phi^+ \text{ and } \eta-\xi\in\Phi^+ . \end{cases}$$

If $(\xi, \eta)>0$, then $(\xi, \eta)=|\lambda|^2$ and so $S_\xi(\eta)=\eta-2\xi$, $S_\eta(\xi)=\xi-\eta$. Therefore the above mapping f is well-defined. If $(\xi, \eta)<0$, then $(\xi, \eta)=-|\lambda|^2$ and so

$S_\xi(\eta)=\eta+2\xi$, $S_\eta(\xi)=\xi+\eta$. Therefore we have easily

$$f^{-1}(\xi;\eta)=\{(\xi;2\xi+\eta),(\xi+\eta;\eta),(\xi+\eta;2\xi+\eta)\}\ . \tag{3.9}$$

This, together with (3.8), implies the assertion, thus completing the proof of the lemma.

We return to the proof of Theorem 1. Taking $\mathfrak{S}$, $\mathfrak{S}^+$ for Φ, Φ^+, let $\Lambda\subset\mathfrak{S}^+$ be as above. Since $k_\xi=k_\lambda$ for $\lambda\in\Lambda$, $\xi\in\mathfrak{S}_\lambda^+$, we have

$$\Big(\sum_{\lambda\in\mathfrak{S}^+}\frac{1}{(\lambda,H)}\lambda,\sum_{\mu\in\mathfrak{S}^+}\frac{k_\mu}{(\mu,H)}\mu\Big)=(\sum_{\lambda\in\Lambda}K(\lambda,H),\sum_{\mu\in\Lambda}k_\mu K(\mu,H))$$
$$=\sum_{\lambda\in\Lambda}k_\lambda|K(\lambda,H)|^2+\sum_{\substack{\lambda,\mu\in\Lambda\\ \lambda\neq\mu}}k_\mu(K(\lambda,H),K(\mu,H))\,.$$

Applying Lemma 7 and Lemma 8, we get (3.4), and this proves Theorem 1.

4. The case where the pair (K, L) is symmetric

4.1. Let $\tilde{\mathfrak{g}}$ be the complexification of $\mathfrak{g}$. For a subspace $\mathfrak{v}$ of $\mathfrak{g}$, we denote by $\tilde{\mathfrak{v}}$ the subspace of $\tilde{\mathfrak{g}}$ spanned by $\mathfrak{v}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ invariant under σ. Put $\mathfrak{h}=\mathfrak{h}^++\mathfrak{h}^-$, where $\mathfrak{h}^+=\mathfrak{k}\cap\mathfrak{h}$ and $\mathfrak{h}^-=\mathfrak{p}\cap\mathfrak{h}$. We denote also by (,) the symmetric $\boldsymbol{C}$-bilinear form on $\tilde{\mathfrak{g}}$ which is the extension of the inner product (,) on $\mathfrak{g}$. Let $\tilde{\mathfrak{r}}$ be the root system of $\tilde{\mathfrak{g}}$ relative to $\tilde{\mathfrak{h}}$. An element $\alpha\in\tilde{\mathfrak{h}}$ belongs to $\tilde{\mathfrak{r}}$, if $\alpha\neq0$ and there exists a non-zero vector $X\in\tilde{\mathfrak{g}}$ such that $[H,X]=(\alpha,H)X$ for any $H\in\tilde{\mathfrak{h}}$. We have the root space decomposition

$$\tilde{\mathfrak{g}}=\tilde{\mathfrak{h}}+\sum_{\alpha\in\tilde{\mathfrak{r}}}\tilde{\mathfrak{g}}_\alpha\,,$$

where $\tilde{\mathfrak{g}}_\alpha$ is the eigenspace belonging to $\alpha\in\tilde{\mathfrak{r}}$. Let τ be the conjugation of $\tilde{\mathfrak{g}}$ with respect to $\mathfrak{g}$. We can choose a Weyl canonical basis $\{E_\alpha;\alpha\in\tilde{\mathfrak{r}}\}$ such that $\tau E_\alpha=E_{-\alpha}$ for each $\alpha\in\tilde{\mathfrak{r}}$ (cf. Serre [8]). We denote also by the same letter σ the conjugation of $\tilde{\mathfrak{g}}$ with respect to $\mathfrak{k}+\sqrt{-1}\,\mathfrak{p}$. Then we have $\sigma(\tilde{\mathfrak{r}})=\tilde{\mathfrak{r}}$ and $\sigma(\tilde{\mathfrak{g}}_\alpha)=\tilde{\mathfrak{g}}_{\sigma\alpha}$. Put $\sigma E_\alpha=\rho_\alpha E_{\sigma\alpha}$ for each $\alpha\in\tilde{\mathfrak{r}}$, and define $\tilde{\mathfrak{r}}_0=\{\alpha\in\tilde{\mathfrak{r}};\sigma\alpha=-\alpha\}$. Then we have easily $|\rho_\alpha|=1$ for any $\alpha\in\tilde{\mathfrak{r}}$ and $\rho_\alpha=\rho_{-\alpha}=\pm1$ for $\alpha\in\tilde{\mathfrak{r}}_0$. Put

$$\tilde{\mathfrak{r}}_0^+=\{\alpha\in\tilde{\mathfrak{r}}_0;\rho_\alpha=1\},\ \tilde{\mathfrak{r}}_0^-=\{\alpha\in\tilde{\mathfrak{r}}_0;\rho_\alpha=-1\}\,.$$

Then we have the following decompositions

$$\tilde{\mathfrak{k}}=\tilde{\mathfrak{h}}^++\sum_{\alpha\in\tilde{\mathfrak{r}}_0^+}\tilde{\mathfrak{g}}_\alpha+\sum_{\alpha\in\tilde{\mathfrak{r}}-\tilde{\mathfrak{r}}_0}\boldsymbol{C}(E_\alpha+\sigma E_{-\alpha})\,, \tag{4.1}$$

$$\tilde{\mathfrak{p}}=\tilde{\mathfrak{h}}^-+\sum_{\alpha\in\tilde{\mathfrak{r}}_0^-}\tilde{\mathfrak{g}}_\alpha+\sum_{\alpha\in\tilde{\mathfrak{r}}-\tilde{\mathfrak{r}}_0}\boldsymbol{C}(E_\alpha-\alpha E_{-\alpha})\,, \tag{4.2}$$

where the last summations in (4.1) and (4.2) run over all unordered pairs $(\alpha,\sigma\alpha)$

such that $\alpha \in \mathfrak{r} - \mathfrak{r}_0$. Put

$$\mathfrak{r}_1 = \{\alpha \in \mathfrak{r};\, \sigma\alpha = \alpha\}\ .$$

The following lemma is an easy consequence of (4.1).

Lemma 9. *$\mathfrak{h}^+$ is maximal abelian subspace of $\mathfrak{k}$, if and only if the set $\mathfrak{r}_1$ is empty.*

In the following, let $\mathfrak{h}^+$ be a maximal abelian subspace of $\mathfrak{k}$. By Lemma 9 we obtain the following lemma.

Lemma 10 (Murakami [6]).

$$\alpha + \sigma\alpha \notin \mathfrak{r} \qquad \textit{for any } \alpha \in \mathfrak{r}\ .$$

Since the group K is compact, we can consider the root system of $\tilde{\mathfrak{k}}$ relative to $\tilde{\mathfrak{h}}^+$, say $\tilde{\Sigma}$. Put $\bar{\alpha} = \frac{1}{2}(\alpha - \sigma\alpha)$ for each $\alpha \in \mathfrak{r}$. By (4.1) and Lemma 9 we have

Lemma 11 (Murakami [6]).

$$\tilde{\Sigma} = \{\bar{\alpha};\, \alpha \in \mathfrak{r}\}\ . \tag{4.3}$$

Lemma 12. *For $\alpha \in \mathfrak{r}$, we have*

$$\frac{(\alpha, \alpha)}{(\bar{\alpha}, \bar{\alpha})} = \begin{cases} 1, & \text{if } \sigma\alpha = -\alpha\,, \\ 2, & \text{if } \sigma\alpha \neq -\alpha\,,\ (\sigma\alpha, \alpha) = 0\,, \\ 4, & \text{if } \sigma\alpha \neq -\alpha\,,\ (\sigma\alpha, \alpha) \neq 0\,. \end{cases} \tag{4.4}$$

Proof. Since $(\sigma\alpha, \sigma\alpha) = (\alpha, \alpha)$ and $\bar{\alpha} \neq 0$, we have

$$\frac{(\alpha, \alpha)}{(\bar{\alpha}, \bar{\alpha})} = \frac{4(\alpha, \alpha)}{(\alpha - \sigma\alpha, \alpha - \sigma\alpha)} = \frac{4}{2 - \dfrac{2(\sigma\alpha, \alpha)}{(\alpha, \alpha)}}\,. \tag{4.5}$$

Since $(\sigma\alpha, \sigma\alpha) = (\alpha, \alpha)$ and $\sigma\alpha \neq \alpha$, we have $\frac{2(\alpha, \sigma\alpha)}{(\alpha, \alpha)} = -2, \pm 1$ or 0, and $\frac{2(\alpha, \sigma\alpha)}{(\alpha, \alpha)} = -2$ if and only if $\sigma\alpha = -\alpha$ (cf. Serre [8]). Suppose $\sigma\alpha \neq -\alpha$. Since $\alpha + \sigma\alpha \notin \mathfrak{r}$ by Lemma 10, we must have $\frac{2(\sigma\alpha, \alpha)}{(\alpha, \alpha)} \geqq 0$ (cf. Serre [8]). Therefore for each $\alpha \in \mathfrak{r}$ we have

$$\frac{2(\sigma\alpha, \alpha)}{(\alpha, \alpha)} = -2,\ 1 \text{ or } 0\,. \tag{4.6}$$

This, together with (4.5), completes the proof.

4.2. We define two K-invariant Riemann metrics g and g' on the quotient space K/L as follows: The metric g is induced from the imbedding $\varphi: K/L\rightarrow S$, $\varphi(kL)=kH$ for $k\in K$. The other metric g' is induced from the K-invariant inner product $(\ ,\ )$ on $\mathfrak{k}$, the restriction of the inner product $(\ ,\ )$ on $\mathfrak{g}$ to $\mathfrak{k}$.

Lemma 13 (Takeuchi-Kobayashi [12]). *If the orthogonal symmetric Lie algebra* $(\mathfrak{g}, \sigma)$ *is irreducible and the pair* (K, L) *is symmetric, then we have*

$$g=(\lambda, H)^2 g' , \tag{4.7}$$

where $\Delta-\Delta_1=\{\lambda\}$.

REMARK. Under the assumptions of Lemma 13, we have $(\xi, H)^2=(\eta, H)^2$ for any $\xi, \eta\in\mathfrak{r}_2^+$.

Let ρ (resp. ρ') be the scalar curvature with respect to the metric g (resp. g'). Under the assumptions of Lemma 13, (4.7) implies

$$\rho=\frac{1}{(\lambda, H)^2}\rho' . \tag{4.8}$$

Suppose that $(\mathfrak{g}, \sigma)$ is irreducible and the pair (K, L) is symmetric. Let θ be the involutive automorphism of K defining the symmetric pair (K, L). Then $\mathfrak{k}=\mathfrak{l}+\mathfrak{m}$, where $\mathfrak{l}$ (resp. $\mathfrak{m}$) is the eigenspace of θ corresponding to the eigenvalue 1 (resp. -1), and $\mathfrak{l}$ is the Lie algebra of L. We have the following decomposition (cf. Helgason [3]):

$$\mathfrak{k}=\mathfrak{k}_0+\mathfrak{k}_1+\cdots+\mathfrak{k}_r ,$$

where each $\mathfrak{k}_j$ is an ideal of $\mathfrak{k}$ invariant under θ, $(\mathfrak{k}_0, \theta)$ is of Euclidean type, and $(\mathfrak{k}_i, \theta)$, $i=1, \cdots, r$, is irreducible of compact type. Put $\mathfrak{l}_j=\mathfrak{k}_j\cap\mathfrak{l}$ and $\mathfrak{m}_j=\mathfrak{k}_j\cap\mathfrak{m}$, $j=0, 1, \cdots, r$. Then $\mathfrak{k}_j=\mathfrak{l}_j+\mathfrak{m}_j$. Let $\mathfrak{b}_j$ be a maximal abelian subspace of $\mathfrak{m}_j$, and $\sum_j$ the restricted root system of $(\mathfrak{k}_j, \theta)$ $(j=0, 1, \cdots, r)$. For each $\mathfrak{b}_j$, we choose a linear order in $\mathfrak{b}_j$. Let $\sum_j^+$ be the set of positive roots in $\sum_j$ with respect to this order.

Lemma 14. *We have*

$$\rho'=\sum_{i=1}^{r}\frac{h_i}{b_i}\sum_{\omega\in\Sigma_i^+} m_\omega|\omega|^2 , \tag{4.9}$$

where $h_j=\dim\mathfrak{m}_j$, $b_j=\dim\mathfrak{b}_j$ $(j=0, 1, \cdots, r)$, *and* m_ω *is the multiplicity of* $\omega\in\sum_i^+$.

Proof. Put $\mathfrak{b}=\mathfrak{b}_0+\mathfrak{b}_1+\cdots+\mathfrak{b}_r$. For $\omega\in\sum_i^+$, $i=1, \cdots, r$, we define the subspace $\mathfrak{m}_\omega$ as follows:

$$\mathfrak{m}_\omega = \{X \in \mathfrak{m};\ ad(H)^2 X = -(\omega, H)^2 X \text{ for any } H \in \mathfrak{b}\}\ .$$

Then we have the decomposition

$$\mathfrak{m} = \sum_{j=0}^{r} \mathfrak{b}_j + \sum_{i=1}^{r} \sum_{\omega \in \Sigma_i^+} \mathfrak{m}_\omega\ .$$

Let $S(\ ,\ )$ be the Ricci tensor of $(K/L, g')$. Since $(\mathfrak{k}_0, \theta)$ is of Euclidean type and $(\mathfrak{k}_i, \theta)$, $i=1, \cdots, r$, is irreducible, there exist constants c_j, $j=0, 1, \cdots, r$, such that

$$S(X, Y) = c_j(X, Y) \quad \text{for any } X,\ Y \in \mathfrak{m}_j\ , \tag{4.10}$$

where we identify the tangent space $T_0(K/L)$ at the origin with $\mathfrak{m}$. Let $\{H_{j\cdot 1}, \cdots, H_{j\cdot b_j}\}$ (resp. $\{X_{\omega\cdot 1}, \cdots, X_{\omega\cdot m\omega}\}$) be an orthonormal basis of $\mathfrak{b}_j$ (resp. $\mathfrak{m}_\omega$) with respect to $(\ ,\ )$. By (4.10) we have

$$\begin{aligned} \rho' &= \sum_{j=0}^{r} \Big(\sum_{p=1}^{bj} S(H_{j\cdot p}, H_{j\cdot p}) + \sum_{\omega \in \Sigma_j^+} \sum_{q=1}^{m_\omega} S(X_{\omega\cdot q}, X_{\omega\cdot q})\Big) \\ &= \sum_{i=1}^{r} c_i h_i \end{aligned} \tag{4.11}$$

because $c_0=0$. Let R be the curvature tensor of $(K/L, g')$. Then we have, (cf. Helgason [3])

$$R(X, Y)Z = -[[X, Y], Z] \text{ for any } X,\ Y,\ Z \in \mathfrak{m}\ .$$

Therefore for $1 \leq i \leq r$, we have

$$\begin{aligned} c_i &= S(H_{i\cdot p}, H_{i\cdot p}) \\ &= \sum_{j=0}^{r} \Big(\sum_{q=1}^{b_j} (R(H_{j\cdot q}, H_{i\cdot p})H_{i\cdot p}, H_{j\cdot q}) \\ &\qquad + \sum_{\omega \in \Sigma_j^+} \sum_{q=1}^{m_\omega} (R(X_{\omega\cdot q}, H_{i\cdot p})H_{i\cdot p}, X_{\omega\cdot q})\Big) \\ &= \sum_{\omega \in \Sigma_i^+} m_\omega (\omega, H_{i\cdot p})^2\ . \end{aligned}$$

So we get

$$\begin{aligned} b_i c_i &= \sum_{p=1}^{b_i} S(H_{i\cdot p}, H_{i\cdot p}) \\ &= \sum_{\omega \in \Sigma_i^+} m_\omega |\omega|^2\ . \end{aligned} \tag{4.12}$$

The formulas (4.11) and (4.12) imply (4.9) in the lemma.

Theorem 2. *If the orthogonal symmetric Lie algebra* $(\mathfrak{g}, \sigma)$ *is irreducible and the pair* (K, L) *is symmetric, then the square of the length of the second fundamental form* $||A||^2$ *is a rational number.*

Proof. By (1.2) it is sufficient to show that ρ is rational. By (4.8) and (4.9) we have

$$\rho=\frac{1}{(\lambda, H)^2}\sum_{i=1}^{r}\frac{h_i}{b_i}\sum_{\omega\in\Sigma_i^+}m_\omega|\omega|^2\,. \tag{4.13}$$

Let $\mathfrak{h}_j$ be a Cartan subalgebra of $\mathfrak{k}_j$ containing $\mathfrak{b}_j$, and $\tilde{\Sigma}_j$ the root system of $\tilde{\mathfrak{k}}_j$ relative to $\tilde{\mathfrak{h}}_j (j=0, 1, \cdots, r)$. Put $\mathfrak{h}^+=\mathfrak{h}_0+\mathfrak{h}_1+\cdots+\mathfrak{h}_r$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}^+$ and $\tilde{\mathfrak{r}}$ the root system of $\tilde{\mathfrak{g}}$ relative to $\tilde{\mathfrak{h}}$. For $i=1, \cdots, r$, let B_i be the Killing form of $\mathfrak{k}_i$. Note that the restriction of the inner product (,) to $\mathfrak{k}_i$ is a positive multiple of $-B_i$, because Σ_i is irreducible and (,) is invariant under Aut($\mathfrak{k}_i$). By the relation between $\tilde{\Sigma}_i$ and Σ_i given by Araki [1] (the proof of Proposition 2.1), for $\omega\in\Sigma_i$ there exists a root $\beta\in\tilde{\Sigma}_i$ such that

$$\frac{-(\beta, \beta)}{(\omega, \omega)}=1, 2 \text{ or } 4\,.$$

By (4.3) there exists a root $\alpha\in\tilde{\mathfrak{r}}$ such that $\beta=\bar{\alpha}$, and we have by (4.4)

$$\frac{(\alpha, \alpha)}{(\beta, \beta)}=1, 2 \text{ or } 4\,.$$

Since $\mathfrak{r}$ is irreducible and the inner product (,) on $\mathfrak{g}$ is invariant under Aut($\mathfrak{g}$), $\frac{-(\alpha, \alpha)}{(\lambda, \lambda)}$ is rational by the same reason as above. Therefore $\frac{|\omega|^2}{|\lambda|^2}$ is rational. By (4.13) it is now sufficient to show that $\frac{|\lambda|^2}{(\lambda, H)^2}$ is rational. Let $\Delta=\{\lambda_1, \lambda_2, \cdots, \lambda_{l+1}=\lambda\}$ and put

$$a_{ij}=(\lambda_i, \lambda_j),\ i, j=1, \cdots, l+1\,,$$
$$A_0=1,\ A_s=|a_{ij}|_{i,j=1,\dots,s},\ s=1, \cdots, l+1\,.$$

Then by induction on j, we have easily $A_j>0, j=0, 1, \cdots, l+1$. Put

$$\xi=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1l+1} \\ \cdots\cdots & & & \\ a_{l1} & a_{l2} & \cdots & a_{ll+1} \\ \lambda_1 & \lambda_2 & \cdots & \lambda_{l+1} \end{vmatrix}\,.$$

Then we have easily $(\lambda_i, \xi)=0\,(i=1, \cdots, l)$, $(\lambda_{l+1}, \xi)=A_{l+1}$ and $(\xi, \xi)=(\lambda_{l+1}, \xi)A_l$. Since H is a multiple of ξ, we have $(\lambda_{l+1}, H)^2=\frac{A_{l+1}}{A_l}$. Since $\mathfrak{r}$ is irreducible, we have $\frac{a_{ij}}{|\lambda|^2}=\frac{(\lambda_i, \lambda_j)}{(\lambda_{l+1}, \lambda_{l+1})}$ and these are rational numbers. Hence we have

$$\frac{|\lambda|^2}{(\lambda, H)^2} = \frac{\frac{1}{|\lambda|^{2l}}A_l}{\frac{1}{|\lambda|^{2(l+1)}}A_{l+1}} = \frac{\begin{vmatrix} \frac{a_{11}}{|\lambda|^2} & \frac{a_{12}}{|\lambda|^2} & \cdots & \frac{a_{1l}}{|\lambda|^2} \\ \cdots\cdots & & & \\ \frac{a_{l1}}{|\lambda|^2} & \frac{a_{l2}}{|\lambda|^2} & \cdots & \frac{a_{ll}}{|\lambda|^2} \end{vmatrix}}{\begin{vmatrix} \frac{a_{11}}{|\lambda|^2} & \frac{a_{12}}{|\lambda|^2} & \cdots & \frac{a_{1l+1}}{|\lambda|^2} \\ \cdots\cdots & & & \\ \frac{a_{l1}}{|\lambda|^2} & \frac{a_{l2}}{|\lambda|^2} & \cdots & \frac{a_{ll+1}}{|\lambda|^2} \\ \frac{a_{l+1\,1}}{|\lambda|^2} & \frac{a_{l+1\,2}}{|\lambda|^2} & \cdots & \frac{a_{l+1\,l+1}}{|\lambda|^2} \end{vmatrix}}$$

and this is a rational number. This completes the proof of Theorem 2.

Corollary. *If the submanifold N is minimal and the pair (K, L) is symmetric, then $||A||^2$ is a rational number.*

Proof. Suppose that $\mathfrak{g}$ decomposes into the direct sum $\mathfrak{g}=\mathfrak{g}_1+\cdots+\mathfrak{g}_r$ of ideals $\mathfrak{g}_i$ invariant under σ and $(\mathfrak{g}_i, \sigma)$ is irreducible. Put $\mathfrak{g}_i=\mathfrak{k}_i+\mathfrak{p}_i$, $S_i=S\cap\mathfrak{p}_i$, $\mathfrak{a}_i=\mathfrak{a}\cap\mathfrak{p}_i$ and $(\mathfrak{r}_i)^+_s=\mathfrak{r}^+_s\cap\mathfrak{p}_i$ $(i=1, \cdots, r, s=1, 2)$, where $\mathfrak{k}_i=\mathfrak{k}\cap\mathfrak{g}_i$ and $\mathfrak{p}_i=\mathfrak{p}\cap\mathfrak{g}_i$. Let $H=a_1H_1+\cdots+a_rH_r$, where $H_i\in S_i\cap\mathfrak{a}_i$ and $(\lambda, H_i)\geqslant 0$ for any $\lambda\in\mathfrak{r}^+$. Let N_i be the orbit of K through H_i. Then by Takeuchi [11] and the remark in 2.3, the submanifold N_i of S_i is a symmetric space and minimal is S_i. Put $n_i=\dim N_i$. By (2.7) we have

$$\begin{aligned} nH &= \sum_{\lambda\in\mathfrak{r}_2^+}\frac{m_\lambda}{(\lambda, H)}\lambda \\ &= \sum_{i=1}^{r}\sum_{\lambda\in(\mathfrak{r}_i)_2^+}\frac{m_\lambda}{(\lambda, a_iH_i)}\lambda \\ &= \sum_{i=1}^{r}\frac{n_i}{a_i}H_i\,. \end{aligned}$$

Therefore we have $a_i=\sqrt{\frac{n_i}{n}}$. Applying Theorem 2 and (2.8), the corollary follows by induction on r.

4.3. We give the table of $||A||^2$ in the following cases:
(1) The orthogonal symmetric Lie algebra $(\mathfrak{g}, \sigma)$ is irreducible.
(2) The pair (K, L) is symmetric.
Here $S'(O(p-1)\times O(q-1))$ is the subgroup of $SO(p)\times SO(q)$ consisting of matrices of the form

$$\begin{pmatrix} \varepsilon & O & & \\ O & A & & \\ & & \varepsilon & O \\ & & O & B \end{pmatrix}, \quad \varepsilon = \pm 1, \quad A \in O(p-1), \quad B \in O(q-1).$$

$(\mathfrak{g}, \sigma)$	N	$\dim N$	$\|A\|^2$
A	$SU(p+q)/S(U(p)\times U(q))$	$2pq$	$2pq(pq-1)$
B	$SO(2n+1)/SO(2)\times SO(2n-1)$	$2(2n-1)$	$4(n-1)(2n-1)$
C	$Sp(n)/U(n)$	$n(n+1)$	$\frac{1}{2}n(n+1)(n-1)(n+2)$
D	(1) $SO(2n)/SO(2)\times SO(2n-2)$ (2) $SO(2n)/U(n)$	$4(n-1)$ $n(n-1)$	$4(n-1)(2n-3)$ $\frac{1}{2}n(n-1)(n+1)(n-2)$
E_6	symmetric space of type EIII	32	32×15
E_7	symmetric space of type EVII	54	54×26
AI	$SO(p+q)/S(O(p)\times O(q))$	pq	$\frac{1}{2}pq\left(\frac{pq(p+q+2)}{p+q}-2\right)$
AII	$Sp(p+q)/Sp(p)\times Sp(q)$	$4pq$	$4pq\left(\frac{2pq(p+q-1)}{p+q}-1\right)$
AIII	$U(n)$	n^2	$\frac{1}{2}n^2(n-1)(n+1)$
BDI	(1) $SO(p)\times SO(q)/S'(O(p-1)$ $\times O(q-1))$ (2) $SO(p)$	$p+q-2$ $\frac{1}{2}p(p-1)$	$2(p-1)(q-1)$ $\frac{1}{8}p(p-1)(p-2)(p+2)$
DIII	$U(2n)/Sp(n)$	$n(2n-1)$	$n(n-1)^2(2n+1)$
CI	$U(n)/O(n)$	$\frac{1}{2}n(n+1)$	$\frac{1}{8}n(n-1)(n+2)^2$
CII	$Sp(n)$	$n(2n+1)$	$n(n-1)(n+1)(2n+1)$
EI	$\mathfrak{k}$ is of type C_4 $\mathfrak{l}$ is of type $C_2\times C_2$	16	$\frac{16\times 25}{3}$
EIV	F_4/Spin (9)	16	16×3
EV	$\mathfrak{k}$ is of type A_7 $\mathfrak{l}$ is of type C_4	27	27×14
EVII	$\mathfrak{k}$ is of type $\boldsymbol{R}\times E_6$ $\mathfrak{l}$ is of type F_4	27	26×9

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