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ON F-PROJECTIVE HOMOTOPY OF SPHERES

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We write F for the real (R), complex (C) or quaternoinic (H) numbers. Let FP^n be the F-projective space of n F-dimensions and

$$h_F: S^{(n+1)d-1} \to FP^n$$

the canonical fibration with fibre S^{d-1}, where $d=\dim_R F$. We work in the topological category of pointed spaces and pointed maps. Given a space X and a positive integer m, we define the F-projective homotopy sets

$$\pi_m^F(X) = \begin{cases} h_F^*[FP^n, X] & \text{if } m=(n+1)d-1 \\ 0 & \text{if } m \not\equiv -1(d) \end{cases}$$

and similarly the stable F-projective homotopy groups

$$\pi_m^{SF}(X) = \begin{cases} h_F^*\{FP^n, X\} & \text{if } m=(n+1)d-1 \\ 0 & \text{if } m \not\equiv -1(d) \end{cases}$$

here $\{X, Y\} = \varinjlim [S^r X, S^r Y]$, the limit maps being induced by suspension.

For small j, $\pi_{n+j}^{(S)F}(S^n)$ has been calculated by Bredon [6], Rees [11], Strutt [13] and Randall [10]. In this note we restrict our attention to the case $F=C$ or H. We calculate the Adams e-invariants of elements in $\pi_m^{(S)F}(S^{nd})$ in §1 and estimate the order of a *canonical* element in $\pi_{(k+n+1)d-1}^{(S)F}(S^{nd})$ for $n=1$ in §2 and $n\equiv 0(M_{k+1}(F))$ in §3 (see §§2, 3 for the definitions of "canonical" and $(k+1)$-th F-James number $M_{k+1}(F)$). For example we show that under some assumptions on k and a prime p, if $n\equiv 0(M_{k+1}(F))$ and $\nu_p(n)=\nu_p(M_{k+1}(F))$, $\pi_{(k+n+1)d-1}^{SF}(S^{nd})$ ($\subset \pi_{(k+1)d-1}^S$, the stable $(k+1)d-1$ stem) contains an element of order $p^{\nu_p(k+1)+1}$, where $\nu_p(q)$ denotes the exponent of p in the prime factorization of q.

1. e-invariants of F-projective elements

It is clear that $\pi_{(m+1)d-1}^F(S^{nd})=\pi_{(m+1)d-1}^{SF}(S^{nd})=0$ for $m<n$. For $m\geqq n$, by cellularity

$$\pi_{(m+1)d-1}^F(S^{nd}) = \bar{h}_F^*[FP_n^m, S^{nd}]$$

and similarly for the stable case, here $FP_n^m = FP^m/FP^{n-1}$ and $\bar{h}_F$ denotes the composition of h_F and the natural projection $FP^m \to FP_n^m$.

We introduce the following notations:

$$\phi_F(x) = \begin{cases} \exp(x)-1 & \text{if } F = C \\ \left\{2\mathrm{sh}\dfrac{\sqrt{x}}{2}\right\}^2 & \text{if } F = H \end{cases}$$

$\left(\mathrm{sh}(x) = \dfrac{\exp(x)-\exp(-x)}{2}\right)$; the rational numbers $\alpha_F(n, j)$ defined by

$$\left\{\frac{\phi_F^{-1}(x)}{x}\right\}^n = \sum_{j=0}^{\infty} \alpha_F(n, j)x^j$$

(ϕ_F^{-1} denotes the inverse function of ϕ_F); e, e_R', the Adams complex and real e-invariants [1];

$$\deg\colon [FP_n^{k+n}, S^{nd}] \ (\text{or } \{FP_n^{k+n}, S^{nd}\}) \to Z$$

maps f to the degree of $S^{nd} = FP_n^n \subset FP_n^{k+n} \xrightarrow{f} S^{nd}$; $\xi = \xi_F(m)$, the underlying complex vector bundle of the canonical F line bundle over FP^m; $z = z_F(m) = \xi - \dfrac{d}{2} \in K(FP^m)$; $t = t_F(m) = (-1)^{d/2+1}c_{d/2}(\xi) \in H^d(FP^m; Z)$ ($d/2$-th Chern class); $\beta = z_C(1) \in K(S^2)$, the Bott generator; $\psi^k\colon K(\ ) \to K(\ )$, the Adams operation; ch: $K(\ ) \to H^*(\ ; Q)$, the Chern character. Then the followings are well known.

$$K(FP^m) = Z[z]/z^{m+1}$$
$$H^*(FP^m; Z) = Z[t]/t^{m+1}$$
$$\mathrm{ch}(z) = \phi_F(t)\,.$$

Now we prove the following.

Theorem 1.1. *For $f \in [FP_n^{k+n}, S^{nd}]$ (or $f \in \{FP_n^{k+n}, S^{nd}\}$), we have*

$$e(\bar{h}_F^*(f)) = -\deg(f)\,\alpha_F(n, k+1)\,.$$

Proof. Consider the following commutative diagram

$$\begin{tikzcd}
S^{(k+n+1)d-1} \arrow[r, "\bar{h}_F"] \arrow[d, "="] & FP_n^{k+n} \arrow[r] \arrow[d, "f"] & FP_n^{k+n+1} \arrow[r] \arrow[d, "\tilde{f}"] & S^{(k+n+1)d} \arrow[d, "="] \\
S^{(k+n+1)d-1} \arrow[r, "\bar{f}"] & S^{nd} \arrow[r, "i"] & C_{\bar{f}} \arrow[r, "j"] & S^{(k+n+1)d}
\end{tikzcd}$$

where the horizontal sequences are cofibrations. Then we have the commutative diagram of the short exact sequences

$$\begin{array}{ccccccccc}
0 \longleftarrow & \tilde{K}(FP_n^{k+n}) & \longleftarrow & \tilde{K}(FP_n^{k+n+1}) & \longleftarrow & \tilde{K}(S^{(k+n+1)d}) & \longleftarrow 0 \\
& \uparrow f^* & & \uparrow \bar{f}^* & & \uparrow = & \\
0 \longleftarrow & \tilde{K}(S^{nd}) & \overset{i^*}{\longleftarrow} & \tilde{K}(C_{\bar{f}}) & \overset{j^*}{\longleftarrow} & \tilde{K}(S^{(k+n+1)d}) & \longleftarrow 0 .
\end{array}$$

Let $a \in K(C_{\bar{f}})$ be such that $i^*(a)=\beta^{nd/2}$. Let $b=j^*(\beta^{(k+n+1)d/2})$. Then

$$\psi^2(a) = d^n a+\lambda b \quad \text{for some } \lambda \in Z ,$$

and

$$e(\bar{f}) = \frac{\lambda}{d^n(d^{k+1}-1)} \in Q/Z .$$

Let

$$\bar{f}^*(a) = \sum_{i=0}^{k+1} a_i z^{i+n} .$$

Then

$$\begin{aligned}
\psi^2 \bar{f}^*(a) &= \sum_{i=0}^{k+1} a_i(\psi^2(z))^{i+n} = \sum_{i=0}^{k+1} a_i(z^2+dz)^{i+n} \\
&= \sum_{j=0}^{k+1} \sum_{i=0}^{k+1} a_i \binom{n+i}{j-i} d^{n+2i-j} z^{n+j}
\end{aligned}$$

and this equals

$$\bar{f}^*\psi^2(a) = \bar{f}^*(d^n a+\lambda b) = d^n \sum_{i=0}^{k+1} a_i z^{n+i}+\lambda z^{k+n+1} ,$$

so that comparing the coefficients of z^{k+n+1} we have

$$\lambda = \sum_{i=0}^{k} a_i \binom{n+i}{k+1-i} d^{n+2i-(k+1)}+d^n(d^{k+1}-1)a_{k+1}$$

and so

$$e(\bar{f}) = \frac{\sum_{i=0}^{k} a_i \binom{n+i}{k+1-i} d^{n+2i-(k+1)}}{d^n(d^{k+1}-1)} . \tag{1.2}$$

Consider the commutative diagram

$$\begin{array}{ccc}
K(FP_n^{k+n}) & \overset{f^*}{\longleftarrow} & K(S^{nd}) \\
\downarrow \mathrm{ch} & & \downarrow \mathrm{ch} \\
H^*(FP_n^{k+n}; Q) & \overset{f^*}{\longleftarrow} & H^*(S^{nd}; Q) .
\end{array}$$

Then

$$f^*(\beta^{nd/2}) = \sum_{i=0}^{k} a_i z^{n+i}$$

and

$$(1.3) \qquad \deg(f)t^n = f^*\mathrm{ch}(\beta^{nd/2}) = \mathrm{ch} f^*(\beta^{nd/2}) = \sum_{i=0}^{k} a_i(\mathrm{ch}(z))^{n+i}$$
$$= \sum_{i=0}^{k} a_i \phi_F(t)^{n+1} .$$

By definition

$$(\phi_F^{-1}(x))^n = \sum_{j=0}^{\infty} \alpha_F(n, j) x^{n+j}$$
$$x = \phi_F^{-1}\phi_F(x)$$

so that

$$t^n = \sum_{j=0}^{\infty} \alpha_F(n, j)\phi_F(t)^{n+j} .$$

Then by (1.3)

$$a_i = \deg(f)\alpha_F(n, i) \quad \text{for } 0 \leqq i \leqq k ,$$

so that by (1.2)

$$(1.4) \qquad e(\tilde{f}) = \frac{\deg(f) \sum_{j=0}^{k} \alpha_F(n, j) \binom{n+j}{k+1-j} d^{n+2j-(k+1)}}{d^n(d^{k+1}-1)} .$$

Next we observe that the function ϕ_F^{-1} satisfies the equation

$$\phi_F^{-1}(x^2+dx) = d\phi_F^{-1}(x) .$$

Then

$$(\phi_F^{-1}(x^2+dx))^n = \sum_{j=0}^{\infty} \alpha_F(n, j) (x^2+dx)^{n+j}$$
$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \alpha_F(n, j) \binom{n+j}{i-j} d^{n+2j-i} x^{n+i}$$

equals

$$(d\phi_F^{-1}(x))^n = d^n \sum_{i=0}^{\infty} \alpha_F(n, i) x^{n+i}$$

so that comparing the coefficients of x^{k+n+1}, we have

$$\sum_{j=0}^{k} \alpha_F(n, j) \binom{n+j}{k+1-j} d^{n+2j-(k+1)} = d^n(1-d^{k+1})\alpha_F(n, k+1)$$

and then by (1.4)

$$e(\tilde{f}) = -\deg(f)\alpha_F(n, k+1) .$$

This completes the proof of Theorem 1.1.

Using KO^*-theory, we can obtain lower bounds of $\deg(f)$ (e.g. [8], [9]), but now we need upper bounds and unfortunately we have not sharp estimation with the exception of the two special cases $n=1$ and $n\equiv 0(M_{k+1}(F))$. In the following two sections we will study these two cases.

2. $\pi^{(S)F}_{(k+n+1)d-1}(S^{nd})$ for $n=1$

For a positive integer q, it is well known that the order of the composition

$$S^{2d-1} \xrightarrow{h_F} FP^1 = S^d \xrightarrow{q} S^d$$

is infinite, so that

$$\deg(f) = 0 \quad \text{for } f\in[FP^{k+1}, S^d] \ (k>0)$$

and so by Theorem 1.1

$$e = 0\colon \pi^F_{(k+2)d-1}(S^d) \longrightarrow Q/Z \ (k>0)\,.$$

By induction on k we know that the rank of $\{FP^{k+n}_n, S^{nd}\}$ is one. We will call a generator of this free part (and its image by $\bar{h}^*_F$) a *canonical* element. Let $f\in\{FP^{k+n}_n, S^{nd}\}$ be a canonical element, then (take $-f$ if necessary)

$$\deg(f) = k_s(FP^{k+n}_n, S^{nd})$$

where the right hand side has been defined in [8] and called the stable James number of the pair (FP^{k+n}_n, S^{nd}). In particuler we have used the notation

$$d_F(k+1) = k_s(FP^{k+1}, S^d)$$

and this has been estimated in [7], [8] and [9].

Proposition 2.1. *For an odd prime p and an integer $l\geqq 1$, e-invariant of a canonical element in $\pi^{SC}_{2pl-1}(S^2)$ (or $\pi^{SH}_{2p+1}(S^4)$) is of order p (or a multiple of p).*

Proof. (i) $F=C$. We have

$$\frac{\phi_C^{-1}(x)}{x} = \frac{\log(1+x)}{x} = \sum_{i=0}^{\infty}\frac{(-1)^i}{i+1}x^i$$

so that

$$\alpha_C(1, k+1) = \frac{(-1)^{k+1}}{k+2}$$

and then for a canonical element $f\in\{CP^{k+1}, S^2\}$

$$e(h^*_C(f)) = (-1)^k\frac{d_C(k+1)}{k+2}\,.$$

Suppose that $k+2=uv$, where u and v are relatively prime integers and not one. Then by [8], u, v and hence uv devide $d_C(k+1)$. Therefore $e(h^*_C(f))=0$. In case with $k+2=2^w$ for $w\geqq 2$, 2^w devides $d_C(2^w-1)$ [8] and hence $e(h_C^*(f))=0$. If $k+2=p^l$ for an odd prime p and a positive integer l, [8] says that $\nu_p(d_C(p^l-1))=l-1$ so that the order of $e(h_C^*(f))$ is p. This completes the proof of Proposition 2.1 for $F=C$.

(ii) $F=H$. We have

$$\frac{\phi_H^{-1}(x)}{x}=\left(\frac{\mathrm{sh}^{-1}\frac{\sqrt{x}}{2}}{\frac{\sqrt{x}}{2}}\right)^2=\sum_{i=0}^{\infty}\frac{(-1)^i}{2^{4i}}\sum_{u+v=i}\frac{(2u)!(2v)!}{(u!)^2(v!)^2(2u+1)(2v+1)}x^i$$

so that

$$\alpha_H(1,k+1)=\frac{(-1)^{k+1}}{2^{4k+4}}\sum_{i+j=k+1}\frac{(2i)!(2j)!}{(i!)^2(j!)^2(2i+1)(2j+1)}\,.$$

Therefore if $2k+3=p$, a prime,

$$\nu_p(\alpha_H(1,k+1))=-1\,.$$

On the other hand by [9]

$$d_H(k+1)\,|\,(2k+2)!(2k)!\cdots 4!$$

so that by Theorem 1.1 for a canonical element $f\in\{HP^{k+1},S^4\}$

$$\nu_p(e(h_H^*(f))=-1\,.$$

This completes the proof of Proposition 2.1.

3. $\pi_{(k+n+1)d-1}^{(S)F}(S^{nd})$ for $n\equiv 0(M_{k+1}(F))$

First we repeat the basic relations of the James number $M_{k+1}(F)$, $\alpha_F(n,j)$ and the coreducibility of FP_n^{k+n} as given in Adams-Walker [2], Atiyah [4] [5], Atiyah-Todd [3] and Sigrist-Suter [12].

Let $M_{k+1}(F)$ be the order of $J(\xi)$ in the J-group $J(FP^k)$ [4].

Lemma 3.1. ([2], [12]) *For a prime p, we have*

$$\text{(i)}\quad \nu_p(M_{k+1}(C))=\begin{cases}\max(r+\nu_p(r)),\ 1\leqq r\leqq \dfrac{k}{p-1} & \text{if } p\leqq k+1\\ 0 & \text{if } p>k+1\,.\end{cases}$$

(ii) $\nu_2(M_{k+1}(H))=\max(2k+1,\,2r+\nu_2(r))$, $1\leqq r\leqq k$,
$\nu_p(M_{k+1}(H))=\nu_p(M_{2k+2}(C))$ *if p odd.*

Lemma 3.2. ([5, p. 143], [3], [12]) *The following three statements are equivalent.*

(i) $n\equiv 0(M_{k+1}(F))$

(ii) *for* $0 \leqq j \leqq k$, $\alpha_F(n, j) \in \begin{cases} Z & \text{if } F=C \text{ or } F=H \text{ and } j \text{ even} \\ 2Z & \text{if } F=H \text{ and } j \text{ odd} \end{cases}$

(iii) FP_n^{k+n} *is coreducible, that is, there exists a retraction* $FP_n^{k+n} \to S^{nd}$.

When above equivalent conditions are satisfied, for a retraction f: $FP_n^{k+n} \to S^{nd}$ we have

$$e(\bar{h}_F^*(f)) = -\alpha_F(n, k+1) . \tag{3.3}$$

Therefore next we have to compute $\alpha_F(n, k+1)$. Remark that f represents a canonical element in the stable category.

Lemma 3.4. ([3], [12]) *Let n be a positive integer, k a non negative integer and p a prime (an odd prime if $F=H$). Then we have*

(i) $\nu_p(\alpha_F(n, j)) \geqq 0$ *for* $0 \leqq j \leqq k$ *if and only if* $\nu_p(n) \geqq \nu_p(M_{k+1}(F))$,

(ii) $\nu_2(\alpha_H(n, j)) \geqq \begin{cases} 0 & j \text{ even} \\ 1 & j \text{ odd} \end{cases}$ *for* $0 \leqq j \leqq k$ *if and only if* $\nu_2(n) \geqq \nu_2(M_{k+1}(H))$,

(iii) *if* $\nu_2(n) \geqq 2j-1$, $\nu_2(n) = 2j + \nu_2(j) + \nu_2(\alpha_H(n, j))$.

In §1 we defined the coefficients $\alpha_C(n, j)$ by the formula

$$\sum_{j=0}^{\infty} \alpha_C(n, j) x^j = \left(\frac{\phi_C^{-1}(x)}{x}\right)^n = \left(\frac{\log(1+x)}{x}\right)^n = \left(\sum_{i=0}^{\infty} \frac{(-1)^i}{i+1} x^i\right)^n .$$

Using the multinomial expansion we find

$$\begin{aligned} \alpha_C(n, j) &= (-1)^j \sum_{s} \frac{n!}{s_0! s_1! \cdots s_j!} \prod_{i=0}^{j} \frac{1}{(i+1)^{s_i}} \\ &= (-1)^j \sum_{s} T(n, j, s), \text{ say,} \end{aligned} \tag{3.5}$$

where the summation extends over all ordered sets $s = (s_0, s_1, \cdots, s_j)$ of non negative integers such that $\sum s_i = n$, $\sum i s_i = j$.

Lemma 3.6. ([3, 6.5]) *Let p be a prime and k a non negative integer. Suppose that $\nu_p(\alpha_C(n, j)) \geqq 0$ for $0 \leqq j \leqq k$. Then*

$\nu_p(T(n, k+1, s)) \geqq 0$ for all sequences s in (3.5), with the following possible exception: if $k+1 = s(p-1)$ with s integral, and if s is the sequence in which $s_0 = n-s$, $s_{p-1} = s$, and all other s_i are zero, we have

$$\nu_p(T(n, k+1, s)) = \nu_p(n) - \nu_p(s) - s .$$

Lemma 3.7. (i) *Let p be a prime (an odd prime if $F=H$), n and k non negative integers. Suppose that $\nu_p(M_{k+1}(F)) \leqq \nu_p(n) < \nu_p(M_{k+2}(F))$. Then $\frac{(k+1)d}{2} = s(p-1)$ for some integer s and*

$$\nu_p(\alpha_F(n, k+1)) = \nu_p(n) - \nu_p(M_{k+2}(F)) .$$

(ii) *If* $\nu_2(M_{k+1}(H))\leqq\nu_2(n)$, $\nu_2(\alpha_H(n, k+1)) = \nu_2(n)-2(k+1)-\nu_2(k+1)$.

Proof. By (3.1)

$$\nu_2(M_{k+1}(H))\geqq 2k+1$$

so that (ii) follows from (3.4).

(i) for $F=C$ follows from (3.1), (3.5) and (3.6) immediately.

We define the rational numbers $d_i(n)$ by

$$\sum_{i=0}^{\infty} d_i(n)y^i = \left(\frac{\mathrm{sh}^{-1}y}{y}\right)^{2n}$$

then

$$d_{2i}(n) = 2^{2i}\alpha_H(n, i),\ d_{2i+1} = 0\,. \tag{3.8}$$

Recall that $\mathrm{sh}^{-1}y=\log(y+\sqrt{1+y^2})$. The power series of $y+\sqrt{1+y^2}$ is of the form $1+g(y)$, where $g(y)$ has the inverse $g^{-1}(x)=x-\dfrac{1}{2}\sum_{i=2}^{\infty}(-1)^i x^i$. We have

$$\sum_{i=0}^{\infty} d_i(n)y^{i+2n} = (\mathrm{sh}^{-1}y)^{2n} = (\log(1+g(y)))^{2n} = \sum_{i=0}^{\infty}\alpha_C(2n, i)g(y)^{i+2n}\,.$$

Put $y=g^{-1}(x)$. Then for non negative integer j we have

$$\sum_{i=0}^{j} d_i(n)\sum_{s}\frac{(i+2n)!}{s_1!s_2!\cdots}\frac{(-1)^{i+j}}{2^{i+2n-s_1}} = \alpha_C(2n, j) \tag{3.9}$$

where the summation $\sum_{s}$ extends over all ordered sets $s=(s_1, s_2, \cdots)$ of non negative integers such that $\sum s_u=i+2n$, $\sum us_u=j+2n$. Hence for an odd prime p and a positive integer m we have

(3.10) $\nu_p(d_i(n))\geqq 0$ *for* $0\leqq i\leqq m$ *if and only if*
$\nu_p(\alpha_C(2n, j))\geqq 0$ *for* $0\leqq j\leqq m$.

If these equivalent conditions are satisfied, (3.9) says that $\nu_p(d_{m+1}(n))$ or $\nu_p(\alpha_C(2n, m+1))<0$ implies $\nu_p(d_{m+1}(n))=\nu_p(\alpha_C(2n, m+1))$. Therefore

(3.11) if $\nu_p(\alpha_C(2n, j))\geqq 0$ for $0\leqq j\leqq 2k+1$ and $\nu_p(\alpha_C(2n, 2k+2))<0$, then $\nu_p(\alpha_H(n, k+1)) = \nu_p(d_{2k+2}(n)) = \nu_p(\alpha_C(2n, 2k+2))$.

Suppose that $\nu_p(M_{k+1}(H))\leqq\nu_p(n)<\nu_p(M_{k+2}(H))$ for an odd prime p. Then by (3.4)

$$\nu_p(\alpha_H(n, j))\geqq 0 \quad \text{for } 0\leqq j\leqq k$$

and by (3.8)

$$\nu_p(d_j(n))\geqq 0 \quad \text{for } 0\leqq j\leqq 2k+1$$

and by (3.10)

$$\nu_p(\alpha_C(2n, j)) \geqq 0 \quad \text{for } 0 \leqq j \leqq 2k+1$$

so that by (3.1) and (3.11) we know that $2k+2=s(p-1)$ with s integral and

$$\nu_p(\alpha_C(2n, 2k+2)) = \nu_p(2n)-\nu_p(s)-s = \nu_p(n)-\nu_p(M_{k+2}(H)) < 0 .$$

This implies (i) for $F=H$ and completes the proof of Lemma 3.7.

Now we will estimate the order of the e-inavriant of a canonical element. Let $\sharp a$ denote the order of an element a of a module.

Proposition 3.12. *Suppose that* $n \equiv 0(M_{k+1}(F))$*) *and let* $f\colon FP_n^{k+n} \to S^{nd}$ *be a retraction.*

(i) *Let p be a prime (an odd prime if $F=H$) and suppose that* $\nu_p(M_{k+1}(F)) \leqq \nu_p(n) < \nu_p(M_{k+2}(F))$. *Then*

$$\nu_p(\sharp e(\bar{h}_F^*(f))) = \nu_p(M_{k+2}(F)) - \nu_p(n) .$$

Moreover, in case $k \equiv 1$ (4) and $(F, p)=(C, 2)$, considering f as a stable map (or if $n \equiv 0$ (4)), we have

$$\nu_2(\sharp e_R'(\bar{h}_C^*(f))) = \nu_2(M_{k+2}(C)) - \nu_2(n) + 1 .$$

(ii) *If* $\nu_2(M_{k+1}(H)) \leqq \nu_2(n) < 2(k+1) + \nu_2(k+1)$,

$$\nu_2(\sharp e(\bar{h}_H^*(f))) = 2(k+1) + \nu_2(k+1) - \nu_2(n) .$$

Moreover in case $k \equiv 0$ (2) and $n \equiv 0$ (2), we have

$$\nu_2(\sharp e_R'(\bar{h}_H^*(f))) = 2(k+1) + \nu_2(k+1) - \nu_2(n) + 1 .$$

Proof. (3.3), (3.7) and the fact

$$e = 2e_R'\colon \pi_{8q+r}(S^{8q}) \to Q/Z \quad \text{if } r \equiv 3(8) \quad [1, 7. 14]$$

imply Proposition 3.12.

Suppose that $\nu_p(M_{k+1}(F)) \leqq \nu_p(n) < \nu_p(M_{k+2}(F))$. Then $\dfrac{(k+1)d}{2} = s(p-1)$ with s integral as seen before. Put $s=p^l u$, $u \not\equiv 0(p)$ for integers l, u. Then by (3.1)

$$\nu_p(M_{k+2}(F)) - \nu_p(n) \leqq \nu_p(M_{k+2}(F)) - \nu_p(M_{k+1}(F)) \leqq \begin{cases} l+1 & \text{if } (F, p) \neq (H, 2) \\ \max(l+1, 2) & \text{if } (F,p) = (H,2) . \end{cases}$$

*) Using S-duality and a theorem of Sigrist (Ill. J. Math. 13 (1969), 198–201), we can show that this hypothesis can be removed but then f must be canonical. The same remark is valid for the next corollary.

In the following Corollary 3.13, we will give a condition that implies

$$\nu_p(M_{k+2}(F))-\nu_p(n)=\begin{cases} l+1 & \text{if } (F, p)\neq(H, 2) \\ \max(l+1, 2) & \text{if } (F, p)=(H, 2)\,. \end{cases}$$

Corollary 3.13. *Let p be a prime. Suppose that $n\equiv 0(M_{k+1}(F))$ and $\nu_p(n)=\nu_p(M_{k+1}(F))$. Let $f\colon FP_n^{k+n}\to S^{nd}$ be a retraction.*

(i) *If $(F, p)\neq(H, 2)$ and k satisfies*

$$\frac{(k+1)d}{2}=p^l u(p-1),\ u\not\equiv 0(p),\quad \begin{matrix} u<p^{l+1} & (p \text{ odd}) \\ u<2^l & (p=2) \end{matrix}$$

for some integers u and l, then

$$\nu_p(\#e(\bar{h}_F^*(f)))=l+1\,.$$

(ii) *If k satisfies*

$$k+1=2^l u,\ u\not\equiv 0(2),\ u<2^{l+2}$$

then

$$\nu_2(\#e(\bar{h}_H^*(f)))=\begin{cases} l+1 & \text{if } l\geqq 1 \\ 2 & \text{if } l=0 \end{cases}$$

and moreover in case $k=0$ or 2 and $n\equiv 0\ (2)$ we have

$$\nu_2(\#e_R'(\bar{h}_H^*(f)))=3$$

Proof. Using (3.1) and the fact [3]

$$M_{2k+1}(C)=M_{2k+2}(C)\quad \text{for } k\geqq 1$$

we can prove this Corollary by elementary calculation, so we omit the proof.

Remark. If $n\equiv 0(M_{k+1}(F))$, we have

$$\pi^F_{(k+n+1)d-1}(S^{nd})\xrightarrow[\cong]{}\pi^{SF}_{(k+n+1)(d-1)}(S^{nd})$$

with the exception of $(F, k, n)=(C, 0, 1)$, $(C, 1, 2)$ or $(H, 0, 1)$. For these three cases, we list up the results without proof.

Proposition 3.14.

$$\pi_3^C(S^2)=\{k^2\eta;\ k\in Z\}$$

$$\pi_7^C(S^4)=\left\{k^2\nu+\frac{k(k-1)}{2}\delta+6l\delta;\ k\in Z,\ l=0 \text{ or } 1\right\}$$

$$\pi_7^H(S^4)=\left\{k^2\nu+\frac{k(k-1)}{2}\delta;\ k\in Z\right\}$$

$$\pi_3^{S\,C}(S^2) = \pi_1^s = Z_2$$
$$\pi_7^{S\,C}(S^4) = \pi_7^{S\,H}(S^4) = \pi_3^s = Z_{24}\,.$$

where $\pi_3(S_2)=Z=\{\eta\}$ *and* $\pi_7(S^4)=Z\oplus Z_{12}=\{\nu\}\oplus\{\delta\}$.

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