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Osaka J. Math.
14 (1977), 117-123

A REMARK ON THE MINLOS-POVZNER TAUBERIAN THEOREM

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(Received November 26, 1975)

In a study of the spectral theory of a random difference operator we utilized without proof a Tauberian theorem on the Laplace transform of a Stieltjes measure supported by $(-\infty, \infty)$ ([1]; Lemma 2). This is also used in [2] and [3]. In the present note we first prove a nontrivial modification of the Minlos-Povzner Tauberian theorem ([4]; Appendix) and then, as its consequence (Corollary 2), derive the above-stated Tauberian theorem on the bilateral Laplace transform.

1. Let $\Phi_t(\beta)$ and $\phi_t(\xi)$ be functions on $(0, \infty)$ related by, for each t,

$$\exp\{p(t)\Phi_t(\beta)\} = \int_0^\infty \exp\{p(t)(\beta\xi-\phi_t(\xi))\}d\xi \tag{1.1}$$

where $p(t)$ is a non-decreasing function tending to infinity as $t\to\infty$.

Theorem. (i) *If $\Phi_t(\beta)$ converges to a function $\Phi(\beta)$ as $t\to\infty$, $\phi_t(\xi)$ is a non-decreasing function for each t such that there exists $\varepsilon(t)$ satisfying $\lim_{t\uparrow\infty}\varepsilon(t)=0$, $\lim_{t\uparrow\infty}\frac{\log\varepsilon(t)}{p(t)}=0$, and $\lim_{t\uparrow\infty}|\phi_t(0+)-\phi_t(\varepsilon(t))|=0$, then $\phi_t(\xi)$ has a limit at every regular point ξ ([4]) of $\Phi(\beta)$ and*

$$\lim_{t\uparrow\infty}\phi_t(\xi) = \sup_{\beta>0}\{\beta\xi-\Phi(\beta)\}\ .$$

(ii) *If a non-deereasing function $\phi_t(\xi)$ converges to $\phi(\xi)$ uniformly in any finite interval and thete exists a function $c(\beta)$ such that $\Phi_t(\beta)<c(\beta)$ for any β and t, then $\Phi_t(\beta)$ has a limit and*

$$\lim_{t\uparrow\infty}\Phi_t(\beta) = \sup_{\xi>0}\{\beta\xi-\phi(\xi)\}\ .$$

For the proof of the first assertion we prepare four Lemmas.

Lemma 1. *For any $\varepsilon>0$*

$$\beta\xi-\phi_t(\xi)\leqq(\beta-\gamma)\xi+K(\gamma;t,\varepsilon),\ \xi>\varepsilon,\ \beta>0,\ \gamma>0 \tag{1.2}$$

where $K(\gamma;t,\varepsilon)$ is such that $\lim_{\varepsilon\downarrow0}\lim_{t\uparrow\infty}K(\gamma;t,\varepsilon)=\Phi(\gamma)$

Proof. For $\xi \geqq \varepsilon$

$$\exp\{p(t)\Phi_t(\gamma)\} \geqq \int_{\xi-\varepsilon}^{\xi} \exp\{p(t)(\gamma\zeta - \phi_t(\zeta)\} d\zeta$$
$$\geqq \varepsilon \exp\{p(t)(\gamma(\xi-\varepsilon) - \phi_t(\varepsilon))\} .$$

Therefore

$$-\phi_t(\xi) \leqq -\gamma\xi + \Phi_t(\gamma) - \frac{\log \varepsilon}{p(t)} + \gamma\varepsilon . \tag{1.3}$$

Adding $\beta\xi$ to each hand side and putting $K(\gamma; t, \beta) = \Phi_t(\gamma) - \frac{\log \varepsilon}{p(t)} + \varepsilon\gamma$, we have the Lemma because of the assumption that $\lim_{t\uparrow\infty} \Phi_t(\gamma) = \Phi(\gamma)$ and $\lim_{t\uparrow\infty} p(t) = \infty$.

Lemma 2. *There exists a constant c such that $-\phi_t(0+) > c$ for all sufficiently large t.*

Proof.
$$\exp\{p(t)\Phi_t(\beta)\} = \int_0^b \exp\{p(t)(\beta\xi - \phi_t(\xi))\} d\xi$$
$$+ \int_b^\infty \exp\{p(t)(\beta\xi - \phi_t(\xi)\} d\xi$$
$$\equiv J_1(b, \beta, t) + J_2(b, \beta, t) .$$

By making use of (1.2), for $\beta < \gamma$

$$J_2(b, \beta, t) \leqq \int_b^\infty \exp\{p(t)((\beta-\gamma)\xi + K(\gamma; t, \varepsilon))\} d\xi$$
$$= \frac{\exp\{p(t)((\beta-\gamma)b + K(\gamma; t, \varepsilon))\}}{(\gamma-\beta)p(t)} ,$$

then

$$J_1(b, \beta, t) \geqq \exp\{p(t)\Phi_t(\beta)\} \left[1 - \frac{\exp\{p(t)((\beta-\gamma)b + K(\gamma; t, \varepsilon) - \Phi_t(\beta))\}}{(\gamma-\beta)p(t)}\right]$$

By taking sufficiently large t, it holds that

$$(\beta-\gamma)b + K(\gamma; t, \varepsilon) - \Phi_t(\beta) < 0$$

for all sufficiently large t. So we have

$$J_1(b, \beta, t) \geqq \frac{1}{2} \exp\{p(t)\Phi_t(\beta)\} \tag{1.4}$$

for all sufficiently large t. On the other hand

$$J_1(b, \beta, t) \leqq \int_0^b \exp\{p(t)(\beta\xi - \phi_t(0+))\} d\xi$$
$$= \frac{\exp\{\beta b - \phi_t(0+)\}}{\beta p(t)} [1 - \exp\{-p(t)\beta b\}] .$$

Combining this with (1.4), we have

$$p(t)\Phi_t(\beta)\leqq\log\frac{2(1-\exp\{-p(t)\beta b\})}{p(t)}+p(t)\{\beta b-\phi_t(0+)\}\ .$$

Then

$$-\phi_t(0+)\geqq\Phi_t(\beta)-\beta b+o(1)\quad\text{as } t\to\infty\ .$$

Therefore we get our Lemma.

Lemma 3. *$\sup_{\xi>0}\{\beta\xi-\phi_t(\xi)\}$ has a limit as $t\to\infty$ and $\lim_{t\uparrow\infty}\sup_{\xi>0}\{\beta\xi-\phi_t(\xi)\}$ $=\Phi(\beta)$.*

Proof. We have

$$\begin{aligned}J_1(a,\beta,t)&\leqq a\exp\{P(t)\sup_{\xi>0}(\beta\xi-\phi_t(\xi))\\&\exp\{-p(t)\sup_{\xi>0}(\beta\xi-\phi_t(\xi))\}J_2(a,\beta,t)\\&\leqq\exp\{-cp(t)\}\frac{\exp\{p(t)((\beta-\gamma)a+K(\gamma;t,\beta))}{(\gamma-\beta)p(t)}\end{aligned}$$

for $\beta<\gamma$.
Taking sufficiently large a, it holds that

$$-c+(\beta-\gamma)a+K(\gamma;t,\varepsilon)<0$$

for all sufficiently large t. Therefore

$$\exp\{p(t)\Phi_t(\beta)\}\leqq\exp\{p(t)\sup_{\xi>0}(\{\beta\xi-\phi_t(\xi)\}\left[a+\frac{1}{(\gamma-\beta)p(t)}\right]$$

for all sufficiently large t, that is,

$$\Phi_t(\beta)+o(1)\leqq\sup_{\xi>0}\{\beta\xi-\phi_t(\xi)\}\quad\text{as } t\to\infty\ .$$

On the other hand we have, by (1.3),

$$\Phi_t(\beta)+\beta\varepsilon(t)-\frac{\log\varepsilon(t)}{p(t)}\geq\sup_{\varepsilon(t)\leqq\xi}\{\beta\xi-\phi_t(\xi)\}\ .$$

From the assumption with respect to $\varepsilon(t)$, it follows that

$$|\sup_{\xi>0}\{\beta\xi-\phi_t(\xi)\}-\sup_{\varepsilon(t)\leqq\xi}\{\beta\xi-\phi_t(\xi)\}|=o(1)\quad\text{as } t\to\infty\ .$$

Hence

$$\Phi_t(\beta)+o(1)\geqq\sup_{\xi>0}\{\beta\xi-\phi_t(\xi)\}\quad\text{as } t\to\infty\ .$$

Lemma 4. *Let ξ be an R-point* ([4]) *and β_ξ be a subordinate to ξ* ([4]), *then there exists $S_t(\beta_\xi)$ such that $\lim_{t\uparrow\infty}\phi_t(S_t(\beta_\xi))=\beta_\xi\xi-\Phi(\beta_\xi)$ and $\lim_{t\uparrow\infty}S_t(\beta_\xi)=\xi$.*

Proof. At first we note that there exists $A(\beta)<\infty$ such that

$$\sup_{0<\xi}\{\beta\xi-\phi_t(\xi)\} = \sup_{0<\xi<A(\beta)}\{\beta\xi-\phi_t(\xi)\}$$

for all sufficiently large t. We can verify this by making use of (1.2) and Lemma 2 in a similar way as the proof of Lemma 5 in [4]. Therefore we can define $\bar{S}_t(\beta_\xi)$ such that

$$|\sup_{0<\zeta}\{\beta_\xi\zeta-\phi_t(\zeta)\}-\{\beta_\xi\bar{S}_t(\beta_\xi)-\phi_t(\bar{S}_t(\beta_\xi))\}|<\frac{1}{t}$$

and

$$\bar{S}_t(\beta_\xi)<A(\beta_\xi)$$

for all sufficiently large t. Since $\{\bar{S}_n(\beta_\xi)\}_{n:\text{integer}}$ is bounded, it has a subsequence which converges. We write it $S_n(\beta_\xi)$ and define $S_t(\beta_\xi)\equiv S_{[t]}(\beta_\xi)$, then from Lemma 3 it follows that

$$\lim_{t\uparrow\infty}\{\beta_\xi S_t(\beta_\xi)-\phi_t(S_t(\beta_\xi))\} = \Phi(\beta_\xi)\,.$$

Putting $\bar{\xi}=\lim_{t\uparrow\infty} S_t(\beta_\xi)$, we have

$$\lim_{t\uparrow\infty} \phi_t(S_t(\beta_\xi)) = \beta_\xi\bar{\xi}-\Phi(\beta_\xi)\,.$$

In the case that $\bar{\xi}\neq 0$, by (1.3) for any γ

$$\phi_t(S_t(\beta_\xi))\geqq\gamma S_t(\beta_\xi)-K(\gamma;\, t,\, \varepsilon)$$

for all sufficiently large t. Therefore for any γ $\beta_\xi\bar{\xi}-\Phi(\beta_\xi)\geqq\gamma\bar{\xi}-\Phi(\gamma)$, which means

$$\beta_\xi\bar{\xi}-\Phi(\beta_\xi) = \sup_{0<\gamma}\{\gamma\bar{\xi}-\Phi(\gamma)\}\,.$$

Since β_ξ is subordinate to ξ we get $\bar{\xi}=\xi$. In the case that $\bar{\xi}=0$ we can define $S_t(\beta_\xi)$ with additional condition $S_t(\beta_\xi)\geqq\varepsilon(t)$ because $|\sup_{0<\zeta}\{\beta_\xi\zeta-\phi_t(\zeta)\}-\sup_{\varepsilon(t)\leqq\zeta}\{\beta_\xi\zeta-\phi_t(\zeta)\}|=o(1)$. Then we get $\bar{\xi}=\xi$ in the same way as the case $\bar{\xi}\neq 0$.

Now we can get the first assertion of our theorem by making use of our lemmas in the same way as in [4].

We turn to the proof of second assertion of our theorem.

Proof of (ii). From (1.3) and the assumptions it follows that

$$\beta\xi-\phi(\xi)\leqq(\beta-\gamma)\xi+c(\gamma)+\gamma\varepsilon\,.$$

Since the right hand side tends to $-\infty$ as $\xi\to\infty$ for fixed $\beta<\gamma$, there exists $A'(\beta)$ such that

$$\sup_{0<\xi}\{\beta\xi-\phi(\xi)\} = \sup_{0<\xi\leqq A'(\beta)}\{\beta\xi-\phi(\xi)\}\,.$$

On the other hand,

$$\exp\{p(t)\Phi_t(\beta)\} = J_1(l, \beta, t)+J_2(l, \beta, t)$$

$$< l \exp\{p(t) \sup_{0<\xi<l} (\beta\xi-\phi_t(\xi))\} + \frac{\exp\left\{p(t)\left((\beta-\gamma)l+c(\gamma)+\gamma\varepsilon+\frac{\log\varepsilon}{p(t)}\right)\right\}}{p(t)(\gamma-\beta)}$$

for $l>\varepsilon$.

Taking sufficiently large l

$$(\beta-\gamma)+c(\gamma)+\gamma\varepsilon+\frac{\log\varepsilon}{p(t)}<0$$

for all sufficiently large t. Put $\delta(t, l)= \sup_{0<\xi\leqq l} |\phi_t(\xi)-\phi(\xi)|$, then we have

$$\Phi_t(\beta) \leqq \sup_{0<\xi<l} \{\beta\xi-\phi(\xi)\} + \delta(t, l)+o(1), \quad \text{as } t\to\infty .$$

Taking $l>A'(\beta)$ in advance

$$\Phi_t(\beta) \leqq \sup_{0<\xi} \{\beta\xi-\phi(\xi)\} + o(1), \quad \text{as } t\to\infty .$$

The converse inequality also follows from (1.3):

$$\Phi_t(\beta) \geqq \sup_{\varepsilon<\xi<A'(\beta)} \{\beta\xi-\phi_t(\xi)\} - \beta\xi+\frac{\log\varepsilon}{p(t)}$$

$$> \sup_{\varepsilon<\xi} \{\beta\xi-\phi(\xi)\} - \beta\varepsilon+\frac{\log\varepsilon}{p(t)} - \delta(t, A'(\beta)) .$$

Now take $\frac{1}{p(t)}$ in the place of ε, then

$$\Phi_t(\beta) \geqq \sup_{\frac{1}{p(t)}<\xi} \{\beta\xi-\phi(\xi)\} + o(1), \quad \text{as } t\to\infty .$$

We arrive at the second statement of our theorem.

2. We now come to the proof of those results on the Laplace transform.

Corollary 1. *Let $\rho(\lambda)$ be a non-decreasing function on $(-\infty, 0]$ with $\rho(-\infty)=0$ and*

(2.1) $$\kappa(t) \equiv \int_{-\infty}^{0} e^{-t\lambda}\rho(\lambda)d\lambda ,$$

then following two conditions (2.2) *and* (2.3) *are equivalent*:

(2.2) $$\lim_{\lambda\downarrow-\infty} \frac{\log\rho(\lambda)}{|\lambda|^{\alpha}} = -A, \ \alpha>1, \ A>0 ,$$

(2.3) $$\lim_{t\uparrow\infty} \frac{\log\kappa(t)}{t^{\gamma}} = B, \ \gamma>1, \ B>0 ,$$

where α, γ, A, B *are related by*

$$\gamma=\frac{\alpha}{\alpha-1}\left(\alpha=\frac{\gamma}{\gamma-1}\right),\quad B=(\alpha-1)\alpha^{\alpha/1-\alpha}A^{1/-\alpha}$$
$$(A=(\gamma-1)\gamma^{\gamma/1-\gamma}B^{1/1-\gamma})\,.$$

Corollary 2. *Let* $\rho(\lambda)$ *be a non-decreasing function on* $(-\infty,\infty)$ *with* $\rho(-\infty)=0$ *and* $k(t)$ *be its Laplace transform*:

$$k(t)=\int_{-\infty}^{\infty}e^{-t\lambda}d\rho(\lambda)<\infty\,, \tag{2.4}$$

then (2.2) *and* (2.3) *in which we take* $k(t)$ *in the place of* $\kappa(t)$, *are equivalent.*

Corollary 2 is immediate from Corollary 1. If $k(t)$ is finite, then $\lim_{c\uparrow\infty}e^{ct}\rho(-c)=0$ and so

$$\int_{-\infty}^{\infty}e^{-t\lambda}d\rho(\lambda)=\rho(0)+t\int_{-\infty}^{0}e^{-t\lambda}\rho(\lambda)d\lambda+\int_{0}^{\infty}e^{-t\lambda}d\rho(\lambda)\,.$$

Since the last term decreases as $t\to\infty$ we have

$$\frac{\log k(t)}{t^{\gamma}}\sim\frac{\log\left[t\int_{-\infty}^{0}e^{-t\lambda}\rho(\lambda)d\lambda\right]}{t^{\gamma}}\sim\frac{\log\kappa(t)}{t^{\gamma}}$$

$t\to\infty$, $\gamma>0$. Hence we conclude Corollary 2 from Corollary 1.

Now we will give the proof of Corollary 1.

Proof. (2.2)$\Rightarrow$(2.3) Put $\Phi_t(\beta)=\frac{1}{t^{\alpha}}\log\kappa(\beta t^{\alpha-1})$ and $-\phi_t(\xi)=\frac{1}{t^{\alpha}}\{\log\rho(-\xi t)+\log t\}$, then $\Phi_t(\beta)$ and $\phi_t(\xi)$ are related by

$$\exp\{t^{\alpha}\Phi_t(\beta)\}=\int_0^{\infty}\exp\{t^{\alpha}(\beta\xi-\phi_t(\xi))\}d\xi\,.$$

It follows from (2.3) that $\phi_t(\xi)$ converges to $\xi^{\alpha}A$ uniformly on each finite interval. Since $\rho(\lambda)$ is non-decreasing, $\phi_t(\xi)$ is non-decreasing in ξ for each t. Therefore to apply the second part of our theorem, we have only to verify the existence of $c(\beta)$ such that $\Phi_t(\beta)<c(\beta)$ for each β and t. For any $\varepsilon>0$ there exists $c>0$ such that

$$\rho(-\lambda)<c\{\exp\{-(A-\varepsilon)\lambda^{\alpha}\},\ \lambda>0\,.$$

Therefore

$$\exp\{t^{\alpha}\Phi_t(\beta)\}=\int_0^{\infty}\exp\{\beta t^{\alpha-1}\lambda\}\rho(-\lambda)d\lambda$$
$$<\int_0^{\infty}c\exp\{\beta t^{\alpha-1}\lambda-(A-\varepsilon)\lambda^{\alpha}\}d\lambda$$

$$< c\int_0^{\eta t}\exp\{\beta t^{\alpha-1}\lambda\}\,d\lambda + c\int_{\eta t}^{\infty}\exp\{-[(A-\varepsilon)\lambda^{\alpha-1}-\beta t^{\alpha-1}]\lambda\}\,d\lambda\,.$$

Taking $\eta^{\alpha-1}=\dfrac{\beta+1}{A-\varepsilon}$

$$\exp\{t^{\alpha}\Phi_t(\beta)\} \leqq \frac{c\exp\{\eta\beta t^{\alpha}\}}{\beta t^{\alpha-1}} + c\int_0^{\infty}\exp\{-[(A-\varepsilon)\eta^{\alpha-1}-\beta]t^{\alpha-1}d\lambda\}\,d\lambda$$

$$\leqq \frac{c\exp\{\eta\beta t^{\alpha}\}}{t^{\alpha-1}} + \frac{c}{t^{\alpha-1}}\,,$$

which means

$$\Phi_t(b) < \eta(\beta) + o(1) \quad \text{as } t\to\infty\,.$$

Since η depends only on β (not on t) the existence of $c(\beta)$ desired is assured. Now it follows from our theorem that

$$\lim_{t\uparrow\infty}\Phi_t(\beta) = \sup_{\xi>0}\{\beta\xi-\xi^{\alpha}A\} = (\alpha-1)\alpha^{\alpha/1-\alpha}A^{1/1-\alpha}B^{\alpha/\alpha-1}\,.$$

Putting $\beta=1$ and $t^{\alpha-1}=x$, we arrive at (2.3).

(2.3)$\Rightarrow$(2.2) Put $\Phi_t'(\beta)=\dfrac{1}{t^{\gamma}}\log\kappa(\beta t)$ and $-\phi_t'(\xi)=\dfrac{1}{t^{\gamma}}\{\log(-\xi t^{\gamma-1})+\log t^{\gamma-1}\}$, then we can see in a similar way as above that they have the relation (1.1) with $p(t)=t^{\gamma}$:

$$\exp\{t^{\gamma}\Phi_t'(\beta)\} = \int_0^{\infty}\exp\{t^{\gamma}(\beta\xi-\phi_t'(\xi))\}\,d\xi\,.$$

It follows from (2.2) that $\lim_{t\uparrow\infty}\Phi_t'(\beta)=\beta^{\gamma}B$. $\phi_t'(\xi)$ is clearly non-decreasing in ξ for each t. We can easily see that $\varepsilon(t)=\dfrac{1}{t^{\gamma-1}}$ satisfies all conditions of the first part of our theorem because $\gamma>1$. Therefore the theorem applies and

$$\lim_{t\uparrow\infty}\phi_t'(\xi) = \sup_{\beta>0}\{\beta\xi-\beta^{\gamma}B\} = (\gamma-1)\gamma^{\gamma/1-\gamma}B^{1/1-\gamma}\,\xi^{\gamma/\gamma-1}$$

Putting $\xi=1$ and $-t^{\gamma-1}=\lambda$, we get (2.2).

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