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THE FIRST COHOMOLOGY GROUP OF A COMMUTATIVE RING

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Given a commutative ring K , let $\mathcal{U}(K)$ be the group of units of K . If G is an arbitrary group, represented by ring automorphisms of K , let $D(K, G)$ be the trivial crossed product of K with G , and k be the subring of K consisting of all elements left fixed by G .

This note is devoted to perform a description of the first cohomology group $H^1(G, \mathcal{U}(K))$ as the group of ring automorphisms of $D(K, G)$ leaving K pointwise fixed, modulo inner ones. This result holds under the assumption that K is its own centralizer in $D(K, G)$. Traducing this condition in terms of the G -action on K , the description of $H^1(G, \mathcal{U}(K))$ mentioned above provides, in the Galois case, the first four terms of the exact sequence associated to a Galois extension [2, 5.5, p. 31].

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Throughout this paper, all rings will be assumed to have units, and all modules will be unitary. A subring will contain the unit element of the ring, and a morphism of rings will preserve unit elements. The general notations are given below.

Let k be a commutative ring, and K be any k -algebra. A (left) K -module A , together with a operation $(x, y) \mapsto x.y$, which is associative, with unit element, and k -1-bilinear, will be called, for shortness, a k -algebra over K . The definition of *morphism of K -algebra over K* is obvious.

If G is a group and K is a commutative ring, given a representation of G by ring automorphisms of K , let $D(K, G) = D$ denote the *trivial crossed product* of K with G . This means that D is the free K -module $K^{(G)}$ generated by G , with multiplication defined by the formula

$$(aw_s)(bw_t) = as(b)w_{st}, \quad a, b \in K \quad \text{and} \quad s, t \in G,$$

where $(w_s)_{s \in G}$ is the canonical basis of $K^{(G)}$. If k is the subring ${}^G K$ of K consisting of all elements left fixed by G , and $w : G \rightarrow D$ is the map $s \mapsto w_s$, then D is a k -algebra over K and w is a multiplicative morphism. The pair (D, w)

is characterized by the following universal mapping property: if A is any k -algebra over K and $m : G \rightarrow A$ is any multiplicative morphism, there exists a unique morphism of k -algebras over K $\varphi : D \rightarrow A$ such that $\varphi w = m$ (φ is defined as the K -module morphism $w_s \mapsto m(s)$, $s \in G$). Note that the ring structure of D has w_1 for its identity element, and that the map $x \mapsto xw_1$ imbeds K as a subring of D .

Let k be a commutative ring, and A be a k -algebra. If M is a two-sided A -module, ${}^A M$ will denote the k -submodule $\{x \in M; ax = xa, a \in A\}$. Note that if A is a subalgebra of a k -algebra B , then ${}^A B$ is just the centralizer of A in B ; in particular, ${}^A A$ is the center of A . If α and β are k -algebra endomorphisms of A , the two-sided A -module with additive group equal to the additive group of M and actions $axb = \alpha(a)x\beta(b)$, $a, b \in A$ and $x \in M$, will be denoted ${}_{\alpha}M_{\beta}$. Finally, set $\mathcal{J}_{\alpha}(M) = {}^A_{\alpha}M_1$.

Henceforth, unless otherwise specified, K is a commutative ring, G is a group represented by ring automorphisms of K , and $k = {}^G K$.

The following result will be needed for the announced description of the first cohomology group.

(1) Proposition. K is its own centralizer in D (i. e. ${}^K D = K$) if and only if $\mathcal{J}_s(K) = 0$, for $s \in G$ and $s \neq 1$.

Proof. Explicitly, $\mathcal{J}_s(K) = \{a \in K; as(x) = xa, x \in K\}$.

Sufficiency. Let $a \in D$, $a = \sum_{s \in G} a_s w_s$, commute with all elements of K .

Since $(w_s)_{s \in G}$ is a basis of D , as a K -module, and $(s(x)w_1)w_s = s(x)w_s = w_s(xw_1)$, for $s \in G$ and $x \in K$, from the relation

$$a(xw_1) = (xw_1)a, x \in K,$$

it follows that

$$a_s s(x) = xa_s, s \in G \text{ and } x \in K.$$

Therefore, $a_s \in \mathcal{J}_s(K)$, for all $s \in G$, and so, $a = a_1 w_1$.

Necessity. Given $s \in G$, if $a \in \mathcal{J}_s(K)$, then

$$(aw_s)(xw_1) = (xw_1)(aw_s), x \in K,$$

and so, $aw_s \in {}^K D = K$. Hence, if $s \neq 1$, $a = 0$.

In the rest of the paper, it will be assumed that $\mathcal{J}_s(K) = 0$, for $s \in G$ and $s \neq 1$. Then,

(2) Corollary. D is a central k -algebra.

Proof. If a is in the center of D , then $a = cw_1$, for some $c \in K$, because K is its own centralizer in D . But,

$$w_s(cw_1) = (cw_1)w_s, s \in G,$$

and so,

$$s(c) = c, s \in G.$$

Since c is left fixed by all elements of G , $c \in k$.

If k is a commutative ring, K is a k -algebra and A is a k -algebra over K , $\mathcal{O}_K(A)$ will denote the group of (k -algebra over K) automorphisms of A , and $\mathcal{I}_K(A)$ will denote the subgroup of inner automorphisms of A . If u is a unit of A , which defines an inner automorphism σ of A , an element $x \in A$ is left fixed by σ if and only if u commutes with x . Thus, $u \in {}^K A$, and it is proved the following

(3) Corollary. *If σ_u is the inner automorphism of D defined by a unit u of K , $x \mapsto uxu^{-1}$, then $\mathcal{I}_K(D) = \{\sigma_u; u \in \mathcal{U}(K)\}$.*

Passing to the cohomological context, if $f \in Z^1(G, \mathcal{U}(K))$, let $\rho(f)$ be the endomorphism of the K -module structure of D defined by

$$\rho(f)(w_s) = f(s)w_s, s \in G.$$

Since $(f(s)w_s)_{s \in G}$ is a basis of D , because $f(s) \in \mathcal{U}(K)$, for all $s \in G$, $\rho(f)$ is a K -module automorphism of D . The fact that f is a cocycle translates into the formula

$$\rho(f)(w_s w_t) = \rho(f)(w_s) \rho(f)(w_t), s, t \in G,$$

and so, $\rho(f) \in \mathcal{O}_K(D)$. An easy computation, using the basis $(w_s)_{s \in G}$, shows that the map $\rho : Z^1(G, \mathcal{U}(K)) \rightarrow \mathcal{O}_K(D)$ is an injective group morphism. Moreover, given $\alpha \in \mathcal{O}_K(D)$, from the relation

$$w_s(xw_1) = (s(x)w_1)w_s, s \in G \text{ and } x \in K,$$

it follows that

$$(\alpha(w_s)w_s^{-1})(xw_1) = (xw_1)(\alpha(w_s)w_s^{-1}), s \in G \text{ and } x \in K.$$

Hence, $\alpha(w_s)w_s^{-1}$ commutes with K (elementwise), for all $s \in G$. Now, applying (1), for each $s \in G$ there is an element $f(s) \in K$, uniquely determined, such that $\alpha(w_s)w_s^{-1} = f(s)w_1$; but $f(s) \in \mathcal{U}(K)$, since $\alpha(w_s)w_s^{-1} \in \mathcal{U}(D)$. A simple calculation shows that f is a cocycle, and so, $\rho(f) = \alpha$. Therefore, ρ is a surjective map. Finally, given $u \in \mathcal{U}(K)$, if f_u is the coboundary defined by u (i. e. $f_u(s) = us(u)^{-1}$, $s \in G$), then $\rho(f_u) = \sigma_u$. Hence, applying (3),

$$\rho(B^1(G, \mathcal{U}(K))) = \mathcal{I}_K(D),$$

group monomorphism $H^1(G, \mathcal{U}(K)) \rightarrow \mathcal{O}_k(D) / \mathcal{I}_k(D)$.

Now, it will be assumed that G is a finite group, and that K is a Galois extension of k (relative to G), in the sense of Auslander-Goldman [1, Appendix, p. 396]. The last requirement holds if and only if K is a separable k -algebra, and $\mathcal{J}_s(K) = 0$, for $s \in G$ and $s \neq 1$ (that is, the general hypothesis of this paper) [3, prop. 2, p. 344]. The assumption that K is a Galois extension of k guarantees that D is an Azumaya k -algebra (cf. (2) and [1, 5.1, p. 380]), and so, the Rosenberg-Zelinsky theorem yields a group monomorphism $\lambda : H^1(G, \mathcal{U}(K)) \rightarrow \mathcal{P}(k)$, $\bar{f} \mapsto \text{cls}(\mathcal{J}_{\rho(\bar{f})}(D))$. If $\varepsilon : \mathcal{P}(k) \rightarrow H^0(G, \mathcal{P}(K))$ is the group morphism $\text{cls}(P) \mapsto \text{cls}(K \otimes_k P)$, then $\text{Im} \lambda = \text{Ker } \varepsilon = \{\text{cls}(P); K \otimes_k P \simeq K, \text{ as } K\text{-modules}\}$.

\subseteq . Given $\alpha \in \mathcal{O}_k(D)$, a general result [5, 3.4, pp. 1111–2] guarantees that the map $\mu : D \otimes_k \mathcal{J}_\alpha(D) \rightarrow {}_\alpha D$, $x \otimes a \mapsto a \cdot x$, is an isomorphism of two-sided D -modules. Supposing that $\alpha \in \mathcal{O}_K(D)$, it follows that $\mathcal{J}_\alpha(D) \subseteq {}^K D = K$. Thus, if $\iota : K \otimes_k \mathcal{J}_\alpha(D) \rightarrow D \otimes_k \mathcal{J}_\alpha(D)$ is the K -module morphism deduced from the imbedding of K into D , $\mu \iota$ induces a K -module morphism $\nu : K \otimes_k \mathcal{J}_\alpha(D) \rightarrow K$. It is clear that ν is a monomorphism, because μ and ι are monomorphisms. Moreover, given $x \in K$, it can be written in the form $x = \sum_{1 \leq i \leq n} a_i x_i$, with $a_i \in \mathcal{J}_\alpha(D)$ and $x_i \in D$, $1 \leq i \leq n$, since μ is an epimorphism (that is, $D = \mathcal{J}_\alpha(D) \cdot D$). Hence, if $x_i = \sum_{s \in G} x_{i,s} w_s$, with $x_{i,s} \in K$, $1 \leq i \leq n$ and $s \in G$, then $x = \sum_{1 \leq i \leq n} a_i x_{i1} \in \mathcal{J}_\alpha(D)$. K ; and so, ν is an epimorphism.

\supseteq . Let P be a finitely generated, projective k -module, of rank one, such that $K \otimes_k P \simeq K$, as K -modules. If f is such an isomorphism, the map $\varphi : D \otimes_k P \rightarrow D$, defined by $\varphi(w_s \otimes x) = f(1 \otimes x)w_s$, $s \in G$ and $x \in P$, is a K -module isomorphism (note that $w_s \otimes x$, with $s \in G$ and $x \in P$, are generators of $D \otimes_k P$, as a K -module). Moreover, since $\varphi((w_s \otimes x)w_t) = f(1 \otimes x)w_{st} = \varphi(w_s \otimes x)w_t$, $s, t \in G$ and $x \in P$, φ is an isomorphism of right D -modules. Therefore, the argument of [4, 3.7 (proof), p. 1113] provides a k -algebra automorphism of D , say α , as follows: If g is a free generator of $D \otimes_k P$, as a right D -module, for each $x \in D$ there is an element $\alpha(x) \in D$, uniquely determined, such that $xg = g\alpha(x)$. It can be assumed that $g \in \iota(K \otimes_k P)$; for example, take $g = \varphi^{-1}(w_1)$, and apply the commutativity of the diagram

$$\begin{array}{ccc}
 K \otimes_k P & \xrightarrow{f} & K \\
 \downarrow \iota & & \downarrow \\
 D \otimes_k P & \xrightarrow{\varphi} & D
 \end{array}$$

Thus, $xg = gx$, for all $x \in K$, since K is contained in the centralizer of $\iota(K \otimes_k P)$ in $D \otimes_k P$. Hence, $\alpha \in \mathcal{O}_K(D)$, and it is proved the following

(7) Proposition. *Let G be a finite group. If K is a separable k -algebra (equivalently, K is a Galois extension of k , relative to G), then the sequence*

$$1 \longrightarrow H^1(G, \mathcal{U}(K)) \xrightarrow{\lambda} \mathcal{P}(k) \xrightarrow{\varepsilon} H^0(G, \mathcal{P}(K))$$

is exact.

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