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SEPARABLE EXTENSIONS AND FROBENIUS EXTENSIONS

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As in our previous paper [2] we say that a ring Λ with 1 is a separable extension of a subring Γ which contains the same 1 if the map $\pi: \Lambda\otimes_{\Gamma}\Lambda\rightarrow\Lambda$ such that $\pi(x\otimes y)=xy$ splits as two sided Λ-module. There has been a problem whether a separable extension is a Frobenius extension. Recently, K. Nakane has given an affirmative answer to this problem in [8] under the condition that Λ is centrally projective over Γ in the sense of K. Hirata [4] and $\mathfrak{m}\Gamma\neq\Gamma$ holds for every maximal ideal $\mathfrak{m}$ of a central subring R of Γ such that $\Lambda=\Gamma\otimes_{R}\Omega$ with Ω finitely generated projective over R. He also proved that if Λ is Γ-centrally projective and separable over Γ, Λ is a quasi-Frobenius extension of Γ. In this paper we shall show that the last condition can be omitted (Theorem 2). Next we consider the opposite situatuon, that is, $\Lambda\otimes_{\Gamma}\Lambda$ is Λ-centrally projective and Γ is a Γ-Γ-direct summand of Λ. In this case we can also see that Λ is a Frobenius extension of Γ if we assume the finitely generated projectivity of Λ_{Γ} or ${}_{\Gamma}\Lambda$ (Theorem 4).

1. Separable extensions

Throughout this paper we assume that all rings have the identity elements and all subrings contain the same 1 as the over ring. Furthermore whenever we say that M is a Γ-Γ-module or a two sided Γ-module for a ring Γ, we assume that M is unitary and associative, that is, $(xm)y=x(my)$ for all $x, y\in\Gamma$ and $m\in M$.

Let Γ be a ring and M a Γ-Γ-module. Then, according to K. Hirata [4] we say that M is centrally projective over Γ, if M is isomorphic to a direct summand of a finite direct sum of the copies of Γ as two sided Γ-module. The next lemma is due to K. Hirata. But since we need it in this paper so often, we shall state here.

Lemma 1 (*Prop.* 5.2 [4]). *If a two sided Γ-module M is centrally projective over Γ, M^{Γ} is finitely generated projective over C and $M\cong\Gamma\otimes_{C}M^{\Gamma}$ by the map: $x\otimes m\rightarrow xm$ and $Hom\ ({}_{\Gamma}M_{\Gamma}, {}_{\Gamma}M_{\Gamma})\cong Hom\ ({}_{C}M^{\Gamma}, {}_{C}M^{\Gamma})$, where $M^{\Gamma}=\{m\in M \mid xm=mx \text{ for every } x\in\Gamma\}$ and C is the center of Γ.*

The next theorem is an immediate consequence of Lemam 1. But it attracts our interests to itself.

Theorem 1. *Let M be an arbitrary centrally projective Γ-Γ-module. Then, $\Omega = Hom\,({}_{\Gamma}M, {}_{\Gamma}M)$ is an H-separable extension of $\Gamma/\mathfrak{a}\Gamma$, where $\mathfrak{a}$ is the annihilator ideal of M^{Γ} in C.*

Proof. Since M is isomorphic to a direct summand of $\Gamma\oplus\cdots\oplus\Gamma$ as Γ-Γ-module, Hom $({}_{\Gamma}M, {}_{\Gamma}M)$ is also a direct summand of a finite direct sum of the copies of Hom $({}_{\Gamma}\Gamma, {}_{\Gamma}M)$, which is isomorphic to M as Γ-Γ-module. Hence, Ω is centrally projective over Γ, as M is so. Then, $\Omega = \Omega^{\Gamma}\otimes_C\Gamma$, where $\Omega^{\Gamma} = \mathrm{Hom}({}_{\Gamma}M_{\Gamma}, {}_{\Gamma}M_{\Gamma}) \cong \mathrm{Hom}\,({}_{C}M^{\Gamma}, {}_{C}M^{\Gamma})$. But Hom $({}_{C}M^{\Gamma}, {}_{C}M^{\Gamma})$ is central separable over $C/\mathfrak{a}$, since M^{Γ} is C-finitely generated projective. Thus Ω is H-separable over $C/\mathfrak{a}\otimes_C\Gamma$, as $\Omega = \Omega^{\Gamma}\otimes_{C/\mathfrak{a}}C/\mathfrak{a}\otimes_C\Gamma$.

Lemma 2. *A two sided Γ-module M is centrally projective over Γ if and only if there exist $f_j\in Hom\,({}_{\Gamma}M_{\Gamma}, {}_{\Gamma}\Gamma_{\Gamma})$ and $m_j\in M^{\Gamma}$, $j=1, 2, \cdots, n$, such that $m=\Sigma f_j(m)m_j$ for every $m\in M$.*

Proof. M is centrally projective over Γ if and only if there exist Γ-Γ-homomorphisms f of M to $\Gamma\oplus\cdots\oplus\Gamma$, the direct sum of n copies of Γ for some n, and g of $\Gamma\oplus\cdots\oplus\Gamma$ to M such that $gf=1_M$. Assume that such f and g exist, and let $f_j=\pi_jf$, where π_j is the jth projection of $\Gamma\oplus\cdots\oplus\Gamma$ to Γ, and g_j the restriction of g to the jth direct summand Γ of $\Gamma\oplus\cdots\oplus\Gamma$. Then, g_j is given by the multiplication of some m_j in M^{Γ}, since g_j is in Hom $({}_{\Gamma}\Gamma_{\Gamma}, {}_{\Gamma}M_{\Gamma})$, which is isomorphic to M^{Γ}. Then $\Sigma f_j(m)m_j = \Sigma g_jf_j(m) = gf(m) = 1_M(m) = m$. Conversely, assume that there exist such $f_j\in\mathrm{Hom}\,({}_{\Gamma}M_{\Gamma}, {}_{\Gamma}\Gamma_{\Gamma})$ and $m_j\in M^{\Gamma}$. Then, if we define f and g as follows;

$$f(m) = (f_1(m), f_2(m), \cdots, f_n(m)), \quad g((x_1, x_2, \cdots, x_n)) = \Sigma\, x_jm_j$$

then f is a Γ-Γ-map of M to $\Gamma\oplus\cdots\oplus\Gamma$ and g is a Γ-Γ-map of $\Gamma\oplus\cdots\oplus\Gamma$ to M such that $gf=1_M$. Hence M is centrally projective over Γ.

Let R be a commutative ring, Γ an R-algebra and A a finitely generated projective R-module. Denote $M=\Gamma\otimes_R A$. Then M is a centrally projective Γ-Γ-module. Let $\bar{f}_j\in\mathrm{Hom}\,({}_RA, {}_RR)$ and $a_j\in A$ be such that $a=\Sigma\bar{f}_j(a)a_j$ for every $a\in A$. Then, clearly $f_j=1_{\Gamma}\otimes\bar{f}_j$ and $1\otimes a_j$ satisfy the condition of Lemma 2. Let m be an arbitrary in M^{Γ}. Then for any $x\in\Gamma$ and every j, $xf_j(m)=f_j(xm)$ $=f_j(mx)=f_j(m)x$, and we see that $f_j(m)\in C$, the center of Γ. Thus we see that $M^{\Gamma}=C\otimes_R A$. By this remark, we get.

Lemma 3. *Let R be a commutative ring. Then if A is a finitely generated projective R-module and Γ is an R-algebra with its center C, $\Gamma\otimes_R A$ is centrally projective over Γ and $(\Gamma\otimes_R A)^{\Gamma}=C\otimes_R A$.*

Proposition 1. *Let* Λ *be a ring and* Γ *a subring of* Λ. *Then* Λ *is an* H-*separable extension of* Γ *if and only if* $1\otimes 1\in\Delta(\Lambda\otimes_\Gamma\Lambda)^\Lambda$ *in* $\Lambda\otimes_\Gamma\Lambda$ *where* $\Delta=V_\Lambda(\Gamma)$, *the commutor subring of* Γ *in* Λ.

Proof. Λ is H-separable over Γ if and only if $\Lambda\otimes_\Gamma\Lambda$ is centrally projective over Λ. This is the case if and only if there exist

$$\varphi_j\in\mathrm{Hom}\,({}_\Lambda\Lambda\otimes_\Gamma\Lambda_\Lambda,\ {}_\Lambda\Lambda_\Lambda)\quad\text{and}\quad\delta_j\in(\Lambda\otimes_\Gamma\Lambda)^\Lambda\qquad j=1,2,\cdots,n$$

such that $\Sigma\,\varphi_j(1\otimes 1)\delta_j=1\otimes 1$, since $1\otimes 1$ generates $\Lambda\otimes_\Gamma\Lambda$ as two sided Λ-module. On the other hand, since $\mathrm{Hom}\,({}_\Lambda\Lambda\otimes_\Gamma\Lambda_\Lambda,\ {}_\Lambda\Lambda_\Lambda)$ is isomorphic to Δ by the map: $\varphi\to\varphi(1\otimes 1)$, each Λ-Λ-map φ of $\Lambda\otimes_\Gamma\Lambda$ to Λ is given by the multiplication of some $d\in\Delta$. Hence the above φ_j and δ_j exist if and only if there exist $d_j\in\Delta$ and $\delta_j\in(\Lambda\otimes_\Gamma\Lambda)^\Lambda$ such that $1\otimes 1=\Sigma\,d_j\delta_j$, i.e., $1\otimes 1\in\Delta(\Lambda\otimes_\Gamma\Lambda)^\Lambda$.

Now, let a ring Λ be left finitely generated projective over a subring Γ of it. Then there exist $f_j\in\mathrm{Hom}\,({}_\Gamma\Lambda,\ {}_\Gamma\Gamma)$ and $z_j\in\Lambda$, $j=1,2,\cdots,n$, such that $x=\Sigma\,f_j(x)z_j$ for every $x\in\Lambda$. On the other hand, we have Λ-Λ-isomorphisms

$$\Lambda\otimes_\Gamma\Lambda\to\mathrm{Hom}\,(\Gamma_\Gamma,\ \Lambda_\Gamma)\otimes_\Gamma\Lambda\to\mathrm{Hom}\,(\mathrm{Hom}\,({}_\Gamma\Lambda,\ {}_\Gamma\Gamma)_\Gamma,\ \Lambda_\Gamma)$$

such that the composition σ of them is given by $\sigma(x\otimes y)(f)=xf(y)$ for every $f\in\mathrm{Hom}\,({}_\Gamma\Lambda,\ {}_\Gamma\Gamma)$. Then we have a commutative diagram of Λ-Λ-maps

$$\begin{tikzcd}
\Lambda\otimes_\Gamma\Gamma \arrow[rr, "\sigma"] \arrow[dr, "\pi"'] & & \mathrm{Hom}\,(\mathrm{Hom}\,({}_\Gamma\Lambda,\ {}_\Gamma\Gamma)_\Gamma,\ \Lambda_\Gamma) \arrow[dl, "\Psi"] \\
& \Lambda &
\end{tikzcd}$$

with $\Psi(\psi)=\Sigma\,\psi(f_j)z_j$, $\pi(x\otimes y)=xy$, for $\psi\in\mathrm{Hom}\,(\mathrm{Hom}\,({}_\Gamma\Lambda,\ {}_\Gamma\Gamma)_\Gamma,\ \Lambda_\Gamma)$ and $x,y\in\Lambda$, because $\Psi\sigma(x\otimes y)=\Sigma\,\sigma(x\otimes y)(f_j)z_j=x\,\Sigma\,f_j(y)z_j=xy=\pi(x\otimes y)$. From this fact, we obtain

Proposition 2. *Let a ring* Λ *be left finitely generated projective over a subring* Γ, *and* f_j *and* z_j *be as above. Then,* Λ *is a separable extension of* Γ *if and only if there exists a* Λ-Γ-*homomorphism* h *of Hom* $({}_\Gamma\Lambda,\ {}_\Gamma\Gamma)$ *to* Λ *such that* $\Sigma\,h(f_j)z_j=1$.

Proof. Λ is separable over Γ if and only if there exists $\Sigma\,x_i\otimes y_i$ in $(\Lambda\otimes_\Gamma\Lambda)^\Lambda$ such that $\pi(\Sigma\,x_i\otimes y_i)=1$. But σ is an isomorphism and induces a one to one correspondence between $(\Lambda\otimes_\Gamma\Lambda)^\Lambda$ and $\mathrm{Hom}\,({}_\Lambda\mathrm{Hom}\,({}_\Gamma\Lambda,\ {}_\Gamma\Gamma)_\Gamma,\ {}_\Lambda\Lambda_\Gamma)$. Hence there exists $\Sigma\,x_i\otimes y_i\in(\Lambda\otimes_\Gamma\Lambda)^\Lambda$ with $\pi(\Sigma\,x_i\otimes y_i)=1$ if and only if there exists an $h\in\mathrm{Hom}\,({}_\Lambda\mathrm{Hom}\,({}_\Gamma\Lambda,\ {}_\Gamma\Gamma)_\Gamma,\ {}_\Lambda\Lambda_\Gamma)$ with $\Psi(h)=1$, i.e., $\Sigma\,h(f_j)z_j=1$.

Let Λ be a separable extension of Γ such that Λ is centrally projective over Γ. Then, there exist $f_j\in\mathrm{Hom}\,({}_\Gamma\Lambda_\Gamma,\ {}_\Gamma\Gamma_\Gamma)$ and $d_j\in\Delta$ as Lemma 2 and

$h \in \mathrm{Hom}\,({}_{\Lambda}\mathrm{Hom}\,({}_{\Gamma}\Lambda, {}_{\Gamma}\Gamma)_{\Gamma}, {}_{\Lambda}\Lambda_{\Gamma})$ with $\Sigma\, h(f_j)\, d_j = 1$ by Proposition 2. Then we see that $h(\mathrm{Hom}\,({}_{\Gamma}\Lambda_{\Gamma}, {}_{\Gamma}\Gamma_{\Gamma})) \subset \Delta$. In fact, let f be an arbitrary in $\mathrm{Hom}\,({}_{\Gamma}\Lambda_{\Gamma}, {}_{\Gamma}\Gamma_{\Gamma})$ and r in Γ. Since $(r \circ f)(x) = f(xr) = f(x)r = (fr)(x)$ for every $x \in \Lambda$, $r \circ f = fr$. Then, $rh(f) = h(r \circ f) = h(fr) = h(f)r$ for any $r \in \Gamma$, since h is a Λ-Γ-map. Therefore $h(f) \in \Delta$. Thus h induces a left Δ-map $\bar{h}$ of $\mathrm{Hom}\,({}_{\Gamma}\Lambda_{\Gamma}, {}_{\Gamma}\Gamma_{\Gamma})$ to Δ, if we restrict h to $\mathrm{Hom}\,({}_{\Gamma}\Lambda_{\Gamma}, {}_{\Gamma}\Gamma_{\Gamma})$. Clearly, $\Sigma\, \bar{h}(f_j)\, d_j = \Sigma\, h(f_j)\, d_j = 1$. On the other hand, $\Lambda = \Gamma \otimes_C \Delta$ by Lemma 1, where C is the center of Γ. Then, since

$$\mathrm{Hom}\,({}_{\Gamma}\Gamma \otimes_C \Delta_{\Gamma}, {}_{\Gamma}\Gamma_{\Gamma}) \cong \mathrm{Hom}\,({}_C\Delta, {}_C\mathrm{Hom}\,({}_{\Gamma}\Gamma_{\Gamma}, {}_{\Gamma}\Gamma_{\Gamma})) \cong \mathrm{Hom}\,({}_C\Delta, {}_CC)$$

as Δ-Δ-map, we have a Δ-Δ-isomorphism ν of $\mathrm{Hom}\,({}_C\Delta, {}_CC)$ to $\mathrm{Hom}({}_{\Gamma}\Lambda_{\Gamma}, {}_{\Gamma}\Gamma_{\Gamma})$ such that $\nu(f)(rd) = rf(d)$ for $r \in \Gamma$ and $d \in \Delta$. Let $\bar{f}_j = \nu^{-1}(f_j)$ for every j. Then, $\Sigma \bar{f}_j(d)\, d_j = d$ for any $d \in \Delta$. Let $h' = h\nu$. Then h' is a left Δ-map of $\mathrm{Hom}\,({}_C\Delta, {}_CC)$ to Δ, and $\Sigma\, h'(\bar{f}_j) d_j = \Sigma\, h(\nu(\bar{f}_j))\, d_j = \Sigma\, h(f_j) d_j = 1$. This implies that Δ is a separable C-algebra by virtue of Proposition 2.

From this remark we obtain

Theorem 2. *Let Λ be a separable extension of Γ such that Λ is centrally projective over Γ. Then we have*

1) *Δ is a separable C-algebra where C is the center of Γ, and Λ is a Frobenius extension of Γ.*

2) *Λ is a centrally projective H-separable extension of Γ' and Γ' is a separable extension of Γ, where $\Gamma' = V_{\Lambda}(V_{\Lambda}(\Gamma))$.*

Proof. 1). Δ is a separable C-algebra by the above remark. Hence, Δ is a Frobenius C-algebra by Theorem 4.2 [1]. Then, since $\Lambda \cong \Gamma \otimes_C \Delta$, Λ is a Frobenius extension of Γ (see Theorem 3 [9]). 2). Let C' be the center of Δ. Then, since $V_{\Lambda}(C') \cong \Gamma' \otimes_{C'} \Delta$, $V_{\Lambda}(C') = \Gamma'\Delta \supset \Gamma\Delta = \Lambda$, and $\Lambda = V_{\Lambda}(C')$. Then we see that C' is the center of Λ. Then $\Lambda \cong \Gamma' \otimes_{C'} \Delta$, Λ is centrally projective and H-separable over Γ'. Next, since $\Lambda = \Gamma' \otimes_{C'} \Delta$, Γ' is a Γ'-Γ'-direct summand, consequently, a Γ-Γ-direct summand of Λ, which is centrally projective over Γ. Thus Γ' is centrally projective over Γ, and $\Gamma' = V_{\Gamma'}(\Gamma) \otimes_C \Gamma$ by Lemma 1. But $V_{\Gamma'}(\Gamma) = \Gamma' \cap V_{\Lambda}(\Gamma) = V_{\Lambda}(\Delta) \cap \Delta = C'$, which is a separable C-algebra as Δ is a separable C-algebra. Hence Γ' is a separable extension of Γ.

Now, we can see that Nakane's theorem in [8] can be obtained under a weaker condition concerning separability.

Corollary 1. *Let Γ and Ω be R-algebras with Ω finitely generated projective over R and C the center of Γ. Suppose $\mathfrak{m}C \neq C$ holds for every maximal ideal $\mathfrak{m}$ of R. Then, $\Lambda = \Gamma \otimes_R \Omega$ is a separable extension of Γ if and only if Ω is a separable R-algebra.*

Proof. The 'if' part is clear by Prop. 2.7 [2]. Suppose Λ is separable over Γ. Then, $C\otimes_R\Omega$ is separable over C by Lemma 3 and Theorem 2. Then, Ω is separable over R by Nakane's results (see Theorem [8]).

2. Strong Frobenius and symmetric extensions

In case Λ is an algebra over a commutative ring R, Λ is called a symmetric R-algebra if Λ is Λ-Λ-isomorphic to $\mathrm{Hom}\,({}_R\Lambda, {}_RR)$. In case of ring extension it is impossible to introduce such a notion. But we can consider the case where $\Lambda|\Gamma$ has the next condition;

(s.F.1) ${}_\Lambda\Lambda_{\Gamma-\Delta}\cong{}_\Lambda\mathrm{Hom}\,({}_\Gamma\Lambda, {}_\Gamma\Gamma)_{\Gamma-\Delta}$ and ${}_\Gamma\Lambda$ is finitely generated projective.

In this case we shall call that Λ is a strong Frobenius extension of Γ. This condition is equivalent to

(s.F.r) ${}_{\Delta-\Gamma}\Lambda_\Lambda={}_{\Delta-\Gamma}\mathrm{Hom}\,(\Lambda_\Gamma, \Gamma_\Gamma)_\Lambda$ and Λ_Γ is finitely generated projective.

The above equivalence can be deduced if we take the dual modules again. In case Λ is an R-algebra, Λ is a strong Frobenius R-algebra if and only if Λ is a symmetric R-algebra. Moreover, if Λ is centrally projective over Γ, the condition (s.F.1) (resp. (s.F.r)) implies

$$ {}_\Lambda\Lambda_\Lambda\cong{}_\Lambda\mathrm{Hom}\,({}_\Gamma\Lambda, {}_\Gamma\Gamma)_\Lambda \text{ (resp. } {}_\Lambda\Lambda_\Lambda\cong{}_\Lambda\mathrm{Hom}\,(\Lambda_\Gamma,\Gamma_\Gamma)_\Lambda) $$

where $\mathrm{Hom}\,({}_\Gamma\Lambda, {}_\Gamma\Gamma)_{\Gamma\otimes\Delta}$ is given by $(f(r\otimes d))(x)=f(dx)r$ for $r\in\Gamma$, $d\in\Delta$, $x\in\Lambda$ and $f\in\mathrm{Hom}\,({}_\Gamma\Lambda, {}_\Gamma\Gamma)$. Hence in this case, we shall call that Λ is a symmetric extension of Γ.

Most parts of the next Lemma is well known (see Theorem 3 [9] and Theorem 35 [7] for example).

Lemma 4. *If Ω is a symmetric (resp. Frobenius or quasi-Frobenius) algebra over a commutative ring R, then $\Lambda=\Gamma\otimes_R\Omega$ is a symmetric (resp. Frobenius or quasi-Frobenius) extension of Γ for any R-algebra Γ.*

Proof. We shall prove in the case of symmetric algebra. Suppose Ω is Ω-Ω-isomorphic to $\mathrm{Hom}\,({}_R\Omega, {}_RR)$. Then

$$ \begin{aligned} {}_{\Gamma\otimes\Omega}\mathrm{Hom}\,({}_\Gamma\Gamma\otimes_R\Omega, {}_\Gamma\Gamma)_{\Gamma-\Delta} &\cong{}_{\Gamma\otimes\Omega}\mathrm{Hom}\,({}_R\Omega, {}_R\mathrm{Hom}\,({}_\Gamma\Gamma, {}_\Gamma\Gamma))_{\Gamma-\Omega} \\ &\cong{}_{\Gamma\otimes\Omega}\mathrm{Hom}\,({}_R\Omega, {}_R\Gamma)_{\Gamma-\Omega}\cong{}_{\Gamma\otimes\Omega}\mathrm{Hom}\,({}_R\mathrm{Hom}\,({}_R\Omega, {}_RR), {}_R\Gamma)_{\Gamma-\Omega} \\ &\cong{}_{\Gamma\otimes\Omega}\mathrm{Hom}\,({}_RR, {}_R\Gamma)\otimes_R\Omega_{\Gamma-\Omega}\cong{}_{\Gamma\otimes\Omega}\Lambda_{\Gamma-\Omega} \end{aligned} $$

since Ω is R-finitely generated projective. Hence, we see ${}_\Lambda\mathrm{Hom}\,({}_\Gamma\Lambda, {}_\Gamma\Gamma)_\Lambda\cong{}_\Lambda\Lambda_\Lambda$, ${}_\Gamma\Lambda$ is finitely generated projective. Thus Λ is a symmetric extension of Γ. By the same method we can prove in the case of Frobenius algebra.

REMARK. If we use Lemma 1.1 [11], we can prove that if Λ_i are R-algebras and left quasi-Frobenius extensions of R-subalgebras Γ_i respectively, and if the natural map: $\Gamma_1 \otimes_R \Gamma_2 \to \Lambda_1 \otimes_R \Lambda_2$ is a monomorphism, then $\Lambda_1 \otimes_R \Lambda_2$ is also a left quasi-Frobenius extension of $\Gamma_1 \otimes_R \Gamma_2$.

The most parts of the next theorem are immediate consequences of Lemma 2 and Theorem 35 [5].

Theorem 3. *Let a ring Λ be centrally projective over a subring Γ. Then, Λ is a symmetric (resp. Frobenius or left (or right) quasi-Frobenius) extension of Γ if and only if $\Delta = V_\Lambda(\Gamma)$ is a symmetric (resp. Frobenius or quasi-Frobenius) algebra over C, the center of Γ.*

Proof. Since $\Lambda = \Gamma \otimes_C \Delta$, the 'if' parts have been proved in Lemma 4. Suppose Λ is a symmetric extension of Γ, and let h be a Λ-Λ-isomorphism

$$h\colon {}_\Lambda \mathrm{Hom}\,({}_\Gamma\Lambda, {}_\Gamma\Gamma)_{\Gamma-\Delta} \to {}_\Lambda\Lambda_{\Gamma-\Delta}$$

Then, h induces a Δ-Δ-map $\bar{h}$ of $\mathrm{Hom}\,({}_\Gamma\Lambda_\Gamma, {}_\Gamma\Gamma_\Gamma)$ to Δ as is shown in the previous section. Clearly $\bar{h}$ is a monomorphism since h is so. Let d be an arbitrary in Δ. Then there exists an f in $\mathrm{Hom}\,({}_\Gamma\Lambda, {}_\Gamma\Gamma)$ with $h(f)=d$. Then $r\circ f = fr$ for every $r \in \Gamma$, since h is a Λ-Γ-map and d is in Δ. Hence f is in $\mathrm{Hom}\,({}_\Gamma\Lambda_\Gamma, {}_\Gamma\Gamma_\Gamma)$, and $\bar{h}$ is an epimorphism. Thus we see that $\mathrm{Hom}\,({}_\Gamma\Lambda_\Gamma, {}_\Gamma\Gamma_\Gamma)$ is Δ-Δ-isomorphic to Δ. On the other hand, as is shown before $\mathrm{Hom}\,({}_\Gamma\Lambda_\Gamma, {}_\Gamma\Gamma_\Gamma)$ is Δ-Δ-isomorphic to $\mathrm{Hom}\,({}_C\Delta, {}_CC)$. Thus we see that $\mathrm{Hom}\,({}_C\Delta, {}_CC)$ is Δ-Δ-isomorphic to Δ, and Δ is a symmetric C-algebra. The same method as above proves in the case of Frobenius extension. Next, suppose that Λ is a left quasi-Frobenius extension of Γ. Then, by Satz 2 [5] there exist Λ-Γ-maps φ_k of $\mathrm{Hom}\,({}_\Gamma\Lambda, {}_\Gamma\Gamma)$ to Λ and Γ-Γ-maps α_k of Λ to Γ with $\Sigma\, \varphi_k(\alpha_k)=1$. But each map φ_k induces a left Δ-map φ_k' of $\mathrm{Hom}\,({}_\Gamma\Lambda_\Gamma, {}_\Gamma\Gamma_\Gamma)$ to Δ, and there exists a left Δ-isomorphism ν of $\mathrm{Hom}\,({}_C\Delta, {}_CC)$ to $\mathrm{Hom}\,({}_\Gamma\Lambda_\Gamma, {}_\Gamma\Gamma_\Gamma)$. Let $\bar{\varphi}_k = \varphi_k'\nu$ and $\bar{\alpha}_k = \nu^{-1}(\alpha_k)$. Then $\Sigma\, \bar{\varphi}_k(\bar{\alpha}_k) = \Sigma\, \varphi_k\nu(\nu^{-1}(\alpha_k)) = \Sigma\, \varphi_k(\alpha_k) = 1$. Therefore, Δ is a quasi-Frobenius C-algebra. (See Theorem 35 [5] and the Bemerkung under it).

3. Application of Morita's results

In sections 1 and 2 we considered the case where Λ is centrally projective over Γ, but in this section we shall consider the case where $\Lambda \otimes_\Gamma \Lambda$ is centrally projective over Λ, i.e., Λ is an H-separable extension of Γ. To do this we shall apply the results of Morita [7].

Lemma 5. *Let Λ be an H-separable extension of Γ, $\Delta = V_\Lambda(\Gamma)$ and $\Omega = Hom\,(\Lambda_\Gamma, \Lambda_\Gamma)$. Then, we have*

1) *$Hom\,({}_\Omega\Lambda, {}_\Omega\Lambda) = V_\Lambda(\Delta)$, thus if $V_\Lambda(V_\Lambda(\Gamma)) = \Gamma$, Λ_Γ has the double centralizer property,*

2) If $V_\Lambda(V_\Lambda(\Gamma))=\Gamma$, ${}_{\Gamma-\Delta}Hom\,({}_\Omega\Lambda, {}_\Omega\Omega)_\Lambda \cong {}_{\Gamma-\Delta}Hom\,(\Lambda_\Gamma, \Gamma_\Gamma)_\Lambda$

Proof. 1). By Prop. 3.3 [4], there exist a ring isomorphism η of $\Lambda\otimes_C\Delta^0$ to $\mathrm{Hom}\,(\Lambda_\Gamma, \Lambda_\Gamma)$ such that $\eta(x\otimes d^0)(y)=xyd$, where C is the center of Λ. Thus ${}_\Omega\Lambda$ is equivalent to ${}_\Lambda\Lambda_\Delta$. Hence, we see

$$\mathrm{Hom}\,({}_\Omega\Lambda, {}_\Omega\Lambda)\cong \mathrm{Hom}\,({}_\Lambda\Lambda_\Delta, {}_\Lambda\Lambda_\Delta)\cong V_\Lambda(\Delta)\,.$$

2). By Theorem 1.1 [7] ${}_\Gamma\mathrm{Hom}\,({}_\Omega\Lambda, {}_\Omega\Omega)_\Omega \cong {}_\Gamma\mathrm{Hom}\,(\Lambda_\Gamma, \Gamma_\Gamma)_\Omega$, and we have

$${}_{\Gamma-\Delta}\mathrm{Hom}\,({}_\Omega\Lambda, {}_\Omega\Omega)_\Lambda \cong {}_{\Gamma-\Delta}\mathrm{Hom}\,(\Lambda_\Gamma, \Gamma_\Gamma)_\Lambda$$

The next theorem is almost due to Theorem 6.1 [7].

Theorem 4. *Let Λ be an H-separable extension of Γ with $V_\Lambda(V_\Lambda(\Gamma))=\Gamma$ and Ω, Δ and C be as in Lemma 5. Then, if Δ is a symmetric (resp. Frobenius or quasi-Frobenius) C-algebra, the following conditions are equivalent;*

1) *Λ is left Γ-finitely generated projective.*
2) *Λ is right Γ-finitely generated projective.*
3) *Λ is a strong Frobenius (resp. Frobenius or quasi-Frobenius) extension of Γ.*

Proof. Suppose Δ is a symmetric C-algebra. Then Ω is a symmetric extension of Λ, since $\Omega=\Lambda\otimes_C\Delta^0$ and Δ^0 is C-symmetric. Hence we have

$${}_\Omega\mathrm{Hom}\,({}_\Lambda\Omega, {}_\Lambda\Lambda)_{\Lambda\otimes\Delta^0}\cong {}_\Omega\Omega_{\Lambda\otimes\Delta^0}, \text{ i.e., } {}_{\Omega-\Delta}\mathrm{Hom}\,({}_\Lambda\Omega, {}_\Lambda\Lambda)_\Lambda \cong {}_{\Omega-\Delta}\Omega_\Lambda$$

Then by Lemma 5 and the above isomorphism, we have

$$\begin{aligned}{}_{\Gamma-\Delta}\mathrm{Hom}\,(\Lambda_\Gamma, \Gamma_\Gamma)_\Lambda &\cong {}_{\Gamma-\Delta}\mathrm{Hom}\,({}_\Omega\Lambda, {}_\Omega\Omega)_\Lambda\\ &\cong {}_{\Gamma-\Delta}\mathrm{Hom}\,({}_\Omega\Lambda, {}_\Omega\mathrm{Hom}\,({}_\Lambda\Omega, {}_\Lambda\Lambda))_\Lambda\\ &\cong {}_{\Gamma-\Delta}\mathrm{Hom}\,({}_\Lambda\Omega\otimes_\Omega\Lambda, {}_\Lambda\Lambda)_\Lambda\\ &\cong {}_{\Gamma-\Delta}\mathrm{Hom}\,({}_\Lambda\Lambda, {}_\Lambda\Lambda)_\Lambda \cong {}_{\Gamma-\Delta}\Lambda_\Lambda\end{aligned}$$

Thus Λ is a strong Frobenius extension of Γ. For the rest of the proof, see Theorem 6.1 [7].

Corollary 2. *Let Λ be an H-separable extension of Γ such that Γ is a Γ-Γ-direct summand of Λ. Then the following conditions are equivalent.*

1) *Λ is left Γ-finitely generated projective.*
2) *Λ is right Γ-finitely generated projective.*
3) *Λ is a strong Frobenius extension of Γ.*

Proof. Since Γ is a Γ-Γ-direct summand of Λ, $V_\Lambda(V_\Lambda(\Gamma))=\Gamma$ by Prop. 1.2 [10] and Δ is C-separable by Prop. 4.7 [4]. Thus Δ is C-symmetric by Theorem 4.2 [1], and we can apply Theorem 4.

The converse of Theorem 4 holds for H-separable extension as follows.

Theorem 5. *Let Λ be an H-separable extension of Γ. Then if Λ is a strong Frobenius (resp. Frobenius or left or right quasi-Frobenius) extension of Γ, Δ is a symmetric (resp. Frobenius or quasi-Frobenius) C-algebra, where $\Delta = V_\Lambda(\Gamma)$ and C is the center of Λ.*

Proof. Suppose Λ is a strong Frobenius extension of Γ. Then there exists a Λ-$(\Delta-\Gamma)$-isomorphism ${}_\Lambda\Lambda_{\Delta-\Gamma} \cong {}_\Lambda\mathrm{Hom}({}_\Gamma\Lambda, {}_\Gamma\Gamma)_{\Delta-\Gamma}$. Then, since ${}_\Gamma\Lambda$ is finitely genereted projective,

$$\begin{aligned}{}_{\Lambda-\Delta}\Lambda\otimes_\Gamma\Lambda_{\Lambda-\Delta} &\cong {}_{\Lambda-\Delta}\mathrm{Hom}(\Gamma_\Gamma, \Lambda_\Gamma)\otimes_\Gamma\Lambda_{\Lambda-\Delta} \cong {}_{\Lambda-\Delta}\mathrm{Hom}(\mathrm{Hom}({}_\Gamma\Lambda, {}_\Gamma\Gamma)_\Gamma, \Lambda_\Gamma)_{\Lambda-\Delta} \\ &\cong {}_{\Lambda-\Delta}\mathrm{Hom}(\Lambda_\Gamma, \Lambda_\Gamma)_{\Lambda-\Delta}\end{aligned}$$

On the other hand, since Λ is H-separable over Γ, there exist $(\Lambda-\Delta)$-$(\Lambda-\Delta)$-isomorphisms

$$\begin{array}{lll}\xi\colon \Lambda\otimes_\Gamma\Lambda \to \mathrm{Hom}({}_C\Delta, {}_C\Lambda) & \xi(x\otimes y)(d) = xdy & \text{for } x, y\in\Lambda \text{ and } d\in\Delta \\ \eta\colon \Lambda\otimes_C\Delta \to \mathrm{Hom}(\Lambda_\Gamma, \Lambda_\Gamma) & \eta(x\otimes d)(y) = xyd & \text{for } x, y\in\Lambda \text{ and } d\in\Delta\end{array}$$

Hence we have $(\Lambda-\Delta)$-$(\Lambda-\Delta)$-isomorphisms

$$\mathrm{Hom}({}_C\Delta, {}_C\Lambda) \cong \Lambda\otimes_\Gamma\Lambda \cong \mathrm{Hom}(\Lambda_\Gamma, \Lambda_\Gamma) \cong \Lambda\otimes_C\Delta$$

Then, taking $\mathrm{Hom}(*, {}_\Lambda\Lambda_\Lambda)$, we obtain Δ-Δ-isomorphisms

$${}_\Delta\mathrm{Hom}({}_\Lambda\mathrm{Hom}({}_C\Delta, {}_C\Lambda)_\Lambda, {}_\Lambda\Lambda_\Lambda)_\Delta \cong {}_\Delta\mathrm{Hom}({}_\Lambda\Lambda_\Lambda, {}_\Lambda\Lambda_\Lambda)\otimes_C\Delta_\Delta \cong {}_\Delta C\otimes_C\Delta_\Delta \cong {}_\Delta\Delta_\Delta$$

since Δ is C-finitely generated projective, and

$${}_\Delta\mathrm{Hom}({}_\Lambda\Lambda\otimes_C\Delta_\Lambda, {}_\Lambda\Lambda_\Lambda)_\Delta \cong {}_\Delta\mathrm{Hom}({}_C\Delta, {}_C\mathrm{Hom}({}_\Lambda\Lambda_\Lambda, {}_\Lambda\Lambda_\Lambda))_\Delta \cong {}_\Delta\mathrm{Hom}({}_C\Delta, {}_C C)_\Delta$$

Thus we see ${}_\Delta\mathrm{Hom}({}_C\Delta, {}_C C)_\Delta \cong {}_\Delta\Delta_\Delta$, which means that Δ is a symmetric C-algebra. In case of Frobenius extension, ${}_\Lambda\Lambda_\Gamma \cong {}_\Lambda\mathrm{Hom}({}_\Gamma\Lambda, {}_\Gamma\Gamma)_\Gamma$ induces

$${}_\Lambda\Lambda\otimes_\Gamma\Lambda_{\Lambda-\Delta} \cong {}_\Lambda\mathrm{Hom}(\mathrm{Hom}({}_\Gamma\Lambda, {}_\Gamma\Gamma)_\Gamma, \Lambda_\Gamma)_{\Lambda-\Delta} \cong {}_\Lambda\mathrm{Hom}(\Lambda_\Gamma, \Lambda_\Gamma)_{\Lambda-\Delta}$$

where ${}_\Lambda\Lambda\otimes_\Gamma\Lambda_{\Lambda-\Delta}$ is iduced by ${}_\Lambda\Lambda_{\Gamma-\Delta}$ and ${}_\Gamma\Lambda_\Lambda$, while in case of right quasi-Frobenius extension ${}_\Lambda\mathrm{Hom}({}_\Gamma\Lambda, {}_\Gamma\Gamma)_\Gamma \langle \oplus_\Lambda(\Sigma^\oplus\Lambda)_\Gamma$ induces

$${}_\Lambda\Lambda\otimes_\Gamma\Lambda_{\Lambda-\Delta} \cong {}_\Lambda\mathrm{Hom}(\mathrm{Hom}({}_\Gamma\Lambda, {}_\Gamma\Gamma)_\Gamma, \Lambda_\Gamma)_{\Lambda-\Delta} \langle \oplus_\Lambda(\Sigma^\oplus\mathrm{Hom}(\Lambda_\Gamma, \Lambda_\Gamma))_{\Lambda-\Delta}$$

Then the same argument as in the case of strong Frobenius extension proves the theorem in both cases.

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