Luh, J.
Osaka J. Math.
7 (1970), 267-274

THE STRUCTURE OF PRIMITIVE GAMMA RINGS

JIANG LUH

(Received November 4, 1969)

1. Introduction

The notion of a Γ-ring was first introduced by Nobusawa [7]. The class of Γ-rings contains not only all rings but also all Hestenes ternary rings. In [7], Nobusawa generalized the Wedderburn-Artin Theorem for simple Γ-rings and for semi-simple Γ-rings. Barnes [1] obtained analogues of the classical Noether-Lasker theorems concerning primary representations of ideals for Γ-rings. The author [5] gave a characterization of primitive Γ-rings with minimal one-sided ideals by means of certain Γ-rings of continuous semilinear transformations. He [6] also established several structure theorems for simple Γ-rings having minimal one-sided ideals. Recently, Coppage and the author [2] introduced the notions of Jacobson radical, Levitzki radical, nil radical for Γ-rings and obtained some basic radical properties and inclusion relations for these radicals together with the prime radical defined by Barnes [1].

The object of this paper is to study the structure of primitive Γ-rings. One of its main results is a generalization of the Jacobson - Chevalley density theorem. This generalizes further a result given by Smiley and Stephenson for Hestenes ternary rings [8].

We refer to [4] for all notions relevent to ring theory.

2. Preliminaries

Let M and Γ be two additive abelian groups. If for all x, y, $z\in M$ and all α, $\beta\in\Gamma$ the conditions

(1) $x\alpha y\in M$

(2) $(x+y)\alpha z=x\alpha z+y\alpha z,$
$x(\alpha+\beta)z=x\alpha z+x\beta z,$
$x\alpha(y+z)=x\alpha y+x\alpha z,$

(3) $(x\alpha y)\beta z=x\alpha(y\beta z)$

are satisfied then we call M *a* Γ-ring.

If these conditions are strengthened to

(1′) $x\alpha y\in M$, $\alpha x\beta\in\Gamma$,
(2′) the same as (2),
(3′) $(x\alpha y)\beta z=x(\alpha y\beta)z=x\alpha(y\beta z)$
(4′) $x\alpha y=0$ for all $x, y\in M$ implies $\alpha=0$, then M is called a Γ-ring in the sense of Nobusawa.

Let M be a Γ-ring. If $S, T\subseteq M$, we write $S\Gamma T$ for the set of finite sums $\Sigma_i s_i\alpha_i t_i$ where $s_i\in S$, $t_i\in T$, $\alpha_i\in T$. A subgroup I of M is a left (right) ideal of M if $M\Gamma I\subseteq I$ ($I\Gamma M\subseteq I$). If I is both a left and a right ideal of M, then I is an ideal of M. A one-sided ideal I is strongly nilpotent if $I^n=I\Gamma I\cdots\Gamma I=0$ for some positive integer n. A non-zero right (left) ideal is minimal if the only right (left) ideals of M contained in I are 0 and I itself. It has been shown that every minimal right ideal which is not strongly nilpotent can be expressed as the form $e\gamma M$, where $\gamma\in\Gamma$, $e\in M$ and $e\gamma e=e$ (see [5] Theorem 3.2).

Let F be the free abelian group generated by the set of all ordered pairs (α, x) where $\alpha\in\Gamma$, $x\in M$. Let K be the subgroup of elements $\Sigma_i m_i\,(\alpha_i, x_i)\in F$, where m_i are integers such that $\Sigma_i m_i\,(x\alpha_i x_i)=0$ for all $x\in M$. Denote by R the factor group F/K and by $[\alpha, x]$ the coset $K+(\alpha, x)$. Clearly every element in R can be expressed as a finite sum $\Sigma_i[\alpha_i, x_i]$. We define multiplication in R by

$$\Sigma_i[\alpha_i, x_i]\cdot\Sigma_j[\beta_j, y_j]=\Sigma_{i,j}[\alpha_i, x_i\beta_j y_j]\,.$$

Then R forms a ring. Furthermore, M is a right R-module with the definition

$$x\Sigma_i[\alpha_i, x_i]=\Sigma_i x\alpha_i x_i\,,\qquad \text{for } x\in M\,.$$

We call the ring R the right operator ring of M. Similarly, we can define the left operator ring L. Every element in L can be expressed as a finite sum $\Sigma_j[x_j, \beta_j]$ where $x_j\in M$, $\beta_j\in\Gamma$. These two operator rings play important roles in studying the structure of Γ-rings. We recall that a Γ-ring M is right primitive if (i) $M\Gamma x=0$ implies $x=0$ and (ii) the right operator ring R of M is a right primitive ring.

Theorem 1. *If M is a right primitive Γ-ring, then the left operator ring of M is a right primitive ring.*

Proof. Let R and L be respectively the right and left operator rings of M.

Let G be a faithful irreducible right R-module. Let A be the free abelian group generated by the set of ordered pairs (g, γ), where $g\in G$, $\gamma\in\Gamma$, and let B be the subgroup of elements $\Sigma_i m_i(g_i, \gamma_i)\in A$ where m_i are integers such that $\Sigma_i m_i g_i[\gamma_i, x]=0$ for all $x\in M$. Denote by H the factor group A/B and, without causing any ambiguity, by $[g, \gamma]$ the coset $B+(g, \gamma)$. Every element in H therefore can be expressed as a finite sum $\Sigma_i[g_i, \gamma_i]$. H forms a right L-module with the definition

$$\Sigma_i[g_i, \gamma_i]\cdot\Sigma_j[x_j, \beta_j] = \Sigma_{i,j}[g_i[\gamma_i, x_j], \beta_j]$$

for $\Sigma_i[g_i, \gamma_i]\in H$ and $\Sigma_j[x_j, \beta_j]\in L$. We claim that H is a faithful irreducible right L-module. Assume $H\,\Sigma_j[x_j, \beta_j]=0$. Then for all $\gamma\in\Gamma$, $g\in G$, we have $\Sigma_j[g[\gamma, x_j], \beta_j]=[g, \gamma]\,\Sigma_j[x_j, \beta_j]=0$, i.e. $g\Sigma_j[\gamma, x_j]\,[\beta_j, x]=0$ for all $x\in M$. By the faithfulness of the R-module of G, $[\gamma, \Sigma_j x_j\beta_j x]=\Sigma_j[\gamma, x_j]\,[\beta_j, x]=0$, so $M\Gamma\Sigma_i x_j\beta_j x=0$. By the condition (i), $\Sigma_j x_i\beta_j x=0$ for all $x\in M$. This means that $\Sigma_j[x_j, \beta_j]=0$ and H is faithful. To see that H is irreducible, let $\Sigma_i[g_i, \gamma_i]$ be an arbitrary non-zero element in H. Then the set $G'=\{\Sigma_i g_i[\gamma_i, x]: x\in M\}$ is a non-zero R-submodule of G. Since G is irreducible, $G'=G$. For any $\Sigma_j[g_j', \gamma_j']\in H$, we may write $g_j'=\Sigma_i g_i[\gamma_i, x_j]$ where $x_j\in M$. Thus $\Sigma_j[g_j', \gamma_j']=\Sigma_j[\Sigma_i g_i[\gamma_i, x_j], \gamma_j']=\Sigma_i[g_i, \gamma_i]\;\Sigma_j[x_j, \gamma_j]\in\Sigma_i[g_i, \gamma_i]L$. Hence H is irreducible and L is a right primitive ring.

3. Irreducible Γ-rings of homomorphisms on groups

Let G and H be non-zero additive abelian groups. If M and Γ are respectively subgroups of Hom (H, G) and Hom (G, H) such that $g\Gamma=H$ and $hM=G$ whenever $0\neq g\in G$ and $0\neq h\in H$, and moreover if $x\alpha y\in M$ and $\alpha x\beta\in\Gamma$ for all $x, y\in M$, then M forms a Γ-ring in the sense of Nobusawa under the composition of mappings. We shall call such a Γ-ring an irreducible Γ-ring of homomorphisms on groups.

A Γ-ring M and a Γ'-ring M' are said to be isomorphic if there exist a group isomorphism θ of M onto M' and a group isomorphism ϕ of Γ onto Γ' such that $(x\alpha y)\theta=(x\theta)\,(\alpha\phi)\,(y\theta)$ for all $x, y\in M$, $\alpha\in\Gamma$. It is clear that M is right primitive if and only if M' is right primitive.

Theorem 2. *A Γ-ring M is a right primitive Γ-ring in the sense of Nobusawa if and only if it is isomorphic to an irreducible Γ-ring of homomorphisms on groups.*

Proof. Necessity. Let M be a right primitive Γ-ring in the sense of Nobusawa with right operator ring R and left operator ring L and let G be a faithful irreducible right R-module, from the proof of Theorem 1, we can construct the faithful irreducible right L-module H. Now, for each $\gamma\in\Gamma$ let $\gamma\phi\in\mathrm{Hom}\,(G, H)$ defined by $g(\gamma\phi)=[g, \gamma]$. Clearly ϕ is a group homomorphism of Γ into $\mathrm{Hom}\,(G, H)$. Moreover, if $\gamma_1\phi=\gamma_2\phi$, then $[g, \gamma_1-\gamma_2]=0$ i.e. $g[\gamma_1-\gamma_1, x]=0$ for all $g\in G$, $x\in M$. By the faithfulness of G as an R-module, $[\gamma_1-\gamma_2, x]=0$ for all $x\in M$. Consequently $M(\gamma_1-\gamma_2)\,M=0$ and, by the condition (4′) in the definition of Γ-ring in the sense of Nobusawa, $\gamma_1=\gamma_2$. Thus ϕ is a group isomorphism of Γ onto $\Gamma'=\Gamma\phi$.

Likewise, for each $x\in M$, let $x\theta$ be the mapping of H into G defined by $\Sigma_i[g_i, \gamma_i](x\theta)=\Sigma_i g_i[\gamma_i, x]$. It can be shown easily that $x\theta\in\mathrm{Hom}\,(H, G)$ and that θ is a group homomorphism of M into Hom (H, G). We claim that θ is one-to-one. Indeed, if $x\theta=y\theta$, where $x, y\in M$, then $g[\gamma, x-y]=g[\gamma, x]-g[\gamma, y]=0$ for all $g\in G$, $\gamma\in\Gamma$. Again by the faithfulness of G, $[\gamma, x-y]=0$ for all $\gamma\in\Gamma$, or equivalently that $M\Gamma(x-y)=0$. Hence $x=y$ and θ is a group isomorphism of M onto $M'=M\theta$. It is easy to see that the Γ-ring M is isomorphic to the Γ'-ring M'.

It remains to show that M' is an irreducible Γ'-ring of homomorphisms on groups. Let $0\neq g\in G$. Since $gR=G$, every element in H can be expressed as $\Sigma_j[g\Sigma_i[\gamma_{ij}, x_{ij}], \beta_j]=g(\gamma\phi)$ where $\gamma_{ij}, \beta_j\in\Gamma$, $x_{ij}\in M$ and $\gamma=\Sigma_{i,j}\gamma_{ij}x_{ij}\beta_j$. Hence $H=g\Gamma'$. Now, let h be an arbitrary non-zero element in H. Then $h=g(\gamma\phi)=[g, \gamma]$ for some $\gamma\in\Gamma$. It follows that $h(x\theta)=[g, \gamma](x\theta)=g[\gamma, x]$ for all $x\in M$. Thus hM' is a non-zero R-submodule of G and hence $hM'=G$.

Sufficiency. We may assume that M is an irreducible Γ-ring of homorphisms on groups, and that $0\neq\Gamma\subseteq$ Hom (G, H), $0\neq M\subseteq$ Hom (H, G) where H and G are abelian groups with the property that for any $0\neq g\in G$ and $0\neq h\in H$, $g\Gamma=H$ and $hM=G$. Clearly, $M\Gamma x=0$ for $x\in M$ implies $x=0$. For $g\in G$ and $\Sigma_i[\gamma_i, x_i]\in R$, the right operator ring of M, we define composition

$$g\Sigma_i[\gamma_i, x_i] = \Sigma_i(g\gamma_i)x_i\,.$$

This composition is well defined. For if $\Sigma_j[\gamma_i, x_i]=\Sigma_j[\beta_j, y_j]$ in R, then $\Sigma_i x\gamma_i x_i-\Sigma_j x\beta_j y_j=0$ for all $x\in M$. By noting that $g\in g\Gamma M$, we obtain $\Sigma_i(g\gamma_i)x_i-\Sigma_j(g\beta_j)y_j=g(\Sigma_i\gamma_i x_i-\beta_j y_j)\in g\Gamma M(\Sigma_i\gamma_i x_i-\Sigma_j\beta_j y_j)=0$, so $g\Sigma_i[\gamma_i, x_i]=g\Sigma_j[\beta_j, y_j]$. Clearly G forms an irreducible right R-module. Moreover, if $\Sigma_i[\gamma_i, x_i]\in R$ and if $G\Sigma_i[\gamma_i, x_i]=0$, then $HM\Sigma_i[\gamma_i, x_i]=G\Gamma M\,\Sigma_i[\gamma_i, x_i]=G\,\Sigma_i[\gamma_i, x]=0$, and hence $M\Sigma_i[\gamma_i x_i]=0$. Consequently, $\Sigma_i[\gamma_i, x_i]=0$ and G is a faithful R-module. Thus, R is a right primitive ring and M is a right primitive Γ-ring in the sense of Nobusawa.

Observe the definition of irreducible Γ-rings of homomorphisms on groups. We can easily see that M is irreducible Γ-rings of homomorphisms on groups if and only if Γ is a irreducible Γ'-ring of homomorphisms on groups, where $\Gamma'=M$. Thus from Theorem 2, we immediately have the following

Corollary. *Let M be a Γ-ring. Then M is a right primitive Γ-ring in the sense of Nobusawa if and only if Γ is a right primitive Γ'-ring in the sense of Nobusawa, where $\Gamma'=M$.*

4. Chevalley-Jacobson density theorem

Let G and H be non-zero right vector spaces over division rings Δ and Δ' respectively, and let σ be an isomorphism of Δ onto Δ'. A group N of semilinear transformations (associated with σ) of G into H is said to be dense if, for every positive integer n and every n linearly independent elements $g_1, g_2,$

$\cdots, g_n$ in G and every n elements $h_1, h_2, \cdots, h_n$ in H, there exists $x \in N$ such that $g_i x = h_i, i=1, 2, \cdots, n$.

Now, if Γ is a dense group of semilinear transformations (associated with σ) of G into H and M is a dense group of semilinear transformations (associated with σ^{-1}) of H into G, and if the compositions of mappings $x\alpha y \in M$ and $\alpha x \beta \in \Gamma$ for all $x, y \in M$, $\alpha, \beta \in \Gamma$, then M forms a Γ-ring in the sense of Nobusawa under the composition of mappings. We shall call such a Γ-ring a dense Γ-ring of semilinear transformations.

Following is a generalization of the well known Chevalley-Jacobson density theorem.

Theorem 3. *Let M be a Γ-ring. Then M is a right primitive Γ-ring in the sense of Nobusawa if and only if it is isomorphic to a dense Γ-ring of semilinear transformations.*

Proof. Sufficiency. It is an immediate consequence of Theorem 2, since a dense Γ-ring of semilinear transformations evidently is an irreducible Γ-ring of homomorphisms on groups.

Necessity. We assume that M is a right primitive Γ-ring in the sense of Nobusawa. According to the proof of Theorem 1 we can construct a faithful irreducible right R-module G and a faithful irreducible right L-module H, where R and L are respectively the right operator ring and the left operator ring of M. Set $\Delta = \mathrm{Hom}_R(G, G)$ and $\Delta' = \mathrm{Hom}_L(H, H)$. By Schur's Lemma, Δ and Δ' are division rings.

First, we shall show that Δ and Δ' are isomorphic. For $\delta \in \Delta$, we define the mapping $\delta^\sigma: H \to H$ by

$$(\Sigma_i[g_i, \gamma_i])\delta^\sigma = \Sigma_i[g\delta_i, \gamma_i]$$

for $\Sigma_i[g_i, \gamma_i] \in H$. Here δ^σ is well defined. For, if $\Sigma_i[g_i, \gamma_i] = \Sigma_j[g_j', \gamma_j']$ then for all $x \in M$, $\Sigma_i g_i[\gamma_i, x] = \Sigma_j g_j'[\gamma_j, x]$, and hence $\Sigma_i(g_i\delta)[\gamma_i, x] = (\Sigma_i g_i [\gamma_i, x])\delta = (\Sigma_j g_j'[\gamma_j', x])\delta = \Sigma_j(g_j'\delta)[\gamma_j', x]$. Thus $\Sigma_i[g_i\delta, \gamma_i] = \Sigma_j[g_j'\delta, \gamma_j']$ as we desired. Clearly, δ^σ preserves addition. Moreover, for $\Sigma_i[g_i, \gamma_i] \in H$ and $\Sigma_j[x_j, \beta_j] \in L$, we have $(\Sigma_i[g_i\ \gamma_i]\Sigma_j[x_j, \beta_j])\delta^\sigma = (\Sigma_{i,j}[g_i[\gamma_i, x_j], \beta_j])\delta^\sigma = (\Sigma_{i,j} [g_i, \gamma_i x_j \beta_j])\delta^\sigma = \Sigma_{i,j}[g_i\delta, \gamma_i x_j \beta_j] = \Sigma_{i,j}[g_i\delta, \gamma_i][x_j, \beta_j] = (\Sigma_i[g_i, \gamma_i]\delta^\sigma)\Sigma_j[x_j, \beta_j]$. Hence $\delta^\sigma \in \Delta'$. It can be easily verified that $\sigma: \delta \to \delta^\sigma$ is a monomorphism of Δ into Δ'. To show that σ is an onto mapping, we note that since H is a faithful irreducible right L-module and G is a faithful irreducible right R-module there exist $g_0 \in G$ and $\gamma_0 \in \Gamma$ such that $\{[g_0, \gamma]: \gamma \in \Gamma\} = H$ and $\{g_0[\gamma_0, x]: x \in M\} = G$. Let δ' be an arbitrary element in Δ' and $[g_0, \gamma_0]\delta' = [g_0, \gamma_1]$, where $\gamma \in \Gamma$. Let $\delta: G \to G$ be defined by $(g_0[\gamma_0, x])\delta = g_0[\gamma_1, x]$ for $x \in M$. This is well defined. In fact, if $g_0[\gamma_0, x] = g_0[\gamma_0, y]$, then, for any $\gamma \in \Gamma$, $[g_0, \gamma_1][x, \gamma] = ([g_0, \gamma_0]\delta')[x, \gamma] = ([g_0, \gamma_0][x, \gamma])\delta' = [g_0, \gamma_0][y, \gamma])\delta' = ([g_0, \gamma_1]\delta')[y, \gamma] = [g_0, \gamma_0][y, \gamma]$

and hence, by the construction of H, $g_0[\gamma_1 x\gamma, z]=g_0[\gamma_1 y\gamma, z]$ for all $\gamma\in\Gamma$, $z\in M$. It follows that $(g_0[\gamma_1, x]-g_0[\gamma_1, y])R=0$. Since G is a faithful irreducible right R-module, $g_0[\gamma_1, x]=g_0[\gamma_1, y]$. Clearly $\delta\in\Delta$ and $\delta^\sigma=\delta'$. Therefore $\Delta\simeq\Delta'$.

In the proof of Theorem 2 we have known already that the Γ-ring M is isomorphic to a Γ'-ring M', where Γ' is a subgroup of Hom (G, H) and M' is a subgroup of Hom (H, G). More precisely, two group isomorphisms $\theta: M\rightarrow M'$ and $\phi: \Gamma\rightarrow\Gamma'$ exist such that $\Sigma_i[g_i, \gamma_i](x\theta)=\Sigma_i g_i[\gamma_i, x]$ and $g(\gamma\phi)=[g, \gamma]$ for all $g_i, g\in G$, $\gamma_i, \gamma\in\Gamma$, $x\in M$.

Now we consider G and H as right Δ-vector space and right Δ'-vector space respectively. For any $g\in G$, $\delta\in\Delta$ and $\gamma\in\Gamma$, we have $(g\delta)(\gamma\phi)=[g\delta, \gamma]=[g, \gamma]\delta^\sigma=(g(\gamma\phi))\delta^\sigma$ and $([g, \gamma]\delta^\sigma)(x\theta)=[g\delta, \gamma](x\theta)=g\delta[\gamma, x]=(g[\gamma, x])\delta=([g, \gamma](x\theta))\delta$. Thus $\gamma\phi$ and $x\theta$ are semilinear transformations (associated with σ and σ^{-1} respectively).

It remains to show the density property for Γ'. The density property for M' can be obtained similarly. We shall show that for any n Δ-independent elements $g_1, g_2, \cdots, g_n\in G$ and any n elements $h_1, h_2, \cdots, h_n\in H$ there exists $\gamma\in\Gamma$ such that $g_i(\gamma\phi)=h_i$, $i=1, 2, \cdots, n$. We proceed by induction on n.

From Theorem 2, the assertion is obviously true for $n=1$. Now we assume that the assertion is true for $n-1$. We want first to show the existence of $\gamma\in\Gamma$ such that $g_i(\gamma\phi)=0$ for $i<n$ and $g_n(\gamma\phi)\neq 0$. Suppose such a $\gamma\in\Gamma$ does not exist. Then, for any $\gamma\in\Gamma$, $g_i(\gamma\phi)=0$, $1\leq i\leq n-1$, implies $g_n(\gamma\phi)=0$. Thus for any $h\in H$, by the induction hypothesis, there exists $\gamma_0\in\Gamma$ such that $g_1(\gamma_0\phi)=h$ and $g_i(\gamma_0\phi)=0$, $1<i\leq n-1$. If also $g_1(\gamma_1\phi)=h$ and $g_i(\gamma_1\phi)=0$, $1<i\leq n-1$, for some $\gamma_1\in\Gamma$, then since $g_i((\gamma_0-\gamma_1)\phi)=0$, for $1\leq i\leq n-1$ it follows that $g_n((\gamma_0-\gamma_1)\phi)=0$, i.e. $g_n(\gamma_0\phi)=g_n(\gamma_1\phi)$. Hence the mapping $\psi: H\rightarrow H$ defined by $h\psi=g_n(\gamma_0\phi)$ whenever $g_1(\gamma_0\phi)=h$ and $g_i(\gamma_0\phi)=0$ for $1<i<n$, is well defined. It is easy to see that ψ preserves addition. Let us recall that g_0 is an element in G with $\{[g_0, \gamma]: \gamma\in\Gamma\}=H$. Let $[g_0, \gamma]\in H$ and $\Sigma_i[x_i, \gamma_i]\in L$. Then $[g_0, \gamma]\psi=g_n(\gamma_0\phi)$ for some $\gamma_0\in\Gamma$, where $g_1(\gamma_0\phi)=[g_0, \gamma]$ and $g_i(\gamma_0\phi)=0, 2\leq i\leq$, $n-1$. Thus, $([g_0, \gamma]\psi)\ \Sigma_i[x_i, \gamma_i]=(g_n(\gamma_0\phi))\ \Sigma_i[x_i, \gamma_i]=[g_n, \gamma_0]\ \Sigma_i[x_i, \gamma_i]=g_n(\gamma_1\phi)$, where $\gamma_1=\Sigma_i\gamma_0 x_i\gamma_i$. On the other hand, since $g_1(\gamma_1\phi)=[g_0, \gamma]\ \Sigma_i[x_i, \gamma_i]$ and $g_i(\gamma_i\phi)=0, 2\leq i\leq n-1$, by the definition of ψ, $([g_0, \gamma]\ \Sigma_i[x_i, \gamma_i])\psi=g_n(\gamma_1\phi)$. Consequently, $([g_0, \gamma]\ \Sigma_i[x_i, \gamma_i])\psi=([g_0, \gamma]\psi)\ \Sigma_i[x_i, \gamma_i]$ and hence $\psi\in\Delta'$. Let $\psi=\delta^\sigma$ where $\delta\in\Delta$. Since $g_1\delta-g_n, g_2, \cdots, g_{n-1}$ are Δ-linearly independent, by the induction hypothesis, there exists $\gamma'\in\gamma$ such that $(g_1\delta-g_n)(\gamma'\phi)\neq 0$ and $g_i(\gamma'\phi)=0$ for $1<i<n$. But by the definition of ψ, $(g_1\delta-g_n)(\gamma'\phi)=(g_1\delta)(\gamma'\phi)-g_n(\gamma'\phi)=(g_1(\gamma'\phi))\psi-g_n(\gamma'\phi)=0$, a contradiction. This proves the existence of $\gamma\in\Gamma$ such that $g_n(\gamma\phi)\neq 0$ and $g_i(\gamma\phi)=0$ for $1\leq i<n$. Since $g_n(\gamma\phi)L=H$, there exists $\gamma_n\in\Gamma$ such that $g_n(\gamma_n\phi)=h_n$, and $g_i(\gamma_n\phi)=0$ for $1\leq i<n$.

Likewise, there exist $\gamma_i \in \Gamma, 1 \leq i < n$, such that $g_i(\gamma_i \phi) = h_i$ and $g_j(\gamma_i \phi) = 0$ for $i \neq j$. Now let $\gamma = \gamma_1 + \gamma_2 \cdots + \gamma_n$. Then $g_i(\gamma \phi) = h_i, 1 \leq i \leq n$ as we desired. This completes the proof of the theorem.

We recall the definition of Hestense ternary rings. Let G and H be additive abelian groups. M and Γ be subgroups of $\mathrm{Hom}(H, G)$ and $\mathrm{Hom}(G, H)$ respectively. If there is a mapping $*$ of M onto Γ such that a $b^*\ c \in M$ whenever $a, b, c \in M$ then M is called a Hestenes ternary ring. The set of all finite sums $\Sigma_i a_i^* b_i$ with $a_i, b_i \in M$ form a ring R and the set of all finite sums $\Sigma_i c_i d_i^*$ with $c_i, d_i \in M$ form a ring L. Clearly M is a right R-module and is a left L-module. If M is irreducible as a R-module and as an L-module then M is called an irreducible Hestenes ternary ring. Evidently, if M is an irreducible Hestenes ternary ring then M is a right primitive Γ-ring in the sense of Nobusawa and the rings R and L are respectively the right operator ring and the left operator ring of M. Therefore Theorem 3 generalizes further the extension of the Chevalley-Jacobson density theorem given by Smiley and Stephenson (see [8, 9]).

5. Primitive Γ-rings with non-zero socles

In [6], we have introduced the notion of socles for Γ-rings. The right (left) socle $S_r(S_l)$ of a Γ-ring M is the sum of all minimal right (left) ideals of M. In the case M has no minimal right (left) ideals, the right (left) socle of M is defined to be 0. It has been shown that if M is an one-sided primitive Γ-ring having minimal one-sided ideals then M is a two sided primitive and its right socle and left socle coincide (see [5, Theorem 4.2] and [6, Theorem 4.3]).

In this section we shall present a characterization for primitive Γ-ring with non-zero socle which is different from the one given in [5].

Theorem 4. *A Γ-ring M in the sense of Nobusawa is primitive with non-zero socle if and only if it is isomorphic to a dense Γ'-ring M' of semi-linear transformations containing non-zero semilinear transformations of finite rank. Moreover, the socle of M' is the set of semilinear transformations of finite rank contained in M'.*

Proof. Necessity. Assume that M is a primitive Γ-ring in the sense of Nobusawa with non-zero socle. According to Theorem 3, M can be regarded as a dense Γ-ring of semilinear transformations. Let G and H be vector spaces over division rings Δ and Δ', $\sigma\colon \Delta \rightarrow \Delta'$ be an isomorphism, M be a dense group of semilinear transformations of H into G (associated with σ^{-1}) and Γ be a dense group of semilinear transformations of G into H (associated with σ). Let $e\gamma M$ be a minimal right ideal of M, where $e \in M$, $\gamma \in \Gamma$ and $e\gamma e = e$. We claim that e is a rank 1, for otherwise, there would exist $h_1, h_2 \in H$ such that $h_1 e$ and $h_2 e$ are Δ-linearly independent. By the density property of Γ and M, there would exists $\gamma_0 \in \Gamma$ such that $h_1 e \gamma_0 = 0$ and $h_2 e \gamma_0 \neq 0$ and $h_2 e \gamma_0 M = G$.

Since $e\gamma M$ is minimal and $h_1 e\gamma(e\gamma_0 M)=h_1 e\ \gamma_0 M=0$, the right ideal $\{x\in e\gamma M: h_1 x=0\}=e\gamma M$, i.e. $h_1 e\gamma M=0$. Particularly, $h_1 e=h_1 e\gamma e=0$, a contradiction. Thus M contains non-zero semilinear transformations of finite rank. In addition, since the socle S of M is the sum of minimal right ideals, every element in S is of finite rank.

Sufficiency. Assume that M is a dense Γ-ring of semilinear transformations on vector spaces G and H described above, and assume that M contains semilinear transformations of finite rank. By density property, M contains semilinear transformations of rank 1. Let $a\in M$ be of rank 1, and let $Ha=\langle g_1\rangle$, the subspace of G generated by g_1. Consider $I=\{x\in M: Hx\subseteq\langle g_1\rangle\}$, a left ideal of M. We claim that I is minimal. Let $0\neq x_1\in I$. Then $Hx_1=\langle g_1\rangle$ and $h_1 x_1=g_1$ for some $h_1\in H$. By the density property of Γ, there exists $\gamma_1\in\Gamma$ such that $g_1\gamma_1=h_1$. Thus $g_1=g_1\gamma_1 x_1$. Now let x be an arbitrary element in I. For any $h\in H$, there exists $\delta\in\Delta$ such that $hx=g_1\delta=(g_1\gamma_1 x_1)\delta=(g_1\delta)\gamma_1 x_1=hx\gamma_1 x_1$. Hence $x=x\gamma_1 x_1\in M\Gamma x_1$, so $I=M\Gamma x_1$ for every $0\neq x_1\in I$. Thereofre I is a minimal left ideal containing a, a is in the socle of M, and M has a non-zero socle S.

The argument just used shows that every element in M of rank 1 is in S. But the density property of M and Γ insures that every element in M of finite rank is a sum of finitely many elements in M of rank 1. Therefore S contains all elements in M of finite rank. This completes the proof.

NORTH CAROLINA STATE UNIVERSITY AT RALEIGH

References

[1] W.E. Barnes: *On the Γ-rings of Nobusawa,* Pacific J. Math. **18** (1966), 411–422.

[2] W.E. Coppage and J. Luh: *Radicals of gamma rings* (to appear).

[3] M.R. Hestenes: *A ternary algebra with applications to matrices and linear transformations,* Arch. Rational Mech. Anal. **11** (1962), 138–194.

[4] N. Jacobson: Structure of Rings, Amer. Math. Soc. Colloquium Publ. 37, Providence, 1964.

[5] J. Luh: *On primitive Γ-rings with minimal one-sided ideals,* Osaka J. Math. **5** (1968), 165–173.

[6] ——— : *On the theory of simple Γ-rings,* Michigan Math. J. **16** (1969), 65–75.

[7] N. Nobusawa: *On a generalization of the ring theory,* Osaka J. Math. **1** (1964), 81–89.

[8] M.F. Smiley: *An introduction to Hestenes ternary rings,* Amer. Math. Monthly **76** (1969), 245–248.

[9] R.A. Stephenson: *Jacobson structure theory for Hestenes ternary rings,* Ph. D. dissertation, University of California, Riverside, 1968.