Murata, K., Kurata, Y. and Marubayashi, H.
Osaka J. Math.
6 (1969), 291-301

A GENERALIZATION OF PRIME IDEALS IN RINGS

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(Received February 6, 1969)

Introduction

In [2], van der Walt has defined s-prime ideals in noncommutative rings and obtained analogous results of McCoy [1] for s-prime ideals. In the present paper, we shall give a generalized concept of prime ideals, called f-prime ideals, by using some family of ideals, and obtain analogous results in [2]. If our family of ideals is, in particular, the set of principal ideals of the ring, the f-prime ideals coincide with the prime ideals and conversely. In addition, if we take multiplicatively closed systems as kernels, the f-prime ideals coincide with the s-prime ideals.

1. f-prime ideals and the f-radical of an ideal

Let R be an arbitrary (associative) ring. Throughout this paper, the term "ideals" will always mean "two-sided ideals in R".

For each element a of R, we shall associate an ideal $f(a)$ which is uniquely determined by a and satisfies the following conditions:

(I) $a \in f(a)$, and

(II) $x \in f(a)+A \Rightarrow f(x) \subseteq f(a)+A$ for any ideal A.

The principal ideal (a) generated by a is an example of the $f(a)$, and this is the case of [2]. Moreover there are other interesting examples of the $f(a)$. For example, let Q be any subset of R. If we define, for each element a of R, $f(a)=(a, Q)$, the ideal generated by a and Q, then it is easy to see that $f(a)$ satisfies the above conditions. If, in particular, Q is the empty set, then the $f(a)$ coincides with the principal ideal (a).

REMARK. As is easily seen, the following four conditions are equivalent:

(i) For any element a of R, $f(a)=(a)$,

(ii) $f(0)=0$,

(iii) For any ideal A, $x \in A \Rightarrow f(x) \subseteq A$,

(iv) For any element a of R, $x \in (a) \Rightarrow f(x) \subseteq (a)$.

DEFINITION 1.1. A subset S of R is called an *f-system* if S contains an

m-system S^*, called the *kernal* of S, such that $f(s)\cap S^*\neq\phi$ for every element s of S. ϕ is also defined to be an f-system.

We note that every s-m-system in the sense of [2] is an f-system and also every m-system is an f-system with kernel itself. In the sequel we shall denote by $S(S^*)$ the f-system S with kernel S^*, whenever it be convenient. We also note that if $S(S^*)$ is an f-system, then $S=\phi$ if and only if $S^*=\phi$.

DEFINITION 1.2. An ideal P is said to be *f-prime* if its complement $C(P)$ in R is an f-system.

R is evidently an f-prime ideal. Obviously an s-prime ideal in the sense of [2] is a prime ideal in the sense of [1], and it follows from Lemma 1.4 below that if we assume $f(a)=(a)$ for every element a in R, then prime ideals are nothing but f-prime ideals. But it can be shown that this is not always true with a suitable choice of $f(a)$.

EXAMPLE 1.3. Consider the ring $\boldsymbol{Z}$ of integers. Let P be the ideal (p^2) and let S^* be the m-system $\{q, q^2, q^3,\cdots\}$, where p and q are different prime numbers. If we put $f(a)=(a, q)$ for each element a in $\boldsymbol{Z}$, then the complement $C(P)$ of P in $\boldsymbol{Z}$ is an f-system with kernel S^*. Hence P is an f-prime ideal, but not a prime ideal. This also shows that an f-prime ideal need not be an s-prime ideal, in general.

Lemma 1.4. *For any f-prime ideal P,*
$f(a_1)f(a_2)\cdots f(a_n)\subseteq P \Rightarrow a_i\in P$ *for some i.*

Proof. It is evident from the definition of f-systems.

Lemma 1.5. *Let $S(S^*)$ be an f-system in R, and let A be an ideal in R which does not meet S. Then A is contained in a maximal ideal P (in the class of all ideals, each of) which does not meet S. The ideal P is necessarily an f-prime ideal.*

Proof. If S is empty, the assertion is trivial, and so suppose that S is not empty. The existence of P follows from Zorn's lemma. We now show that $C(P)$ is an f-system with kernel S^*+P. For any element a of $C(P)$, the maximal property of P implies that $f(a)+P$ contains an element s of S, and thus we can choose an element s^* in $f(s)\cap S^*$. Since $f(s)$ is contained in $f(a)+P$, we can write $s^*=a'+p$ where a' in $f(a)$ and p in P. Then $a'=s^*-p$ is contained in $f(a)\cap(S^*+P)$, which completes the proof of the lemma.

DEFINITION 1.6. The *f-radical* $r(A)$ of an ideal A will be defined to be the set of all elements a of R with the property that every f-system which contains a contains an element of A.

Theorem 1.7. *The f-radical of an ideal A is the intersection of all the f-prime ideals containing A.*

Proof. We show that if P is an f-prime ideal containing A, then $r(A)$ is contained in P. For suppose that $r(A)$ is not contained in P. Then there exists an element x in $r(A)$ not in P. Since $C(P)$ is an f-system, $C(P) \cap A \neq \phi$. But this contradicts the fact that A is contained in P. Hence $r(A)$ is contained in the intersection of all f-prime ideals which contain A.

Conversely, let a be an element of R, but not in $r(A)$. Then there exists an f-system $S(S^*)$ which contains a but does not meet A. There exists, by Lemma 1.5, an f-prime ideal P which contains A and does not meet S. Hence, P does not contain a and a can not be in the intersection of all f-prime ideals containing A. This completes the proof.

Corollary 1.8. *The f-radical of an ideal is an ideal.*

Now, let $S(S^*)$ be an f-system in R and let A be an ideal which does not meet S. It follows from Zorn's lemma that there exists a maximal m-system S_1^* which contains S^* and does not meet A. Let us consider the set $S_1 = \{x \in R \mid f(x) \cap S_1^* \neq \phi\} \cap C(A)$. Then S_1 is an f-system with kernel S_1^* and does not meet A. According to Lemma 1.5, there exists an f-prime ideal P which contains A and does not meet S_1. As is seen in the proof of Lemma 1.5, $C(P)$ is an f-system with kernel $S_1^* + P$, and the maximal property of S_1^* implies that $S_1^* + P = S_1^*$. Hence we have $C(P) = S_1$ by the definition of S_1.

In view of this we make the following definition:

DEFINITION 1.9. An f-prime ideal P is said to be a *minimal f-prime ideal belonging to an ideal A* if P contains A and there exists a kernel S^* for the f-system $C(P)$ such that S^* is a maximal m-system which does not meet A.

It follows from the above consideration that any f-prime ideal P containing A contains a minimal f-prime ideal belonging to A. From Theorem 1.7, we can conclude the following:

Theorem 1.10. *The f-radical of an ideal A coincides with the intersection of all minimal f-prime ideals belonging to A.*

2. Elements f-related to an ideal

We now make the following definition:

Definition 2.1. An element a of R is said to be *(left-)f-related* to an ideal A if, for every element a' in $f(a)$, there exists an element c not in A such that $a'c$ is in A. An ideal B is said to be *(left-)f-related* to A if every element of B is f-related to A. Elements and ideals not f-related to A is called *(left-)f-unrelated* to A.

Elements and ideals right-f-related to A can be similarly defined, but the right hand definitions and theorems will be omitted.

Proposition 2.2. *Let A be an ideal. Then the set S consisting of all elements of R which are f-unrelated to A is an f-system.*

Proof. For every element a in S, we can choose an element a^* in $f(a)$ such that, for every element c not in A, a^*c is not in A. The set S^* which consists of all such elements a^* is multiplicatively closed and hence S is an f-system with kernel S^*.

It is natural to consider that every element of R is f-related to R. Furthermore we shall now assume, in this section, the following condition:

(α) *Each ideal A is f-related to itself.*

It may be remarked that (α) can be stated in the following convenient form:

(α') *0 is f-related to each ideal A.*

For suppose that 0 is f-related to A. Let a be any element in A. Then a is in $A+f(0)$ and hence $f(a)$ is contained in $A+f(0)$. For any element a' in $f(a)$, there exist a'' in A and b'' in $f(0)$ such that $a'=a''+b''$. Since 0 is f-related to A, we can choose an element c not in A such that $b''c$ is in A. Therefore, $a'c=a''c+b''c$ is in A and this means that A is f-related to itself.

Clearly, (α) is fulfilled in case $f(a)=(a)$ for every element a in R. And, it can be proved that, whenever R has no right zero-divisors, R satisfies (α) if and only if $f(a)=(a)$ for every element a in R. But, in case of general rings, this need not be true as is seen from the following example.

EXAMPLE 2.3. Consider a simple module M such that $m_1m_2=0$ for any two elements m_1 and m_2 in M. Let K be a field and let R be the direct sum of M and K as modules. Then R can be made into a commutative ring by defining as

$$(m_1+k_1)(m_2+k_2) = k_1k_2\,,$$

where m_1, m_2 in M and k_1, k_2 in K. As is easily seen, the ideals in R are R, M, K and (0). If we define $f(a)=(a, M)$ for every element a in R, then R satisfies (α), but $f(a)$ does not coincide with (a), since $f(0)=M\neq(0)$.

Proposition 2.4. *Let A be an ideal. Then the f-radical $r(A)$ of A is f-related to A.*

Proof. Let S be as in Proposition 2.2. If $r(A)$ contains an element f-unrelated to A, then, by the definition of the radical, we have $S\cap A\neq\phi$, a contradiction.

It follows from this proof, in terms of relatedness, that the assumption (α) can be also restated as follows: for any ideal A, the f-radical of A is f-related to A.

Let A be an ideal and let S be the f-system consisting of all elements f-

unrelated to A. Then S does not meet the ideal (0), and hence, by Lemma 1.5, there exists a maximal ideal (in the class of all ideals, each of) which does not meet S, or equivalently, a maximal ideal (each of) which is f-related to A. Each such maximal ideal is necessarily an f-prime ideal. In view of this, we put the following:

DEFINITION 2.5. A maximal ideal in the class of all ideals, each of which is f-related to an ideal A, is called a *maximal f-prime ideal belonging to A.*

Proposition 2.6. *Let A be an ideal. Then A is contained in every maximal f-prime ideal belonging to A.*

Proof. Let P be any maximal f-prime ideal belonging to A. Then it is sufficient to show that $A+P$ is f-related to A. Let $a+p$ be any element in $A+P$, where a in A and p in P. Since $a+p$ is in $A+f(p)$, $f(a+p)$ is contained in $A+f(p)$, and hence each element a' in $f(a+p)$ can be written as $a'=a''+p''$, where a'' in A and p'' in $f(p)$. We can choose an element c not in A such that $p''c$ is in A. Then $a'c=a''c+p''c$ is contained in A, which completes the proof.

Since any f-prime ideal containing A contains a minimal f-prime ideal belonging to A, it follows from Proposition 2.6 that every maximal f-prime ideal belonging to A necessarily contains a minimal f-prime ideal belonging to A. The converse is also true in case of [1], but we can provide an example to show that this need not be true in our case.

EXAMPLE 2.7. Let us consider the ideal $A=(xy)$ in the ring $K[x, y]$ of polynomials in two non-commutative indeterminates x and y over a field K. If we define $f(a)=(a)$ for every element a in $K[x, y]$, then the assumption (α) is satisfied and A is f-related to itself. Hence we can consider the maximal f-prime ideal belonging to A. As is easily seen, the ideal (y) is a minimal f-prime ideal belonging to A, but it is f-unrelated to A. Thus, (y) is not contained by any maximal f-prime ideal belonging to A.

Proposition 2.8. *Let A be an ideal. Then every element or ideal which is f-related to A is contained in a maximal f-prime ideal belonging to A.*

Proof. Obviously, an element a is f-related to A if and only if $f(a)$ is f-related to A. So we shall prove the only case of an ideal which is f-related to A. Let B be such an ideal, and let S be the f-system consisting of all elements of R which are f-unrelated to A. Then B does not meet S and hence, by Lemma 1.5, B is contained in a maximal f-prime ideal P belonging to A.

It follows from this proposition that the ideals of R which are f-related to A are spread over the maximal f-prime ideals belonging to A.

DEFINITION 2.9. Let A be an ideal and let b be an element in R. The (*left-*)

f-quotient $A:b$ of A by b will be defined to be the set of all elements x of R such that $f(b)f(x)$ is contained in A. Moreover, for any ideal B, the *(left-)f-quotient of A by B* will be defined as $\cap_{b\in B}(A:b)$, and denoted by $A:B$.

From this definition, we have

(1) $A'\subseteq A'' \Rightarrow A':b\subseteq A'':b$ and $A':B\subseteq A'':B$,

(2) $B'\subseteq B'' \Rightarrow A:B'\supseteq A:B''$,

(3) $(A'\cap A''):b=(A':b)\cap(A'':b)$ and $(A'\cap A''):B=(A':B)\cap(A'':B)$.

We note that $A:b$ may be empty. However, if it is not, it is an ideal containing A. To see this, take an arbitrary element $x+a$ in $(A:b)+A$, where x in $A:b$ and a in A. Then $x+a$ is contained in $f(x)+A$, and so is $f(x+a)$. Hence $f(b)f(x+a)$ is contained in A. That is, $(A:b)+A$ is contained in $A:b$.

DEFINITION 2.10. Let A be an ideal, and let P be any maximal f-prime ideal belonging to A. The *principal f-component A_P of A determined by P* will be defined as follows:

$$A_P=\begin{cases}\cup_{s\notin P}(A:s) & (\text{if} \quad P\neq R)\\ A & (\text{if} \quad P=R).\end{cases}$$

For $P\neq R$, the principal f-component A_P may be empty in certain cases. In case $f(a)=(a)$ for every a in R it is not empty, but, as is seen from Example 2.3, there exists a ring in which (α) is satisfied, and $f(a)$ need not be (a), and A_P is not empty for all A and $P\neq R$.

So we shall assume, in the rest of this paper, the following condition:

(β) *For any ideal A and ideal B not contained in $r(A)$, we have $A:B\neq\phi$.*

For any maximal f-prime ideal P belonging to A, it follows from Proposition 2.6 that P contains A, and hence $r(A)$ is contained in P. If s is not in P, then s does not contained in $r(A)$. Hence, from the assumption (β), $A:s\neq\phi$ and therefore we have $A_P\neq\phi$.

We now show that A_P is an ideal containing A. If $P=R$, the assertion is trivial. Let $P\neq R$ and let x, y be any two elements of A_P. Then there exist s and t in $C(P)$ such that both $f(s)f(x)$ and $f(t)f(y)$ are contained in A. Take two elements s^* in $S^*\cap f(s)$ and t^* in $S^*\cap f(t)$, where S^* is a kernel of $C(P)$. Since S^* is an m-system, s^*zt^* is in S^* (whence is in $C(P)$) for some z in R. Thus $s^*zt^*\in f(s)\cap f(t)$, $f(s^*zt^*)\subseteq f(s)\cap f(t)$. Hence $f(s^*zt^*)f(x+y)\subseteq(f(s)\cap f(t))(f(x)+f(y))\subseteq f(s)f(x)+f(t)f(y)\subseteq A$.

Now let $x=x'+x''$ be any element in A_P+A, where x' in A_P and x'' in A. Then $f(s)f(x')$ is contained in A for some s in $C(P)$. Since x is in $f(x')+A$, $f(x)$ is contained in $f(x')+A$, and hence we have $f(s)f(x)\subseteq f(s)f(x')+f(s)A\subseteq A$. Thus x is in A_P and A is contained in A_P.

For any maximal f-prime ideal P belonging to A, since $A\subseteq A_P\subseteq P$, $A_P=R$ if and only if $A=R$. Furthermore, if P is the only maximal f-prime ideal belonging to A, or equivalently by Proposition 2.8, if its complement $C(P)$ consists of all elements which are f-unrelated to A, then we have $A_P=A$.

Proposition 2.11. *Let A be an ideal, and let P be any maximal f-prime ideal belonging to A. Then the principal f-component A_P is contained in every ideal D such that A is contained in D and that any element of $C(P)$ are f-unrelated to D.*

Proof. If $P=R$, the assertion is trivial. Let $P\neq R$ and let D be any ideal such that A is contained in D and that any element of $C(P)$ are f-unrelated to D. If x is an arbitrary element of A_P, then there exists an element s in $C(P)$ such that $f(s)f(x)\subseteq A$. Since s is f-unrelated to D, we can choose an element s^* in $f(s)$ such that $s^*c\in D$ implies $c\in D$. s^*x is in D and hence x is in D.

We note from Proposition 2.8 that any element of $C(P)$ are f-unrelated to D if and only if any maximal f-prime ideal belonging to D are contained in P.

Theorem 2.12. *Any ideal A is represented as the intersection of all its principal f-components A_P.*

Proof. Since A is contained in every principal f-component of A, it is also contained in their intersection. To prove the converse, let a be an arbitrary element of the intersection of all principal f-components A_P. For any maximal f-prime ideal P belonging to A, $f(s)f(a)\subseteq A$ for some s in $S=C(P)$. Consider the ideal B which consists of all elements b of R such that $f(b)f(a)\subseteq A$. Then B is not contained in P, and hence according to Proposition 2.8, B can not be f-related to A. This means that B contains at least one element b which is f-unrelated to A. Since $f(b)f(a)$ is in A, the f-unrelatedness of b implies that a is in A. The theorem is therefore established.

REMARK. It is natural to define a (left-)f-primal ideal as follows: an ideal A is said to be *(left-)f-primal*, if the set X of the elements, each of which is (left-) f-related to A, forms an ideal. If A is f-primal, X is called the *(left-)adjoint* of A. Then we can prove that the principal f-component of A determined by the maximal f-prime ideal P is contained in the intersection of all f-primal ideals A_λ such that (1) A_λ contains A, and (2) the adjoint of A_λ is contained in P.

3. f-primary decompositions

In this section, we shall consider f-primary decompositions of ideals on the analogy of the primary decompositions of ideals in a commutative Noetherian ring. For this purpose, we assume besides (β), throughout this section, the following condition:

(γ) *If S is an f-system with kernel S^*, and if for any ideal A, $S\cap A$ is not empty, then so is $S^*\cap A$.*

Clearly, this assumption is satisfied in case $f(a)=(a)$ for every element a in R. But, for a suitable choice of $f(a)$, this is not always satisfied as is seen from the following example:

EXAMPLE 3.1. As is seen from Example 1.3, for the ideal $P=(p^2)$ in the ring $\boldsymbol{Z}$ of integers, its complement $S=C(P)$ is an f-system with kernel $S^*=\{q, q^2, q^3, \cdots\}$, where p and q are different prime numbers. Now, let A be the ideal (p), then we have $S\cap A\neq\phi$, though $S^*\cap A=\phi$.

Proposition 3.2. *Let A and B be any two ideals. Then*

(1) $A\subseteq B \Rightarrow r(A)\subseteq r(B)$,
(2) $r(r(A))=r(A)$,
(3) $r(A\cap B)=r(A)\cap r(B)$.

Proof. (1) and (2) follow from the definition of the radical.

It is clear that $r(A\cap B)\subseteq r(A)\cap r(B)$. Conversely, let x be any element in $r(A)\cap r(B)$ and let S be any f-system containing x. Then, there exist two elements a and b in $S\cap A$ and $S\cap B$ respectively. By the assumption (γ), we can choose two elements a^* and b^* in $S^*\cap A$ and $S^*\cap B$ respectively. Since S^* is an m-system, a^*zb^* is in S^* for some element z in R. Therefore $a^*zb^* \in S^*\cap(A\cap B)$, and hence $S\cap(A\cap B)$ is not empty. This means that x is in $r(A\cap B)$, which completes the proof of (3).

DEFINITION 3.3. An ideal Q is called *(left-)f-primary*, if $f(a)f(b)\subseteq Q$ implies that $a\in r(Q)$ or $b\in Q$.

Let us note that, by Lemma 1.4, f-prime ideals are always f-primary ideals. As is easily seen from Definition 3.3, we have

Proposition 3.4. *If Q' and Q'' are f-primary ideals such that $r(Q')=r(Q'')$, then $Q=Q'\cap Q''$ is also an f-primary ideal such that $r(Q)=r(Q')=r(Q'')$.*

Another characterization of f-primary ideals can be given by means of f-quotients.

Proposition 3.5. *An ideal Q is f-primary if and only if $Q:B=Q$ for all ideals B not contained in $r(Q)$.*

Proof. Suppose that Q is f-primary and that B is an ideal not contained in $r(Q)$. We can choose an element b in B but not in $r(Q)$. By the assumption (β), $Q:b$ is not empty, and for any element a in $Q:b$, $f(b)f(a)$ is contained in Q. Since Q is f-primary and b is not in $r(Q)$, a is in Q. Thus $Q:b$ is contained in Q. This shows that $Q=Q:B$, because again by (β) $Q:B$ is an ideal such that $Q\subseteq Q:B \subseteq Q:b$.

Conversely, suppose that $f(a)f(b)$ is contained in Q and that a is not in

$r(Q)$. Then $f(a)$ is not contained in $r(Q)$, and hence we have $Q:f(a)=Q$. For an arbitrary element a' in $f(a)$, $f(a')f(b)\subseteq f(a)f(b)\subseteq Q$, and thus b is in $Q:f(a)=Q$. This proves that Q is f-primary.

If an ideal A can be written as

$$A = Q_1 \cap Q_2 \cap \cdots \cap Q_n,$$

where each Q_i is an f-primary ideal, this will be called an f-primary decomposition of A, and each Q_i will be called the f-primary component of the decomposition. A decomposition in which no Q_i contains the intersection of the remaining Q_j is called irredundant. Moreover, an irredundant f-primary decomposition, in which the radicals of the various f-primary components are all different, is called a normal decomposition. As is easily seen from Proposition 3.4, each f-primary decomposition can be refined into one which is normal.

Besides the assumptions (β) and (γ), we assume, in this section, the following condition:

(δ) *For any f-primary ideal Q, we have $Q:Q=R$.*

Evidently, this assumption is satisfied in case $f(a)=(a)$ for every element a in R. But, for a suitable choice of $f(a)$, this is not all true.

EXAMPLE 3.6. As is seen from Example 1.3, the ideal (p^2) is f-prime and hence is an f-primary ideal in $\boldsymbol{Z}$. Suppose that the assumption (δ) is satisfied for this (p^2). Then we have $f(p^2)\subseteq(p^2)$ and hence $(p^2)=f(p^2)=(p^2)+(q)$, a contradiction.

Now we shall prove, under the assumptions (β), (γ) and (δ), that the number of f-primary components and the radicals of f-primary components of a normal decomposition of A depend only on A and not on the particular normal decomposition considered. This is a main theorem of this section.

Theorem 3.7. *Suppose that an ideal A has an f-primary decomposition, and let*

$$A = Q_1 \cap Q_2 \cap \cdots \cap Q_n = Q_1' \cap Q_2' \cap \cdots \cap Q_m'$$

be two normal decomposions of A. Then $n=m$, and it is possible to number the f-primary components in such a way that $r(Q_i)=r(Q_i')$ for $1\leq i\leq n=m$.

Proof. If A coincides with R, the assertion is trivial. We may suppose therefore that A does not coincide with R, in which case all the f-primary components $Q_1, \cdots, Q_n, Q_1', \cdots, Q_m'$ are proper ideals. Among the radicals $r(Q_1), \cdots, r(Q_n), r(Q_1'), \cdots, r(Q_m')$ take one which is maximal in this set, and we may assume that it is $r(Q_1)$. We now prove that $r(Q_1)$ occurs among $r(Q_1'), \cdots, r(Q_m')$. To prove this it will be enough to show that Q_1 is contained in $r(Q_j')$ for some j.

Suppose that Q_1 is not contained in $r(Q'_j)$ for $1 \leq j \leq m$. Then we have, by Proposition 3.5, $Q'_j : Q_1 = Q'_j$ for $1 \leq j \leq m$, and consequently

$$\begin{aligned} A : Q_1 &= (Q'_1 \cap \cdots \cap Q'_m) : Q_1 \\ &= (Q'_1 : Q_1) \cap \cdots \cap (Q'_m : Q_1) \\ &= Q'_1 \cap \cdots \cap Q'_m \\ &= A . \end{aligned}$$

If $n=1$, then, by the assumption (δ), we have

$$R = Q_1 : Q_1 = A : Q_1 = A ,$$

a contradiction. On the other hand, if $n > 1$, then we have again by (δ)

$$\begin{aligned} A = A : Q_1 &= (Q_1 \cap \cdots \cap Q_n) : Q_1 \\ &= (Q_1 : Q_1) \cap \cdots \cap (Q_n : Q_1) \\ &= Q_2 \cap \cdots \cap Q_n , \end{aligned}$$

since Q_1 is not contained in $r(Q_i)$ for $2 \leq i \leq n$. This is a contradiction. Now we may arrange that Q_i and Q'_j so that $r(Q_1) = r(Q'_1)$.

We shall use an induction on the number n of f-primary components. If $n=1$, then $A = Q_1 = Q'_1 \cap \cdots \cap Q'_m$, and moreover if $m > 1$, then Q_1 is not contained in $r(Q_1')$ for $2 \leq j \leq m$. Since

$$R = Q_1 : Q_1 = (Q'_1 : Q_1) \cap \cdots \cap (Q'_m : Q_1) ,$$

we have $R = Q'_2 = Q'_3 = \cdots = Q'_m$, by Proposition 3.5, a contradiction. Similarly, $m=1$ implies that $n=1$, and in this case the assertion is trivial.

Let us now assume that $n \leq m$. We shall show that $n=m$ and by a suitable ordering $r(Q_i) = r(Q'_i)$ for $1 \leq i \leq n = m$. Assume that these results are valid for ideals which may be represented by fewer than n f-primary components. Put $Q = Q_1 \cap Q'_1$, then by Proposition 3.4, Q is an f-primary ideal such that $r(Q) = r(Q_1) = r(Q'_1)$. Also $Q_i : Q = Q_i$ for $2 \leq i \leq n$, and $Q_1 : Q = R$. For the first relation follows from the fact that Q is not contained in $r(Q_i)$, while the second follows from $R = Q_1 : Q_1 \subseteq Q_1 : Q$. Consequently $A : Q = Q_2 \cap \cdots \cap Q_n$, and an exactly similar argument shows that $A : Q = Q'_2 \cap \cdots \cap Q'_m$. Hence, we have

$$Q_2 \cap \cdots \cap Q_n = Q'_2 \cap \cdots \cap Q'_m ,$$

and moreover both decompositions are normal. Thus by the induction hypothesis we have $n-1 = m-1$, that is, $n=m$. Furthermore, by a suitable ordering we have $r(Q_i) = r(Q'_i)$ for $2 \leq i \leq n = m$. This completes the proof.

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