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A DECOMPOSITION THEOREM IN A MULTIPLICATIVE SYSTEM

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1. Introduction

In [2], K. Murata has introduced the notion of an accessible join-generator system in multiplicative lattices. The condition for multiplicative lattices to have such systems is strictly weaker than the usual ascending chain condition for elements. Nevertheless, we can obtain, under that condition, many important results on the decompositions of elements in multiplicative lattices. In [3], Murata has considered a multiplicative system S as a lattice-theoretic interpretation of one-sided ideals of any ring-system, and discussed on the decomposition of φ-closed elements of S under the assumption of the existence of some restricted accessible join-generator systems.

In the present paper, we shall generalize locally the concept of φ-closed elements of S assuming the existence of the same accessible join-generator systems. We define φ_c-closed elements of S, and investigate the decomposition of such elements as a meet of some prime elements of S.

2. Preliminary

Let S be a partly ordered set with the greatest element e and the least element 0. We assume that the multiplication is defined in S which satisfies the following conditions[1].

P_1: e and a are composable with respect to the multiplication and $ea \geq a$, $ae \geq a$ for every element a of S.

P_2: 0 and a are composable with respect to the multiplication and $0a = a0 = 0$ for every element a of S.

P_3: S is the set-union of $L = \{a\,;\ ea = a, a \in S\}$ and $R = \{a\,;\ ae = a, a \in S\}$.

1) cf. [3]

P_4: L forms a cl-semigroup.[2]

P_5: R forms a cl-semigroup.

It is then easy to see that (1) $ab \leq b$ for any two elements a, b of L and (2) $ab \leq a$ for any two elements a, b of R.

In the following, we shall denote by T the intersection of L and R. Then it is clear that T is a complete cl-subsemigroup of both L and R. We suppose, throughout this paper, that L and R has accessible join-generator systems $\sum_L$ and $\sum_R$ respectively, which have the following two conditions (*) and (**).

(*) The product xy of any two elements x, y of $\sum_L$ is expressible as a join of a finite number of elements of $\sum_L$. Similarly for $\sum_R$.

(**) $\sum_L \supseteq \sum_L e$[3], $\sum_L e \subseteq \sum_R$, $\sum_R \supseteq e\sum_R$, $e\sum_R \subseteq \sum_L$.

From the condition (**), it is clear that $\sum_L \wedge \sum_R$[4] $= \sum_L e = \sum_L \wedge R = \sum_R \wedge L = e\sum_R$. Let $\sum_T$ be the intersection of $\sum_L$ and $\sum_R$, then $\sum_T$ is an accessible join-generator system of T, and it satisfies the condition (*).

An element p of T is said to be prime, if $ab \leq p$ implies $a \leq p$ or $b \leq p$ for any two elements a, b of T. Let c be an element of T. The radical of c is defined as the infimum of all prime elements p_α containing c. In symbols: Rad(c). In particular, the radical of the least element is called the radical of S, which is denoted by Rad(S). A non-zero element a of S is said to be *nilpotent*, if $a^n = 0$ for some positive integer n. A non-void subset of S is said to be *nilpotent-free*, if it has no nilpotent element. Then the following seven conditions are equivalent to one another:[5] (1) Rad$(S)=0$, (2) $\sum_T$ is nilpotent-free, (3) T is nilpotent-free, (4) $\sum_L$ is nilpotent-free, (5) L is nilpotent-free, (6) $\sum_R$ is nilpotent-free and (7) R is nilpotent-free.

3. Lemmas

In this and the next sections we shall assume that Rad$(S)=0$. Let a be an element of L, and c an element of T. Let further $\sum'$ be a subset of $\sum_L$. By $l_c(a; \sum')$, we mean the set of all elements x of $\sum'$ such that $xa=0$ and $x \leq c$. Symmetrically, we can define $r_c(a; \sum')$ for an element a of R, c of T and a subset $\sum'$ of $\sum_R$.

Lemma 1. *Let a be an element of L, and c an element of T. Then*

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2) cf. [1]. We do not assume the existence of multiplicative unity.

3) $\sum_L e$ means the set of the elements xe for every element x of $\sum_L$. Similary for $e\sum_R$, etc.

4) The symbol ($\wedge$) means the intersection.

5) cf. [3; Theorem 10]

$$\sup [l_c(a\,;\ \textstyle\sum_L)] = \sup [l_c(a\,;\ \textstyle\sum_T)]\,,$$

and it is contained in T.

Proof. Since T is complete, there exists $\sup [l_c(a\,; \sum_T)]$ in T. Take any element x of $l_c(a\,; \sum_L)$. Then we have $(xe)a = x(ea) = xa = 0$, $xe \leq c$ and $xe \in \sum_L e = \sum_T$. Therefore $xe \in l_c(a\,; \sum_T)$. Since $x \leq xe$, we obtain $\sup [l_c(a\,; \sum_L)] \leq \sup [l_c(a\,; \sum_T)]$. The converse inclusion is evident. This completes the proof.

Throughout this and the next sections we shall denote $\sup [l_c(a\,; \sum_L)]$ by $l_c^*(a)$. Symmetrically we shall define $\sup [r_c(a\,; \sum_R)]$ and $r_c^*(a)$, where a is an element of R, and c of T.

Lemma 2. *Let a and c be two elements of T. Then*

$$l_c^*(a) = r_c^*(a)\,.$$

Proof. Take an arbitrary element x of $l_c(a\,; \sum_T)$. Then we have $ax \leq x$ and $ax \leq a$. Hence $(ax)^2 \leq xa = 0$. This implies $ax = 0$, because T is nilpotent-free. Therefore $l_c(a\,; \sum_T)$ is contained in $r_c(a\,; \sum_T)$. Similarly $r_c(a\,; \sum_T)$ is contained in $l_c(a\,; \sum_T)$. That is, $l_c(a\,; \sum_T) = r_c(a\,; \sum_T)$, $l_c^*(a) = r_c^*(a)$, q.e.d.

In the following, we shall denote $l_c^*(a)$ $(= r_c^*(a))$ by a_c^*. Now it is easily verified that

$$a_c^* a = a a_c^* = 0\,.$$

DEFINITION. Let c be an element of T. A mapping

$$a \to \varphi_c(a) = r_c^*(l_c^*(a)) = (l_c^*(a))_c^*$$

from L into T is said to be a φ_c-mapping. If $\varphi_c(a) = a$, a is called φ_c-closed. We can define symmetrically a φ_c'-mapping from R into T, and a φ_c'-closed element.

Let a and c be two elements of T such that $a \leq c$. In the following, by c/a we mean the set of all elements x of T such that $a \leq x \leq c$. Then c/a forms a *cl*-semigroup under the inclusion relation $(\leq)$ and the multiplication $x \circ y = xy \cup a$, where $x, y \in c/a$. For, it is easy to see that c/a is a complete lattice under the inclusion relation $(\leq)$, and that it is closed under the multiplication $(\circ)$. Moreover, with respect to this multiplication, the associative law and the infinite distributive law hold for c/a.

An element p of c/a is said to be $(\circ)$-*prime*, if $x \circ y \leq p$ implies $x \leq p$ or $y \leq p$ for any two elements x, y of c/a. For any element x of c/a, $x^{(n)}$ will denote the n-th power of x with respect to $(\circ)$-multiplication.

Then we have $x^{(n)}=x^n \cup a$. An element x, not equal to a, is said to be $(\circ)$-*nilpotent*, if $x^{(m)}=a$ for a suitable positive integer m. A non-void subset A of c/a is called $(\circ)$-*nilpotent-free* if it has no $(\circ)$-nilpotent element.

Lemma 3. *Let a be a φ_c-closed element of T. Then c/a is $(\circ)$-nilpotent-free.*

Proof. Suppose that c/a is not $(\circ)$-nilpotent-free. Then we can take an element t of c/a such that $t>a$ and $t^{(m)}=a$. Since $t^{(m)}=t^m \cup a$ $=a$, we have $t^m \leq a$. Now it is easy to see that $(t \cap a_c^*)^m \leq t^m \leq a$ and $(t \cap a_c^*)^m \leq a_c^*$, and so, $(t \cap a_c^*)^m \leq a \cap a_c^*$. The reflection $aa_c^*=0 \Leftrightarrow a \cap a_c^*=0$ is valid by Lemma 14 in [3]. Hence $a \cap a_c^*=0$. It follows that $t \cap a_c^*$ $=0$, because T is nilpotent-free by Theorem 10 in [3]. Therefore we have $ta_c^*=0$. Now let $t=\bigcup_\alpha x_\alpha$ be the sup-expression of t by a subset $\{x_\alpha\}$ of $\sum_T$. Then $x_\alpha \leq t$ and $x_\alpha a_c^* \leq ta_c^*=0$, $x_\alpha a_c^*=0$ for all α. This shows that x_α is contained in $l_c(a_c^*\,;\,\sum_T)$ for all α. We obtain therefore $a<t=\bigcup_\alpha x_\alpha \leq l_c^*(a_c^*)=(a_c^*)_c^*=\varphi_c(a)$. This contradicts our assumption.

Lemma 4. *Let a be a φ_c-closed element of T. Then c/a has an accessible join-generator system which satisfies the condition* $(*)$.

Proof. By $\sum_{c/a}$ we shall denote the subset $c/a \wedge (\sum_T \cup a)$[6] of c/a. Let $\{x_\alpha\}$ be a non-void subset of $\sum_{c/a}$. Since $a \leq x_\alpha \leq c$ for all α, we have $a \leq \bigcup_\alpha x_\alpha \leq c$, and $\bigcup_\alpha x_\alpha$ is contained in T. This implies $\bigcup_\alpha x_\alpha$ is contained in c/a. Now let g be an arbitrary element of c/a. Then, of course, g is an element of T. Hence g is expressible as a supremum of a subset N of $\sum_T$. Let N_a be the set $N \cup a$. Then $a \leq x \cup a \leq g \cup a \leq c$, and so $x \cup a$ is contained in c/a for every x of N, i.e., N_a is the subset of c/a. On the other hand, it is evident that N_a is a subset of $\sum_T \cup a$. Therefore, N_a is contained in $\sum_{c/a}$. Moreover, $g=g \cup a=(\sup N) \cup a=$ $\sup(N \cup a)=\sup N_a$. Next, we suppose that $x \leq \sup N$ for an element x of $\sum_{c/a}$, and a subset N of $\sum_{c/a}$. Since there exists an element x' and a subset N' of $\sum_T$ such that $x=x' \cup a$ and $N=N' \cup a$, we obtain $x' \leq x \leq$ $\sup N=(\sup N') \cup a$. Now let $a=\sup A$ be the sup-expression of a by a subset A of $\sum_T$. Then we have $x' \leq \sup N' \cup \sup A=\sup(N' \cup A)$[7]$=$ $\sup(N' \vee A)$. Therefore, we can find a finite number of elements $u_1, \cdots, u_m$ of N', and $v_1, \cdots, v_n$ of A which satisfy

$$x' \leq u_1 \cup \cdots \cup u_m \cup v_1 \cup \cdots \cup v_n\,.$$

6) $\sum_T \cup a$ is the set of the elements $x \cup a$ for all x of $\sum_T$.
7) $N' \cup A$ is the set of the elements $x \cup a$ for all x of N' and for all a of A.

It follows that $x' \leq (u_1 \cup \cdots \cup u_m) \cup a$. This implies $x = x' \cup a \leq (u_1 \cup a) \cup \cdots \cup (u_m \cup a)$. Consequently, $\sum_{c/a}$ forms an accessible join-generator system of c/a. Now, we prove that the condition $(*)$ holds for $\sum_{c/a}$ with respect to the multiplication $(\circ)$. Take any two elements g and h of $\sum_{c/a}$. Then, there exist two elements g' and h' of $\sum_T$ such that $g = g' \cup a$ and $h = h' \cup a$, from the fact that g and h are contained in $\sum_T \cup a$. We obtain, therefore, $g \circ h = gh \cup a = (g' \cup a)(h' \cup a) \cup a = g'h' \cup ah' \cup g'a \cup a^2 \cup a = g'h' \cup a$. Now, by the condition $(*)$ of $\sum_T$, there exists a finite number of elements $u_1, \cdots, u_n$ of $\sum_T$ such that $g'h' = u_1 \cup \cdots \cup u_n$. Hence, we have $g'h' \cup a = (u_1 \cup \cdots \cup u_n) \cup a = (u_1 \cup a) \cup \cdots \cup (u_n \cup a)$, and $a \leq u_i \cup a \leq c$, because $g \circ h = g'h' \cup a$ is contained in c/a. This completes the proof.

Let a be an element of T. A prime element p containing a is said to be *minimal prime* of a, if there exists no prime element p' such that $p > p' \geq a$.

4. Main Result

Proposition 1. *Let a and c be two elements of T such that $a \leq c$. If a is φ_c-closed, then a is, in c/a, decomposed into a meet of a finite or infinite number of minimal primes of a.*

Proof. In § 3 we saw that c/a forms a cl-semigroup, and by Lemma 4 it has an accessible join-generator system with the condition $(*)$. Since c/a is $(\circ)$-nilpotent-free by Lemma 3, we obtain $\mathrm{Rad}(c/a) = a$ by using Theorem 1 in [3]. This means that a is decomposed as a meet of all prime elements in c/a. Take now an arbitrary descending chain

$$p_1 \geq p_2 \geq \cdots \geq p_n \geq \cdots$$

of prime elements p_n of c/a. Then it is easy to see that $p = \bigcap_n p_n$ is a minimal prime of a in c/a, and that a is expressible as a meet of minimal primes. This proves our proposition.

Let c be an element of T, and let a be an element of L such that $a \leq c$. Then it is easy to see that $a \leq \varphi_c(a)$. Moreover if b is an element of L such that $a \leq b$, then we have $\varphi_c(a) \leq \varphi_c(b)$. Symmetrically, for $\varphi'_c(a)$, $a \in R$.

Proposition 2. *Let c be any fixed element of T. Then*

$$\varphi_c \varphi_c(a) = \varphi_c(a)$$

for every element a of L.

Proof. Since $\varphi_c(a) \leq c$ and $\varphi_c(a) \in T$, we obtain $\varphi_c(a) \leq \varphi_c \varphi_c(a)$. On the other hand, $l_c^*(a) \leq c$ and $l_c^*(a) \in T$. Hence $\varphi_c'(l_c^*(a)) \geq l_c^*(a)$, $r_c^*[\varphi_c'(l_c^*(a))] \leq r_c^*(l_c^*(a))$. That is $\varphi_c\varphi_c(a) \leq \varphi_c(a)$. Therefore we have $\varphi_c\varphi_c(a) = \varphi_c(a)$, as desired.

Proposition 3. *Let a and c be any two elements of T. Then*
(1) $\varphi_c(a) = \varphi_c'(a)$,
(2) a is φ_c-closed if and only if a is φ_c'-closed.

Proof. (1) is immediate by Lemmas 1 and 2. (2) follows directly from (1).

Theorem. *Let c be any fixed element of T, and let a be an arbitrary element of L. Then $\varphi_c(a)$ is, in $c/\varphi_c(a)$, decomposed into a finite or infinite number of minimal primes of $\varphi_c(a)$.*

Proof. This is immediate by Propositions 1 and 2.

Corollary 1. *Let c be any fixed element of T, and let a be an arbitrary element of L. If the ascending chain condition holds for the elements of $c/\varphi_c(a)$, then $\varphi_c(a)$ is uniquely decomposed (up to the ordering) into a finite number of prime elements of $c/\varphi_c(a)$.*

Corollary 2. *Let a be an element of T. If a is φ_a-closed, a is a prime element.*

Finally we shall suppose that T is modular as a lattice. Let a be an element of L, and let c be an element of T. Take now any lattice-quotient $V(d/b)$ of two elements b, d of T such that $V(d/b)$ is projective to the lattice-quotient $V(c/\varphi_c(a))$. It is then easily verified that the element of d/b which corresponds to a prime element of $c/\varphi_c(a)$ under the Dedekind's isomorphism is prime of d/b. Then, by Corollary 1, if the ascending chain condition holds for the elements of $c/\varphi_c(a)$, b is uniquely decomposed (up to the ordering) into a finite number of prime elements in d/b.

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