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ON NILPOTENT-FREE MULTIPLICATIVE SYSTEMS

Dedicated to Professor K. Shoda on his sixtieth birthday

By

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In his study of multiplicative systems, the author [8], [9] has defined the accessible join-generator system, and utilized it for decompositions of elements of multiplicative lattices (m-lattices) and of ideals of multiplicative systems. The concept[0] of accessible join-generator systems seems to be important to develop the algebraic theory of many sorts of lattices, ideal lattices and other multiplicative systems having no usual finite conditions.

The purpose of the present paper is to study the nilpotent-free multiplicative systems. The fundamental concept in our investigation is some restricted accessible join-generator systems of multiplicative systems.

In §1, we shall consider a complete m-lattice (not necessarily commutative nor associative), and obtain a condition to be nilpotent-free. The subject is largely based on the results of the rings having no nilpotent ideals, which are studied by McCoy [6], Levitzki [5] and Nagata [11]. §2 is concerned with some annihilators of elements in a commutative but not necessarily associative nilpotent-free m-lattice. The results of this section are useful in the next one. §3 treats a group having no solvable normal subgroup. Some properties of such a group which have already studied by the author [10] will be used under a suitable restriction. In §§4 and 5, we shall show the analogous results of §§1 and 2 in the case of an associative but not necessarily commutative multiplicative system. The results in these two sections are applicable to the family of ideals of general ring-systems, and which will be shown in the last section.

Throughout this paper, the symbols $\vee$ and $\wedge$ will denote respectively the set-theoretic union and the intersection. By $\{a\,;\,a$ has property $P\}$ we mean the set of all elements a having property P.

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0) From purely lattice-theoretical stand point of view, R. P. Dilworth and Peter Crawley have recently emphasized the importance of such concept. Cf. [3] and [4].

§ 1. Nilpotent-free m-Lattices

Let K be a complete (upper and lower) m-lattice[1] with the greatest element e and the least element 0, and suppose that $ab \leq a$ and $ab \leq b$ for any two elements a and b of K. It is then clear that 0 is the zero element. We do not assume the greatest element to be multiplicative unity, and the multiplication to be associative or commutative.

An element p of K is said to be prime if whenever $ab \leq p$ for two elements a, b of K, then $a \leq p$ or $b \leq p$. If K has the join-generator system[2] Σ, then an element p of K is prime if (and only if) $xy \leq p$ implies $x \leq p$ or $y \leq p$ for any two elements x, y of Σ. For, let $ab \leq p$ and $b \nleq p$ for two elements a, b of K, and let $a = \sup[X]$, $b = \sup[Y]$ be the sup-expressions of a, b by subsets X, Y of Σ. Then there exists an element y of Y such that $y \nleq p$. Since $p \geq ay \geq xy$, $p \geq x$ for every x of X, we obtain $p \leq a$.

Throughout this and the next sections, we assume that K *has an accessible join-generator system* Σ *with the condition*:[3] (*) *the product* xy *of any two elements* x, y *of* Σ *is expressible as a join of a finite number of elements of* Σ.

By the symbol Σ^* we shall mean the closed system of Σ under the join-operation. Then, by the condition (*), it follows that Σ^* is closed under multiplication. Now it is easy to see that an element p is prime if (and only if) $mm' \leq p$ implies $m \leq p$ or $m' \leq p$ for m, m' of Σ^*.

DEFINITION 1. A subset M of Σ^* is called a *μ-system* if it is closed under multiplication. The void set is to be considered as a μ-system.

Let p be a prime element of K. Then the set $M(p)$ of the elements m of Σ^* such that $m \nleq p$ forms a μ-system.

Lemma 1. *Let M be a μ-system such that $M \wedge J(a) = \emptyset$[4] for an element a of K. Then there exists[5] an element p such that* (1) $a \leq p$, (2) $M \wedge J(p) = \emptyset$, (3) *$p < c$ implies $M \wedge J(c) \neq \emptyset$ and* (4) *p is prime.*

Proof. Let $a \leq \cdots \leq a_\nu \leq \cdots$ be any ascending chain such that $M \wedge J(a_\nu) = \emptyset$ for every ν, and let $a^* = \sup[X]$, where $X = \{x ; x \in \Sigma, x \leq a_\nu$

1) This means that K is a lower-complete cm-lattice. It is easy to see that K forms an integral and residuated lattice. Cf. [2; pp. 200–201].

2) A subset Σ of K is called a join-generator system, if every element of K is expressible as the supremum of a subset of Σ. Cf. [8; p. 105]. Evidently K is one of its own join-generator system.

3) This condition is strictly weaker than the ascending chain condition for the elements of K.

4) $\emptyset$ denotes the void set. $J(a)$ denotes the principal lattice-ideal generated by a.

5) See the footnote 16) of this paper.

for some $\nu\}$. Then it is easily verified that $\sup_{\nu}[a_\nu]=a^*$. We now prove that $M\wedge J(a^*)$ is void. If $M\wedge J(a^*)$ is not void, we can take an element m of M such that $m\leq a^*$. By the condition $(*)$, there exists a finite number of elements x_i of Σ such that $m=x_1\cup\cdots\cup x_t$. Hence we can find $u_{i1},\cdots,u_{i,s(i)}$ of X such that $x_i\leq\bigvee_{j=1}^{s(i)}u_{ij}$. Then $m\leq\bigvee_{i=1}^{t}\bigvee_{j=1}^{s(i)}x_{ij}$ $\leq a_\rho$ for a sufficiently large ordinal number ρ. Hence $M\wedge J(a_\rho)$ is not void. This is a contradiction. Zorn's Lemma assures therefore the existence of an element p satisfying the conditions (1), (2) and (3). It is only necessary to prove the conditions (4). Suppose that $a\not\leq p$ and $b\not\leq p$. Then, by the maximality of p, there exist two elements m and m' such that $m\leq p\cup a$ and $m'\leq p\cup b$. This implies that $mm'\leq(p\cup a)(p\cup b)\leq p\cup ab$. Now if we suppose that $ab\leq p$, then $mm'\leq p$. This is a contradiction. We have therefore $ab\not\leq p$.

DEFINITION 2. A non-zero element a of K is said to be *nilpotent* if $a^{(\rho)}=0$[6] for a suitable whole number ρ. A subset of K is said to be *nilpotent-free* if it has no nilpotent element.

Lemma 2. *Let x be any element of Σ. Then the set $M_x=\{x, x^2, x(x^2), (x^2)x, x(x(x^2)), x((x^2)x), \cdots\}$ of all powers of x is a μ-system. An element x is nilpotent if (and only if) M_x contains the zero.*

Proof. The first part is easy to see. Suppose now that M_x contains 0. Then by using the properties (2), (5) and (6) in [7], we obtain $x^{(\rho)}=0$ for a sufficiently large whole number ρ.

DEFINITION 3. The infimum of the prime elements containing a is called a *radical* of a, and denoted by $Rad(a)$. In particular, the radical of 0 is the infimum of all prime elements of K, and denoted by $Rad(K)$.

Lemma 3. *Let $X(a)$ be the set of the elements x of Σ such that every μ-system containing x contains an element of $J(a)$. Then $X(a)$ contains every element x of Σ such that $x\leq Rad(a)$.*

Proof. Let x be any element of Σ such that $x\leq \mathrm{Rad}(a)$, and let M be any μ-system containing x. If $M\wedge J(a)$ is void, then, by Lemma 1, there exists prime element p such that $a\leq p$ and $M\wedge J(p)=\varnothing$. Since $x\leq p$, x is not contained in M, which is a contradiction. Hence $M\wedge J(a)$ is not void. Therefore x is contained in $X(a)$.

Theorem 1. *The following three conditions are equivalent*:

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6) The power $a^{(\rho)}$ of a is similarly defined as in the case of commutative residuated lattices. Cf. [7; p. 31]. Then the properties (2), (5) and (6) in [7] hold for the elements of K.

(1) *Rad* $(K)=0$,
(2) K *is nilpotent-free,*
(3) Σ *is nilpotent-free.*

Proof. (1)$\Rightarrow$(2): Let a be a nilpotent element of K. Then $a\leq p$ for every prime element p of K. This implies $a\leq \mathrm{Rad}(K)$. Therefore $\mathrm{Rad}(K)\neq 0$. (2)$\Rightarrow$(3) is evident. (3)$\Rightarrow$(1): Suppose now that $\mathrm{Rad}(K)\neq 0$. Then we can take an element x of Σ such that $x\leq \mathrm{Rad}(K)$ and $x\neq 0$. By Lemma 3, x is contained in $X(0)$. This implies that M_x contains 0. Hence by Lemma 2 x is nilpotent. This completes the proof.

REMARK 1. A prime element p is said to be a minimal prime of a if $a\leq p$ and there exists no prime element p' such as $a\leq p'<p$. Then in order that an element p is a minimal prime of an element a, it is necessary and sufficieht that $M(p)$ is maximal in the inclusion ordered family $\mathfrak{M}$ of μ-systems M with $M\wedge J(a)=\emptyset$. For, let $M(p)$ be maximal in $\mathfrak{M}$; and take a maximal element p^* such that $p\leq p^*$ and $M(p)\wedge J(p^*)=\emptyset$. Since $M(p)\subseteq M(p^*)$ and $M(p^*)\wedge J(a)\subseteq M(p^*)\wedge J(p^*)=\emptyset$, we have $M(p)=M(p^*)$ by the maximality of $M(p)$ in $\mathfrak{M}$. Then it is easily verified that $p=p^*$. Hence p is a prime element which contains a. Next we suppose that there exists a prime element p' such that $a\leq p'<p$. Then $M(p')\wedge J(a)$ is void and $M(p)$ is contained in $M(p')$ strictly. This contradicts the maximality of $M(p)$ in $\mathfrak{M}$. Conversely, let p be a minimal prime of a. Then it is easy to see that $M(p)\wedge J(a)$ is void. Take now a maximal μ-system M^* such that $M(p)\subseteq M^*$ and $M^*\wedge J(a)=\emptyset$. (The existence of M^* is assured by Zorn's Lemma.) Again take a maximal element p^* satisfying $a\leq p^*$ and $M^*\wedge J(p^*)=\emptyset$. Since $M(p^*)\supseteq M^*$ and $M(p^*)\wedge J(p)=\emptyset$, $M(p^*)$ would coincide witn M^*. Hence by the first part of this proof, p^* is a minimal prime of a. Since $p\leq p^*$, consequently $p=p^*$, and therefore $M(p)=M^*$, say, $M(p)$ is maxtimal in $\mathfrak{M}$.

It is then proved, as in the case of rings [6], that $\mathrm{Rad}(a)$ is the infimum of the minimal primes of a, and in particular $\mathrm{Rad}(K)$ is the infimum of all minimal primes of K.

REMARK 2. $\mathrm{Rad}(a)=\sup[X(a)]$, where $X(a)$ is similar as in Lemma 3. For, by Lemma 3, we have $\mathrm{Rad}(a)\leq\sup[X(a)]$. Conversely, let x be an element of $X(a)$ and let p be a prime element such that $p\geq a$. If $x\not\leq p$, x is contained in $M(p)$. Hence $M(p)\wedge J(a)$ is not void. On the other hand, $a\leq p$ implies $M(p)\wedge J(a)\subseteq M(p)\wedge J(p)=\emptyset$. This is a contradiction. Therefore $x\leq p$, consequently $\sup[X(a)]\leq\mathrm{Rad}(a)$.

REMARK 3. As is well known, any relatively complemented pseudolattice is distributive. Hence it forms an m-lattice when the multiplication is defined as its meet-operation. Now let T be any relatively (dually)[7] complemented pseudo-lattice with the greatest element e and the least element 0, and suppose that T is the accessible join-generator system. Then the condition (*) is satisfied trivially. Hence by Theorem 1 we obtain $\mathrm{Rad}(T)=0$. Let p_λ be the l-prime elements of T. It is then easily verified that the mapping $a\rightarrow(\cdots, a\cup p_\lambda, \cdots)$ from T into the direct union of sub-lattices $[e, p_\lambda]$ gives a lattice-isomorphism. A lattice with the least element 0 is said to be prime if 0 is prime. Since $[e, p_\lambda]$ is prime, we obtain that T is isomorphic to a subdirect union of a finite or infinite number of prime lattices. In particular, so is a topological lattice with the greatest element and the least element which satisfies the condition P_2 in [8, § 9]. Moreover, by Theorem 49 in [14], any Boolean algebra with the condition P_2 in [8, § 9] is isomorphic to the direct union of two-element Boolean algebras; and which is a special case of the well known Ston's theorem.

§ 2. Commutative Nilpotent-free m-Lattices

In this section we shall suppose that K is commutative. The associative law is not assumed.

Lemma 4. *Let $P(a_1, \cdots, a_n)$ be any product of a finite number of elements $a_1, \cdots, a_n$ of K. Then $P(a_1, \cdots, a_n)=0$ if and only if $a_1\cap\cdots\cap a_n=0$. In particular $ab=0$ if and only if $a\cap b=0$.*

Proof. We obtain that $(a_1\cap\cdots\cap a_n)^{(\rho)}\leq P(a_1, \cdots, a_n)$ for a sufficiently large whole number ρ[8]. Hence $P(a_1, \cdots, a_n)=0$ implies $(a_1\cap\cdots\cap a_n)^{(\rho)}=0$, and implies $a_1\cap\cdots\cap a_n=0$. Since $P(a_1, \cdots, a_n)\leq a_1\cap\cdots\cap a_n$, the converse is immediate, q.e.d.

By $N(a)$ we mean the set of the elements x of Σ which satisfies $ax=0$, and by $Q(a)$ the set of the elements t of K which satisfies $at=0$. Then by the accessibility of Σ we can easily verify that $N(a)$ coincides with the set of the elements y of Σ such that $y\leq t$ for some t of $Q(a)$. Hence we have that $\sup[N(a)]=\sup[Q(a)]$. In the following $\sup[N(a)]$ will be denoted by a^*, and a^{**} by $\varphi(a)$. Then we have

1) $a\leq\varphi(a)$,
2) $\varphi\varphi(a)=\varphi(a)$,
3) $a\leq b$ implies $\varphi(a)\leq\varphi(b)$.

Definition 4. An element a is said to be *φ-closed* if $\varphi(a)=a$.

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7) Cf. [2; p. 147].
8) This is immediate by induction on the whole number n.

Theorem 2. *Any φ-closed element of K is decomposed as a meet of a finite or infinite number of prime elements; and so is $\varphi(a)$ for every element a of K.*

Proof. Let a be a φ-closed element. Suppose that it is not decomposed as a meet of a finite or of an infinite number of prime elements. Then applying Theorem 1 to the interval $[e, a]$, we can find an element t such that $t > a$ and $t^2 \leq a$. Now since $(t \cap a^*)^2 \leq t^2 \cap a^* \leq a \cap a^* = 0$ by Lemma 4, we have that $t \cap a^* = 0$. Hence $ta^* = 0$, hence $t \leq a^{**} = \varphi(a)$, and hence $a \neq \varphi(a)$. This is a contradiction.

Lemma 5. *Suppose that $a \leq b$. If $b \leq \varphi(a)$, then $at = 0$ for every element t with $bt = 0$, and vice versa.*

Proof. This is immediate.

Theorem 3. *$\varphi(ab) = \varphi(a) \cap \varphi(b)$ for any two elements a and b of K.*

Proof. Take an element t such that $s \equiv [\varphi(a) \cap \varphi(b)] \cdot t \neq 0$. Then since $\varphi(a)t \neq 0$, we have that $at \neq 0$ by Lemma 5. If $(ab)t \neq 0$, then of course $\varphi(ab)t \neq 0$. Since $\varphi(ab) \leq \varphi(a) \cap \varphi(b)$, we have $\varphi(a) \cap \varphi(b) \leq \varphi\varphi(ab) = \varphi(ab)$ by Lemma 5. Hence $\varphi(a) \cap \varphi(b) = \varphi(ab)$. Therefore it suffices to prove that $(ab)t \neq 0$. Now we assume that $(ab)t = 0$. Then since $ab \cap t = 0$ by Lemma 4, we have $(a \cap b \cap s)^2 \leq ab \cap s \leq ab \cap t = 0$. Hence $a \cap b \cap s = 0$, hence $a(bs) = 0$, and hence $bs \leq a^*$. On the other hand, since $bs \leq s \leq \varphi(a)$, we obtain $(bs)^2 \leq a^* \varphi(a) = 0$. This implies $bs = 0$, and implies $\varphi(b)s = 0$ by Lemma 5. Hence $\varphi(b) \cap s = 0$ by Lemma 4. Therefore we have that $s = [\varphi(a) \cap \varphi(b)]t = \varphi(b) \cap [\varphi(a) \cap \varphi(b)]t = \varphi(b) \cap s = 0$, a contradiction. This completes the proof.

Corollary. *Let $P(a_1, \cdots, a_n)$ be any product of a finite number of elements $a_1, \cdots, a_n$ of K. Then $\varphi(P(a_1, \cdots, a_n)) = \varphi(a_1) \cap \cdots \cap \varphi(a_n)$. In particular, any product of a finite number of φ-closed elements $a_1, \cdots, a_n$ is equal to $a_1 \cap \cdots \cap a_n$.*

Lemma 6. *$\varphi(a \cup a^*) = e$ for any element a of K.*

Proof. Since $(a \cup a^*)^* \leq a^* \cap \varphi(a) = 0$, we obtain $(a \cup a^*)^* = 0$. Hence $\varphi(a \cup a^*) = e$.

Theorem 4. *The set K_φ of all φ-closed elements of K forms a Boolean algebra under the join $\varphi(a \cup b)$ and the meet $\varphi(a) \cap \varphi(b) = \varphi(ab)$.*

Proof. By Theorem 3 and Lemma 6, K_φ forms a complemented lattice under the join $\varphi(a \cup b)$ and the meet $\varphi(a) \cap \varphi(b) = \varphi(ab)$. Then by Theorems 5 in [2; Chap. XIII] and 3, K_φ is relatively pseudo-complemented. Now it is easy to see that K_φ is pseudo-complemented. Hence by using Glivenko's theorem (see [14]), we complete the proof.

§ 3. Strongly Non-solvable Groups

Let G be a group, and let $\mathfrak{N}$ be the set of all normal subgroups of G. It is then easily verified that $\mathfrak{N}$ forms a lower-complete commutative cm-lattice under the set-inclusion relation and the commutator-multiplication[9]. Evidently G is the greatest element of $\mathfrak{N}$, the unit subgroup is the least element of $\mathfrak{N}$ and the commutator group $[N_1, N_2]$ of two normal subgroups N_1 and N_2 is of course contained in $N_1 \wedge N_2$.

Definition 5. A group is said to be *strongly non-solvable* if and only if it has no solvable normal subgroup except the unit subgroup.

A normal subgroup P of a group G is said to be *prime* if the commutator group $[N_1, N_2]$ of two normal subgroups N_1 and N_2 of G is in P, then at least one of the N_i is in P. It is then easily proved that in order that a normal subgroup P of G is prime, it is necessary and sufficient that G/P has the unique minimal non-abelian normal subgroup[10]. Let N be any non-unit normal subgroup of a prime group G. Then the centralizer $C(N)$ of N is evidently the unit subgroup of G. Hence the centralizer $C^2(N)$ of $C(N)$ is of course the whole group G.

We now consider a family $\mathfrak{F}$ of normal subgroups of G with the following three conditions:

(1°) If a normal subgroup N in $\mathfrak{F}$ is contained in a subgroup generated by $\{N_\alpha\} \subseteq \mathfrak{F}$, N is contained in a subgroup generated by a finite number of normal subgroups in $\{N_\alpha\}$.

(2°) Any normal subgroup of G is generated by a finite or infinite number of normal subgroups in $\mathfrak{F}$.

(3°) The commutator group of any two normal subgroups in $\mathfrak{F}$ is generated by a finite number of normal subgroups in $\mathfrak{F}$.

If the ascending chain condition holds for the normal subgroups of G, the whole m-lattice $\mathfrak{N}$ satisfies these three conditions. Throughout this section, we suppose that there exists the family $\mathfrak{F}$ of normal subgroups of G which satisfies the three conditions (1°), (2°) and (3°).

Theorem 5. *The following conditions are equivalent*:

1) *G is strongly non-solvable.*

2) *The intersection of all prime normal subgroups of G is the unit subgroup.*

9) Cf. [8; p. 104].

10) Cf. [13; p. 377].

3) $\mathfrak{F}$ *has no solvable normal subgroup except the unit subgroup.*

Theorem 6. *The centralizer of any normal subgroup of G is represented as an intersection of a finite or infinite number of prime normal subgroups of G, and so is the normal subgroup N with* $C^2(N)=N$.

Theorem 7. *For any two normal subgroups* N_1 *and* N_2 *of G,*

$$[C^2(N_1), C^2(N_2)] = C^2(N_1) \wedge C^2(N_2).$$

In particular, if $C^2(N_i)=N_i$ $(i=1, 2)$, *then* $[N_1, N_2]=N_1 \wedge N_2$.

Theorem 8. *The set of* $C^2(N)$ *of all normal subgroups N of G forms a Boolean algebra under the join* $C^2(N_1 \cup N_2)$ *and the meet* $C^2([N_1, N_2])$.

Above four theorems are immediate by Theorems 1, 2, 3 and 4, respectively.

Theorem 9. *Let* $\mathfrak{G}$ *be a strongly non-solvable group such that the commutator group of any two normal subgroups with single generators is generated by the set-union of a finite number of normal subgroups with single generators*[11]. *Then the centralizers of all normal subgroups of G form a Boolean algebra under* $C(N_1) \cup C(N_2)$ *and the intersection, which is isomorphic to the lattice of the subsets of a direct union of a finite or infinite number of two-element Boolean algebras.*

Proof. Let $\mathfrak{F}'$ be the family of all normal subgroups with single generators. Then it is easily proved that the conditions (1°) and (2°) hold for $\mathfrak{F}'$, and by the assumption of Theorem the condition (3°) holds for $\mathfrak{F}'$. Hence by Theorem 1, the intersection of all prime normal subgroups of $\mathfrak{G}$ is the unit subgroup. Then it is proved that $\mathfrak{G}$ is isomorphic to a subdirect product[12] of a finite or infinite number of prime groups G_λ: $\mathfrak{G}=\overset{s}{\Pi}{}_{\lambda}^{\otimes} G_\lambda$. Now let N be any normal subgroup of $\mathfrak{G}$. Then $C(N)=C(\bar{N})$, where $\bar{N}$ denotes the "Hülle" of N. For, let x be an arbitrary element of $C(N)$, and y an arbitrary element of $\bar{N}$. Then $y=z_1 \cdots z_\kappa$, where $z_i \in G_i$. Since there exists an element z of N such that $z=z'_1 \cdots z'_{i-1} z_i z'_{i+1} \cdots z'_\kappa$ $(z'_i \in G_i)$, we obtain $z=x^{-1}zx=(x^{-1}z'_1x) \cdots (x^{-1}z'_{i-1}x)(x^{-1}z_ix)(x^{-1}z'_{i+1}x) \cdots (x^{-1}z'_\kappa x)$. This implies $x^{-1}z_ix=z_i$ $(i=1, \cdots, \kappa)$, and implies $x^{-1}yx=y$.

11) This assumption is of course strictly weaker than the ascending chain condition for normal subgroups of $\mathfrak{G}$.

12) A group G is called here a subdirect product of group G_λ, if (1) G is a subgroup of the (restricted) direct product $\Pi_{\lambda}^{\otimes} G_\lambda$ and (2) the λ-component of G is G_λ or the unit group of G_λ for every index λ. Cf. [10]. The theorem in [10] and its proof are incomplete. Combining Theorem 1 and the results of § 2 in [10], we complete the proof of the theorem under the restriction in Theorem 9.

Therefore $C(N)$ is contained in $C(\bar{N})$. It is evident that $C(\bar{N})$ is contained in $C(N)$. Next let $\mathfrak{C}(G_i)$ be the set of all centralizers of normal subgroups of G_i. Then $\mathfrak{C}(G_i)$ is a two-element Boolean algebra, say, $\mathfrak{C}(G_i)=\{G_i, e\}$. For, if N_i is any non-unit subgroup of G_i, then by the primeness of G_i, we have $C(N_i)=e$. Therefore we obtain that the set of the centralizers of the normal subgroups of $\mathfrak{G}$ coincides with the set of all subdirect products of $\Pi_\lambda^{\otimes s} G_\lambda$. Now let N_1 and N_2 be any two normal subgroups of $\mathfrak{G}$. Then, by the above argument, $C(N_i)$ are represented as

$$C(N_1) = \Pi^{\otimes}_{\alpha\in\Lambda} G_\alpha \times \Pi^{\otimes}_{\beta\in\Delta} G_\beta ,$$
$$C(N_2) = \Pi^{\otimes}_{\alpha\in\Lambda} G_\alpha \times \Pi^{\otimes}_{\gamma\in\Gamma} G_\gamma ,$$

where $\Delta\wedge\Gamma=\varnothing$. It is then easy to see that

$$\begin{aligned} C^2(C(N_1)\cup C(N_2)) &= \Pi^{\otimes}_{\alpha\in\Lambda} G_\alpha \times \Pi^{\otimes}_{\beta\in\Delta} G_\beta \times \Pi^{\otimes}_{\gamma\in\Gamma} G_\gamma \\ &= C(N_1)\cup C(N_2) . \end{aligned}$$

Hence by Theorem 8 we complete the proof.

§ 4. Nilpotent-free Lattice-ordered Sytems

In this and the next sections, S is a partly ordered set with the greatest element e and the least element 0. Let further S has the multiplication, binary operation denoted by $(\cdot)$. We now suppose that

P_1: e and a are composable with respect to the multiplication, and $ea\geq a$ and $ae\geq a$ for every a of S.

P_2: 0 and a are composable with repect to the multiplication, and $0a=a0=0$ for every a of S.

P_3: S is the set-union of $L=\{a\,;\ ea=a,\ a\in S\}$ and $R=\{a\,;\ ae=a,\ a\in S\}$.

P_4: L forms a complete (upper and lower) associative multiplicative lattice ((lower complete) cl-semigroup[13]) under $(\cdot)$ and $(\leq)$.

P_5: R forms a lower complete cl-semigroup under $(\cdot)$ and $(\leq)$.

It is then easy to see that (1) e is an idempotent: $e^2=e$, (2) $ab\leq b$ for every two elements a, b of L and (3) $ab\leq a$ for every two elements a, b of R. By T we shall mean the intersection of L and R. Then T is a lower complete cl-subsemigroup of both L and R. Hence the results in § 1 are, of course, applicable to T, if it has an accessible join-generator system.

Lemma 7. *Let Σ_L be any accessible join-generator system of L such that Σ_L contains $\Sigma_L\cdot e$, the set of the elements xe $(x\in\Sigma_L)$. Then $\Sigma_L\cdot e$*

13) Cf. [2; p. 200].

forms an accessible join-generator system of T. If Σ_L satisfies the condition $(*)$ of §1, then so is $\Sigma_L\cdot e$.

Proof. Let $c=\sup[X]$ be the sup-expression of any element c of T by the subset X of Σ_L. Then $c=ce=\sup[Xe]$ is a sup-expression of c by the subset Xe of $\Sigma_L\cdot e$. The accessibility of $\Sigma_L\cdot e$ and the last part of Lemma are immediate.

Lemma 8. *Let Σ_L and Σ_R be two accessible join-generator systems of L and R respectively, and suppose that*

$$[*] \qquad \Sigma_L \supseteq \Sigma_L\cdot e\,,\quad \Sigma_L\cdot e \subseteq \Sigma_R\,,\quad \Sigma_R \supseteq e\cdot\Sigma_R \quad \textit{and} \quad e\cdot\Sigma_R \subseteq \Sigma_L\,.$$

Then $\Sigma_L \wedge \Sigma_R = \Sigma_L\cdot e = \Sigma_L \wedge R = \Sigma_R \wedge L = e\cdot\Sigma_R$.

Proof. This is immediate.

Throughout this and the next sections we suppose that there exist two accessible join-generator systems Σ_L and Σ_R of L and R respectively, which satisfy the conditions $(*)$ in §1 and $[*]$ in Lemma 8. We put $\Sigma_T \equiv \Sigma_L \wedge \Sigma_R$ and $\Sigma \equiv \Sigma_L \vee \Sigma_R$.

Lemma 9[14]. *The following conditions are equivalent to one another.*

(1) *If a, b are elements of T such that $ab\leq p$, then $a\leq p$ or $b\leq p$.*
(2) *If z, z' are elements of Σ_T such that $zz'\leq p$, then $z\leq p$ or $z'\leq p$.*
(3) *If x, y are elements of Σ such that $exeye\leq p$, then $x\leq p$ or $y\leq p$.*
(4) *If x, y are elements of Σ such that $xey\leq p$, then $x\leq p$ or $y\leq p$.*
(5) *If l, l' are elements of L such that $ll'\leq p$, then $l\leq p$ or $l'\leq p$.*
(6) *If r, r' are elements of R such that $rr'\leq p$, then $r\leq p$ or $r'\leq p$.*
(7) *If l is an element of L and r of R such that $ler\leq p$, then $l\leq p$ or $r\leq p$.*
(8) *If l is an element of L an r of R such that $rl\leq p$, then $l\leq p$ or $r\leq p$.*
(9) *If u, u' are elements of Σ_L such that $uu'\leq p$, then $u\leq p$ or $u'\leq p$.*
(10) *If v, v' are elements of Σ_R such that $vv'\leq p$, then $v\leq p$ or $v'\leq p$.*
(11) *If u is an element of Σ_L and v of Σ_R such that $uev\leq p$, then $u\leq p$ or $v\leq p$.*
(12) *If u is an element of Σ_L and v of Σ_R such that $vu\leq p$, then $u\leq p$ or $v\leq p$.*

Proof. This is immediate by the following implications: (1)⇒(2)⇒(3)⇒(4)⇒(5)⇒(9)⇒(1), (4)⇒(6)⇒(10)⇒(1), (4)⇒(8)⇒(12)⇒(9) and (4)⇒(7)⇒(11)⇒(1).

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14) Cf. [1; p. 39], [6; p. 825].

DEFINITION 6. An element p of T which has any one (and therefore all) of the properties stated in Lemma 9 is said to be a *prime* element of T or of S.

DEFINITION 7. Let c be an element of T. The infimum of the primes containing c is said to be a *radical* of c. In particular, the radical of 0 is said to be a radical of S. The radical of c is denoted by $Rad(c)$, and the radical of S by $Rad(S)$.

Lemma 10. *$Rad(c)$ is contained in T for every element c of T. In particular $Rad(S)$ is contained in T.*

Proof. Since T is lower-complete, this is immediate.

A non-zero element a of S is said to be nilpotent if $a^n=0$ for a whole number n. A subset of S is said to be nilpotent-free, if it has no nilpotent element.

Theorem 10. *The following conditions are equivalent to one another*:

(1) $Rad(S)=0$.
(2) Σ_T *is nilpotent-free.*
(3) T *is nilpotent-free.*
(4) Σ_L *is nilpotent-free.*
(5) L *is nilpotent-free.*
(6) Σ_R *is nilpotent-free.*
(7) R *is nilpotent-free.*

Proof. Let c be an element of T, and let $U(c)$ be the set of the elements of Σ_T such that every μ-system, defined by Σ_T, containing x contains an element of the lattice-ideal of T, which is generated by c. Then, by using Lemma 10, we can prove similarly to the proof of Lemma 3 that $\mathrm{Rad}(c)$ is contained in $\sup[U(c)]$. Hence the proof of equivalency of the conditions (1), (2) and (3) is obtained similarly to that of Theorem 1. Let a be an element of L such that $a\neq 0$ and $a^\rho=0$ for a whole number ρ. Then we can take an element x such that $x\leq a$, $x\neq 0$ and $x\in\Sigma_L$. Since $(xe)^\rho=0$, $xa\neq 0$ and $xe\in\Sigma_T$, we obtain that $(2)\Rightarrow(5)$. $(5)\Rightarrow(2)$ is evident. Hence of course (4) is equivalent to (2). Similarly for (6) and (7).

§ 5. φ-Closed Elements

In this section we suppose that $\mathrm{Rad}(S)=0$. Σ_L, Σ_R, Σ_T, etc. are similar as in § 4. We are now going to define φ-closed elements of S, and consider a meet decomposition of φ-closed elements.

Let Σ' be a subset of Σ_L, and a an element of L. We denote by

$l(a;\Sigma')$ the elements x of Σ' such that $xa=0$. $r(a;\Sigma')$ is defined symmetrically for an element a of R and a subset Σ' of Σ_R.

Lemma 11. *Let a be an element of L, and let F be the elements of L such as $fa=0$. Then $l(a;\Sigma_L)$ coincides with the set of the elements x of Σ_L such that $x\leq f$ for some f of F.*

Proof. This is immediate.

Lemma 12. *Let a be any element of L. Then $\sup[l(a;\Sigma_L)]$ is equal to $\sup[l(a;\Sigma_T)]$, and contained in T.*

Proof. Take an element x of Σ_L such that $x\leq\sup[l(a;\Sigma_L)]$. Then there exists a finite number of elements $u_1,\cdots,u_n$ of $l(a;\Sigma_L)$ such that $x\leq u_1\cup\cdots\cup u_n$. Hence $(xe)a=x(ea)=xa\leq\bigvee_{i=1}^{n}u_ia=0$. Since $xe\in\Sigma_T$ and $x\leq xe$, we obtain $\sup[l(a;\Sigma_L)]\leq\sup[l(a;\Sigma_T)]$. The converse inclusion is evident, q.e.d.

$\operatorname{Sup}[l(a;\Sigma_L)]$ is denoted by $l^*(a)$, and $\sup[r(a;\Sigma_R)]$ is denoted by $r^*(a)$.

Lemma 13. *If c is an element of T, then $l^*(c)=r^*(c)$ and $cc^*=c^*c=0$, where $c^*=l^*(c)(=r^*(c))$.*

Proof. Take an element x of Σ_L such that $x\leq l^*(c)$. Since there exist a finite number of elements $x_1,\cdots,x_r$ of $l(c;\Sigma_L)$ such as $x\leq x_1\cup\cdots\cup x_r$, we have $xc\leq\bigvee_{i=1}^{r}x_ic=0$, $xc=0$. Hence $l^*(c)\cdot c=\sup[l(c;\Sigma_L)\cdot c]=0$, and hence $(c\cdot l^*(c))^2=c(l^*(c)\cdot c)l^*(c)=0$. Since $c\cdot l^*(c)$ is contained in L and L is nilpotent-free, we obtain $c\cdot l^*(c)=0$. Symmetrically $r^*(c)\cdot c=0$. Next, we let x be an element of Σ_T such as $x\leq l^*(c)$. Then, by the above argument, we have $cx=0$. Hence x is contained in $r(c;\Sigma_T)$. We obtain therefore $l^*(c)=\sup[x;x\in\Sigma_T,\ x\leq l^*(c)]\leq\sup[r(c;\Sigma_T)]=r^*(c)$. Symmetrically $r^*(c)\leq l^*(c)$, and $l^*(c)=r^*(c)$.

Lemma 14. *Let c be an element of T, and a an element of L. Then the following conditions are equivalent:*

(1) $ca=0$.

(2) $c\cap a=0$.

Proof. This is immediate by $(c\cap a)^2\leq ca\leq c\cap a$.

Lemma 15. *If a is an element of L, then $l^*(a)\cap a=0$.*

Proof. Since $l^*(a)\cdot a=0$ and $l^*(a)\in T$, this is immediate by Lemma 11.

Lemma 16. *Let c be any fixed element of T. If $c\cap g=0$ for an element g of T, then $g\leq c^*$.*

Proof. Take an element x of Σ_T such that $x \leq g$. Then $x \cap c = 0$. Hence $xc = 0$, hence x is contained in $l(c\,;\Sigma_T)$ and hence $x \leq c^*$. We have therefore $g \leq c^*$.

DEFINITION 8. The mapping $a \to \varphi(a) = r^*(l^*(a)) = (l^*(a))^*$ from L into T is called *φ-mapping* of L. An element a of L is said to be *φ-closed* if $\varphi(a) = a$. φ'-mapping of R is defined symmetrically.

It is easy to see that the φ-mapping of L has the following properties:

1) $a \leq \varphi(a)$,
2) $\varphi\varphi(a) = \varphi(a)$,
3) $a \leq b$ implies $\varphi(a) \leq \varphi(b)$.

Symmetrically for φ'-mapping of R.

Lemma 17. $\varphi(c) = \varphi'(c) = c^{**}$ for every element c of T. c is φ-closed if and only if it is φ'-closed.

Proof. This is immediate by Lemma 13.

Theorem 11. *If c is a φ-closed element of T, then c is decomposed as a meet of a finite or infinite number of prime elements of T; and so is $\varphi(a)$ for every element a of L.*

Proof. Let $[e, c]$ be the interval $\{t\,; e \geq t \geq c,\ t \in S\}$. If we suppose that c is not represented as a meet of prime elements of T, then, applying Theorem 10 to the l-semigroup $[e, c]$, we can find an element g of $[e, c] \wedge T$ such that $g > c$ and $c \geq g^2$. Since $(g \cap c^*)^2 \leq c \cap c^* = 0$, we have $g \cap c^* = 0$. Hence $g \leq c^{**} = \varphi(c)$ by Lemma 16. This implies $c \neq \varphi(c)$, which is a contradiction.

Lemma 18. If $c \cap a = 0$ for elements c of T and a of L, then $\varphi(c) \cap a = c \cap \varphi(a) = 0$.

Proof. Since $ca = 0$ by Lemma 14, we have $a \leq r^*(c) = c^*$. Hence $a \cap \varphi(c) \leq c^* \cap \varphi(c) = 0$, $a \cap \varphi(c) = 0$. Next we have $c \leq l^*(a)$, since $ca = 0$. Hence $c \cap \varphi(a) \leq l^*(a) \cap \varphi(a) = l^*(a) \cap (l^*(a))^* = 0$ by Lemma 15. We have therefore $c \cap \varphi(a) = 0$.

Lemma 19. *Let a be an element of L. If $c \in [e, a] \wedge T$, then $ga = 0$ for every element g with $gc = 0$, $g \in T$. Conversely, if* (1) *$a \leq c$ $(c \in T)$ and* (2) *$gc = 0$ for every element g with $ga = 0$, $g \in T$, then $c \leq \varphi(a)$.*

Proof. The first part is easy to see. We now suppose that $l^*(a) \cdot c \neq 0$. Then by Lemma 14 we have $g = c \cap l^*(a) \neq 0$, Now it is evident that g is contained in T and $c \cap g = g \neq 0$. Hence we have that $gc \neq 0$.

On the other hand, we have $ga=(c\frown l^*(a))a\leq l^*(a)a=0$, $ga=0$. This is a contradiction. Hence we have $l^*(a)c=0$, $c\leq(l^*(a))^*=\varphi(a)$.

Lemma 20. *$\varphi(ae)=\varphi(a)$ for any element a of L.*

Proof. Suppose that $xa=0$ for an element x of Σ_L. Then of course $x\cdot ae=0$. Hence $l(a;\Sigma_L)$ is contained in $l(ae;\Sigma_L)$. Conversely, let x be any element of $l(ae;\Sigma_L)$. Then $x\cdot ae=0$. This implies $xa\leq xae=0$, $xa=0$. Hence x is contained in $l(a;\Sigma_L)$. Hence $l(a;\Sigma_L)=l(ae;\Sigma_L)$. We have therefore $l^*(a)=\sup[l(a;\Sigma_L)]=\sup[l(ae;\Sigma_L)]=l^*(ae)$, $\varphi(a)=(l^*(a))^*=(l^*(ae))^*=\varphi(ae)$.

Theorem 12. *$\varphi(ab)=\varphi(a)\frown\varphi(b)$ for any two elements a and b of L.*

Proof. First we suppose that a is an element of T. Then, since $ab\leq a\frown b$, we have $\varphi(ab)\leq\varphi(a\frown b)\leq\varphi(a)\frown\varphi(b)$. Take now an element g of T such that

$$h\equiv\varphi(a)\frown\varphi(b)\frown g\neq 0\,. \tag{1}$$

Then, since $\varphi(a)\frown g\neq 0$, we have $a\frown g\neq 0$ by Lemma 18. It suffices to prove that $ab\frown g\neq 0$. Because, if so, we have $\varphi(a)\frown\varphi(b)\leq\varphi\varphi(ab)=\varphi(ab)$ by Lemma 19. Assume now that $ab\frown g=0$. Then $(a\frown b\frown h)^2\leq ab\frown h\leq ab\frown g=0$, $a\frown b\frown h=0$. Hence $a(b\frown h)=0$. Therefore we have

$$b\frown h\leq r^*(a) = a^*\,. \tag{2}$$

By using (1) we have

$$b\frown h\leq h\leq\varphi(a)\,. \tag{3}$$

Since $\varphi(a)\cdot a^*=0$, $\varphi(a)\frown a^*=0$, we obtain $b\frown h=0$ by (2) and (3). Hence by Lemma 18 we have $0=\varphi(b)\frown h=h\neq 0$, a contradiction.

Next we suppose that a is an element of L. Then by using Lemma 20, we obtain $\varphi(ab)=\varphi(a\cdot eb)=\varphi(ae\cdot b)=\varphi(ae)\frown\varphi(b)=\varphi(a)\frown\varphi(b)$. This completes the proof.

Corollary. *$\varphi(ab)=\varphi(a\frown b)$ for any element a of T and b of L. If both a and b are φ-closed, then $ab=a\frown b$.*

DEFINITION 9. An element q of L is said to be left φ-prime if $ab\leq q$, $a,b\in L$, $\varphi(a)=e$ imply $b\leq q$.

Lemma 21. *An element q of L is left φ-prime if (and only if) $ab\leq q$, $a\in T$, $\varphi(a)=e$, $b\in L$ imply $b\leq q$.*

Proof. Suppose that $ab\leq q$, $a,b\in L$, $\varphi(a)=e$. Since $q\geq ab=a\cdot eb=ae\cdot b$, $ae\in T$, we obtain $\varphi(ae)=\varphi(a)=e$ by Lemma 20. Hence $b\leq q$.

Lemma 22. *Let q be a left φ-prime element. Then $ab \leq q$, $a, b \in L$, $\varphi(a) \geq \varphi(b)$ imply $b \leq q$.*

Proof. Since $l^*(a)b \leq \varphi(l^*(a))\varphi(b) \leq \varphi(l^*(a))\varphi(a) = \varphi(l^*(a)\cdot a) = \varphi(0) = 0$, $l^*(a)b = 0$, we have $(a \cup l^*(a)b = ab \leq q$. Now since $l^*(a \cup l^*(a) \leq l^*(a) \cap (l^*(a))^* = 0$, we have $l^*(a \cup l^*(a)) = 0$. Hence $\varphi(l^*(a \cup l^*(a)) = e$. We get therefore $b \leq q$, as desired.

Theorem 13. *Let q be any left φ-prime element of L. Then the supremum $\bar{q}$ of the elements c such that $c \leq q$, $c \in T$, is also a left φ-prime and φ-closed element; and $\bar{q}$ is decomposed as a meet of a finite or infinite number of prime elements.*

Proof. Suppose that $aq \leq \bar{q}$, where $a, b \in L$ and $\varphi(a) = e$. Then $a\cdot be \leq \bar{q}e = \bar{q} \leq q$. Hence $be \leq q$, hence $be \leq \bar{q}$, and hence $b \leq \bar{q}$. This proves that $\bar{q}$ is left φ-prime. Next, we prove that $\bar{q}$ is φ-closed. Since $\bar{q}\varphi(\bar{q}) \leq \bar{q}$ and $\varphi(\bar{q}) = \varphi\varphi(\bar{q})$, we have $\varphi(\bar{q}) \leq \bar{q}$ by Lemma 22. Hence $\varphi(\bar{q}) = \bar{q}$. The last part of this theorem is an immediate consequence of Corollary to Theorem 12.

§ 6. Nilpotent-free Ring Systems

An associative multiplicative semigroup $\mathfrak{o}$ is called here a ring system, if it has the following two conditions: (1) $\mathfrak{o}$ is an algebra[15] with a void or non-void set $\Gamma = \{\varphi\}$ of finitary operations which does not contain the multiplication and (2) the left and the right distributive laws hold for $\mathfrak{o}$, say, $x\varphi(x_1, \cdots, x_n) = \varphi(xx_1, \cdots, xx_n)$, $\varphi(x_1, \cdots, x_n)x = \varphi(x_1x, \cdots, x_nx)$ for all $x, x_i \in \mathfrak{o}$ and all $\varphi \in \Gamma$. Usual semigroups, associative rings, some near-rings and distributive lattices are included as very special cases.

Now we assume that $\mathfrak{o}$ has the zero element 0. A non-void subset $\mathfrak{a}$ of $\mathfrak{o}$ is called a left ideal of $\mathfrak{o}$, when (1) $\mathfrak{a}$ is closed under all operations in Γ and (2) if $a \in \mathfrak{a}$, then $ra \in \mathfrak{a}$, for all $r \in \mathfrak{o}$. A right ideal of $\mathfrak{o}$ is defined analogousely. A left and right ideal is called a two-sided ideal or shortly an ideal of $\mathfrak{o}$. Let X be a non-void subset of $\mathfrak{o}$. By $[X]^{\Gamma}$ we mean the closed subsystem of X under all operations in Γ. Any element of $[X]^{\Gamma}$ is denoted by $f_{\varphi_1,\cdots,\varphi_m}(x_1, \cdots, x_n)$ or losely by $f(x_1, \cdots, x_n)$. Let X and Y be any two non-void subsets of $\mathfrak{o}$. By XY we mean the set of the elements xy, $x \in X$, $y \in Y$. The product $X\cdot Y$ is defined as the elements of the form $\varphi(x_1y_1, \cdots, x_ny_n)$, where $x_i \in X$, $y_i \in Y$ and $\varphi \in \Gamma$. By $(X)_l$ we mean the left ideal generated by X. Then it is easily verified that $(X)_l = [\mathfrak{o}X \vee [X]^{\Gamma}]^{\Gamma} = [\mathfrak{o}X \vee X]^{\Gamma}$. For any element x of $\mathfrak{o}$, $(x)_l$

15) In the sense of G. Birkhoff. Cf. [2; Foreword on algebra].

is called a principal left ideal of $\mathfrak{o}$. Symmetrically for right ideals $(X)_r$, $(x)_r$ of $\mathfrak{o}$. The (two-sided) ideal generated by X is denoted by (X), which is equal to $[\mathfrak{o}X\mathfrak{o} \vee \mathfrak{o}X \vee X\mathfrak{o} \vee X]^{\Gamma}$. Then we have $(x) = [(x)_l \cdot \mathfrak{o} \vee (x)_l]^{\Gamma}$ for any $x \in \mathfrak{o}$. For, since $[\mathfrak{o}x\mathfrak{o} \vee x\mathfrak{o}]^{\Gamma} = [(\mathfrak{o}x \vee x)\mathfrak{o}]^{\Gamma} = [[\mathfrak{o}x \vee x]^{\Gamma}\mathfrak{o}]^{\Gamma} = [\mathfrak{o}x \vee x]^{\Gamma} \cdot \mathfrak{o} = (x)_l \cdot \mathfrak{o}$, we obtain $(x) = [\mathfrak{o}x\mathfrak{o} \vee x\mathfrak{o} \vee \mathfrak{o}x \vee x]^{\Gamma} = [[\mathfrak{o}x\mathfrak{o} \vee x\mathfrak{o}]^{\Gamma} \vee [\mathfrak{o}x \vee x]^{\Gamma}]^{\Gamma} = [(x)_l \cdot \mathfrak{o} \vee (x)_l]^{\Gamma}$, as desired.

Now let $\mathfrak{L}$ be the left ideals of $\mathfrak{o}$, $\mathfrak{R}$ the right ideals of $\mathfrak{o}$, $\mathfrak{S}$ the set-union of $\mathfrak{L}$ and $\mathfrak{R}$, and $\mathfrak{T}$ the intersection of $\mathfrak{L}$ and $\mathfrak{R}$. Then $\mathfrak{S}$ is, of course, a partly ordered set under the set-inclusion relation. Evidently $\mathfrak{o}$ is the greatest element of $\mathfrak{S}$, and the zero element 0 is the least element of $\mathfrak{S}$. Evidently $0 \cdot \mathfrak{a} = \mathfrak{a} \cdot 0 = 0$ for all $\mathfrak{a}$ of $\mathfrak{S}$. If $\mathfrak{o}$ has the unit element, then $\mathfrak{o} \cdot \mathfrak{a} \supseteq \mathfrak{a}$, $\mathfrak{a} \cdot \mathfrak{o} \supseteq \mathfrak{a}$ for all $\mathfrak{a}$ of $\mathfrak{S}$; and if $\mathfrak{a} \in \mathfrak{L}(\mathfrak{R})$, then $\mathfrak{o} \cdot \mathfrak{a} = \mathfrak{a}$ $(\mathfrak{a} \cdot \mathfrak{o} = \mathfrak{a})$. It is now easy to see that both $\mathfrak{L}$ and $\mathfrak{R}$ form lower-complete *cl*-semigroups under the operation $(\cdot)$ and the set-inclusion relation. Hence, if $\mathfrak{o}$ has the unit element, the conditions P_1, P_2, P_3, P_4 and P_5 hold for $\mathfrak{S}$.

An ideal $\mathfrak{p}$ of $\mathfrak{o}$ is said to be prime if whenever $\mathfrak{p}$ contains a product of two ideals of $\mathfrak{o}$, then $\mathfrak{p}$ contains at least one of the factors. A radical of $\mathfrak{o}$ (or of $\mathfrak{S}$) is defined as the intersection of all prime ideals of $\mathfrak{o}$. A non-zero ideal is called nilpotent when some power of it is the zero ideal.

In the following we suppose that $\mathfrak{o}$ has the unit element.

Let $\Sigma_{\mathfrak{L}}$ and $\Sigma_{\mathfrak{R}}$ be any two accessible join-generator systems of $\mathfrak{L}$ and $\mathfrak{R}$ respectively, which satisfy the conditions $(*)$ in §1 and $[*]$ of Lemma 8 in §4. Now we consider a family $\mathfrak{M}$ (non-void or void) of ideals such that $\mathfrak{M}$ is closed under the multiplication $(\cdot)$. If $\mathfrak{a}$ is an ideal such that $\mathfrak{a}' \subseteq \mathfrak{a}$ implies $\mathfrak{a}' \notin \mathfrak{M}$, then by Lemma 1 there exists[16] an ideal $\mathfrak{p}$ satisfying the following conditions; (1) $\mathfrak{a} \subseteq \mathfrak{p}$, (2) $\mathfrak{b} \subseteq \mathfrak{p}$ implies $\mathfrak{b} \notin \mathfrak{M}$, (3) if an ideal $\mathfrak{c}$ contains $\mathfrak{p}$ strictly, there exists an ideal in $\mathfrak{M}$ which is contained in $\mathfrak{c}$ and (4) $\mathfrak{p}$ is prime. Then, by Theorem 10, the following conditions are equivalent: (1) $\mathfrak{o}$ has the zero radical, (2) $\Sigma_{\mathfrak{L}} \wedge \Sigma_{\mathfrak{R}}$ is nilpotent-free, (3) $\mathfrak{T}$ is nilpotent-free, (4) $\mathfrak{L}$ is nilpotent-free, (5) $\Sigma_{\mathfrak{L}}$ is nilpotent-free, (6) $\mathfrak{R}$ is nilpotent-free and (7) $\Sigma_{\mathfrak{R}}$ is nilpotent-free.

Lemma 23. *The set of all principal left ideals of $\mathfrak{o}$ forms an accessible join-generator system of $\mathfrak{L}$.*

Proof. Suppose that $(a)_l$ is contained in $\bigcup_{\lambda} (a_{\lambda})_l$. Then there exists

16) If we define $\mathfrak{M}$ as a family of ideals such that there exists an ideal $\mathfrak{c}$ contained in $\mathfrak{a} \cdot \mathfrak{b}$ for any two ideals $\mathfrak{a}, \mathfrak{b} \in \mathfrak{M}$, then, whether there exists $\mathfrak{p}$ with (1), (2) and (3) or not, is unknown to the author. Cf. [12; Theorem 14·11].

a finite number of elements $x_1, \cdots, x_n$ such that $a=f(x_1, \cdots, x_n)$, $x_i \in \bigvee_\lambda (a_\lambda)_l$ for some f. Hence $x_i \in (a_{\lambda(i)})_l$, $i=1, \cdots, n$. Hence $x_j \in \bigvee_{i=1}^n (a_{\lambda(i)})_l$. We obtain therefore $(a)_l=(f(x_1, \cdots, x_n))_l \subseteq \bigvee_{i=1}^n (a_{\lambda(i)})_l$. The other condition is easy to see. This completes the proof.

By $\mathfrak{P}_{\mathfrak{L}}$ ($\mathfrak{P}_{\mathfrak{R}}$) we mean the system of the principal left (right) ideals of $\mathfrak{o}$. Then it is easy to verify that these two systems satisfy the condition [$*$] in Lemma 8. Hence, by using Theorem 10, we obtain the following

Theorem 14. *Let $\mathfrak{o}$ be a ring system with the unit element and the zero. Suppose that* ($*$) *the product of any two principal left ideals is finitely generated*[17], *and symmetrically for right ideals. Then the following conditions are equivalent*: (1) $\mathfrak{o}$ *has the zero radical,* (2) $\mathfrak{o}$ *has no nilpotent principal ideal,* (3) $\mathfrak{T}$ *is nilpotent-free,* (4) $\mathfrak{P}_{\mathfrak{L}}$ *is nilpotent-free,* (5) $\mathfrak{L}$ *is nilpotent-free,* (6) $\mathfrak{P}_{\mathfrak{R}}$ *is nilpotent-free and* (7) $\mathfrak{R}$ *is nilpotent-free.*

Throughout the rest of this section, we suppose that $\mathfrak{o}$ has the zero radical.

Let $\mathfrak{a}$ be any left ideal of $\mathfrak{o}$. Then, by Lemma 12, the left ideal $l^*(\mathfrak{a})$ generated by the set-union of the left ideals $\mathfrak{x}$ with $\mathfrak{x}\cdot\mathfrak{a}=0$ ($\mathfrak{x} \in \Sigma_{\mathfrak{L}}$) is equal to the (two-sided) ideal generated by the set-union of the ideals $\mathfrak{y}$ with $\mathfrak{y}\cdot\mathfrak{a}=0$ ($\mathfrak{y} \in \Sigma_{\mathfrak{L}} \wedge \Sigma_{\mathfrak{R}}$). Hence $l^*(\mathfrak{a})$ is a two-sided ideal of $\mathfrak{o}$. The same is true for $r^*(\mathfrak{a})$, where $\mathfrak{a}$ is a right ideal of $\mathfrak{o}$. If $\mathfrak{c}$ is an ideal of $\mathfrak{o}$, we have $l^*(\mathfrak{c})=r^*(\mathfrak{c})$ and $\mathfrak{c}\cdot\mathfrak{c}^*=\mathfrak{c}^*\cdot\mathfrak{c}=0$, where $\mathfrak{c}^*=l^*(\mathfrak{c})=r^*(\mathfrak{c})$. Let $\mathfrak{a}$ be any left ideal of $\mathfrak{o}$. Then the mapping $\mathfrak{a}\rightarrow\varphi(\mathfrak{a})=(l^*(\mathfrak{a}))^*$ from $\mathfrak{L}$ into $\mathfrak{T}$ is said to be φ-mapping of $\mathfrak{L}$. A left ideal $\mathfrak{a}$ is said to be φ-closed if $\varphi(\mathfrak{a})=\mathfrak{a}$. The φ-mapping satisfies $\varphi(\mathfrak{a}\cdot\mathfrak{b})=\varphi(\mathfrak{a})\frown\varphi(\mathfrak{b})$ for any two left ideals $\mathfrak{a}$, $\mathfrak{b}$ of $\mathfrak{o}$ (Theorem 12). In particular, if $\mathfrak{a}$ is an ideal, $\varphi(\mathfrak{a}\cdot\mathfrak{b})=\varphi(\mathfrak{a}\frown\mathfrak{b})$. If $\mathfrak{a}$, $\mathfrak{b}$ are φ-closed ideals, then $\mathfrak{a}\cdot\mathfrak{b}=\mathfrak{a}\frown\mathfrak{b}$. φ'-mapping of $\mathfrak{R}$ and φ'-closed ideals are defined in a similar way. Then $\varphi(\mathfrak{c})=\varphi'(\mathfrak{c})=\mathfrak{c}^{**}$ for any ideal $\mathfrak{c}$ of $\mathfrak{o}$, and $\mathfrak{c}$ is φ-closed if and only if it is φ'-closed. In such a case we shall say that $\mathfrak{c}$ is closed. Then by Theosem 11 any closed ideal of $\mathfrak{o}$ is decomposed as an intersection of a finite or infinite number of prime ideals of $\mathfrak{o}$; and so is $\varphi(\mathfrak{a})$ for all left ideals $\mathfrak{a}$ of $\mathfrak{o}$.

If $\mathfrak{o}$ satisfies the condition ($*$) in Theorem 14, we can argue similarly as above, by using $\mathfrak{P}_{\mathfrak{L}}$ and $\mathfrak{P}_{\mathfrak{R}}$ instead of $\Sigma_{\mathfrak{L}}$ and $\Sigma_{\mathfrak{R}}$ respectively. In this case we use "$\varphi^p(\mathfrak{a})$" instead of "$\varphi(\mathfrak{a})$", and say "principally φ-closed" or shortly "p-φ-closed" instead of "φ-closed". Then any p-φ-closed ideal is decomposed as an intersection of a finite or infinite number of

17) This is essential in our argument. In fact, we can find an example which shows that the conditions (2) and (3) are not equivalent, if this condition does not hold. This condition is of course weaker than the ascending chain condition for the left (right) ideals of $\mathfrak{o}$.

prime ideals of $\mathfrak{o}$; and so is $\varphi^p(\mathfrak{a})$ for all left ideals $\mathfrak{a}$ of $\mathfrak{o}$. Therefore we obtain the

Theorem 15. *Let $\mathfrak{o}$ be a ring system with the unit element and the zero, and suppose that $\mathfrak{o}$ has the zero radical. If the ascending chain condition holds for ideals of $\mathfrak{o}$, then, for any left ideal $\mathfrak{a}$ of $\mathfrak{o}$, $\varphi^p(\mathfrak{a})$ is uniquely decomposed as an intersection of a finite number of prime ideals of $\mathfrak{o}$. In particular, so is any p-φ-closed two-sided ideal of $\mathfrak{o}$.*

A left ideal $\mathfrak{q}$ of $\mathfrak{o}$ is said to be left φ-prime if whenever a product of two left ideals $\mathfrak{a}$ and $\mathfrak{b}$ with $\varphi(\mathfrak{b})=\mathfrak{o}$ is in $\mathfrak{q}$, then, $\mathfrak{a}$ is in $\mathfrak{q}$. Then, by Theorem 13, the ideal $\bar{\mathfrak{q}}$ generated by the two-sided ideals contained in $\mathfrak{q}$ is closed and left φ-prime; and $\bar{\mathfrak{q}}$ is decomposed into an intersection of a finite or infinite number of prime ideals of $\mathfrak{o}$.

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Note: Recently I have received a letter from Prof. M. P. Drazin, in which he kindly pointed out the incompleteness of the proof of Theorem in [10]. By using Theorem 1 of this paper, we shall be able to correct the proof under the condition in Theorem 9 of this paper.