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On $\bigtriangledown$-Polynomials of Links

By Fujitsugu HOSOKAWA

In 1934 H. Seifert [3] proved that the Alexander polynomial $\triangle(t)$ of a knot is a symmetric polynomial of even degree and $|\triangle(1)|=1$. And moreover he proved that for a given symmetric polynomial $f(t)$ of even degree, if $|f(1)|=1$, then there exists a knot which has $f(t)$ as its Alexander polynomial.

In 1953 G. Torres [1, 2] proved that for the Alexander polynomial of a link of multiplicity μ, whose components are $X_1, X_2, \cdots, X_\mu$, it holds

$$\triangle(t_1, t_2, \cdots, t_\mu) = (-1)^\mu t_1^{\nu_1} t_2^{\nu_2} \cdots t_\mu^{\nu_\mu} \triangle(t_1^{-1}, t_2^{-1}, \cdots, t_\mu^{-1})$$

for suitably chosen integers $\nu_1, \nu_2, \cdots, \nu_\mu$. Further he proved that it holds

$$\triangle(t_1, t_2, \cdots, t_{\mu-1}, 1) = (t_1^{l_1}, t_2^{l_2}, \cdots, t_{\mu-1}^{l_{\mu-1}}-1)\triangle(t_1, t_2, \cdots, t_{\mu-1}) \quad \text{for } \mu>2$$

and

$$\triangle(t_1, 1) = (t_1^{l_1}-1)\triangle(t_1)/(t_1-1) \qquad \text{for } \mu=2$$

where l_i is the linking number of X_i and X_μ for $i=1, 2, \cdots, \mu-1$, and that it holds

$$\triangle(1, 1, \cdots, 1) = 0 \qquad \text{for } \mu>2 \qquad \text{and}$$
$$\triangle(1, 1) = l_1 \qquad \text{for } \mu=2\,.$$

In this paper we shall introduce a polynomial, the $\bigtriangledown$-*polynomial* of a link, deduced simply from the Alexander polynomial. For this $\bigtriangledown$-polynomial $\bigtriangledown(t)$ we shall prove the following properties:

1) $\bigtriangledown(t)$ is a symmetric polynomial of even degree.
2) $|\bigtriangledown(t)|$ is determined uniquely by the linking numbers between all pairs of μ components of the link of multiplicity μ.
3) If a polynomial $f(t)$ is a symmetric polynomial of even degree, then there exists a link of any given multiplicity μ such that its $\bigtriangledown$-polynomial coincides with the polynomial $f(t)$.

1. We shall call a polynomial $f(t)$ *symmetric* (*skew symmetric*) if $f(t)=t^\nu f(t^{-1})$ ($f(t)=-t^\nu f(t^{-1})$) for a suitably chosen integer ν. Then integer $n-m$ will be called the *reduced degree* of a polynomial

$$f(t) = a_l t^l + a_{l-1}t^{l-1} + \cdots + a_n t^n + \cdots + a_m t^m + \cdots + a_1 t + a_0$$

if $a_l = a_{l-1} = \cdots = a_{n+1} = 0$, $a_n \neq 0$, $a_m \neq 0$ $(n \geq m)$ and $a_{m-1} = \cdots = a_1 = a_0 = 0$. Then we have the following

Lemma.[1] *Let $f(t)$ and $F(t)$ be symmetric polynomials of even reduced degree and let $g(t)$ and $G(t)$ be skew symmetric polynomials such that*

$$F(t) = tf(t) + (t-1)g(t),$$
$$G(t) = (1-t)f(t) + g(t) \quad .$$

Then the difference of reduced degrees of $f(t)$ and $g(t)$ is an odd integer.

2. Let κ be a link of multiplicity μ in a 3-dimensional euclidean space E^3 and let $X_1, X_2, \cdots, X_\mu$ be the components of κ. The Alexander polynomial $\triangle(t_1, t_2, \cdots, t_\mu)$ of κ may be defined as usual [1].

Now we shall define the ∇-*polynomial* $\nabla(t)$ of a link of multiplicity μ as follows:

$$\nabla(t) = \triangle(t, \cdots, t)/(1-t)^{\mu-2} \qquad \text{for} \quad \mu \geq 2,$$
$$\nabla(t) = \triangle(t) \qquad \text{for} \quad \mu = 1,$$

where $\triangle(t, \cdots, t)$ is the reduced polynomial[2] obtained by putting $t_1 = t_2 = \cdots = t_\mu = t$ in the Alexander polynomial $\triangle(t_1, t_2, \cdots, t_\mu)$ of κ.

G. Torres showed that

$$\triangle(t_1, t_2, \cdots, t_\mu) = (-1)^\mu t_1^{\nu_1} t_2^{\nu_2} \cdots t_\mu^{\nu_\mu} \triangle(t_1^{-1}, t_2^{-1}, \cdots, t_\mu^{-1})$$

for suitably chosen integers $\nu_1, \nu_2, \cdots, \nu_\mu$ [1].

From this it is easy to see that the ∇-polynomial is symmetric. Then we have the following

Theorem 1. *The ∇-polynomial of a link κ is a symmetric polynomial of even degree.*

Proof. We have only to prove that the degree of the ∇-polynomial is even. To prove this we shall use a mathematical induction on the multiplicity of κ. In the case of multiplicity 1, i.e. in the case of a knot, our theorem is evidently true [3].

Now suppose that our theorem is true for the case of multiplicity $\mu-1$.

Let $\bar{\kappa}$ be a regular projection[3] of a link κ of multiplicity μ on a plane E^2 in E^3 and let $\bar{X}_1, \bar{X}_2, \cdots, \bar{X}_\mu$ be the projections of $X_1, X_2, \cdots, X_\mu$ respectively on E^2. We may confine κ in a cube Q in E^3. We choose

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1) See [4].

2) A reduced polynomial is by definition a polynomial with non vanishing constant term.

3) See p. 60 of [1].

arbitrarily two components $\bar{X}_i$ and $\bar{X}_j$ of $\bar{\kappa}$ and take out parts of arcs of $\bar{X}_i$ and $\bar{X}_j$ from the cube Q, running parallel to each other in the same direction as shown in Fig. 1.

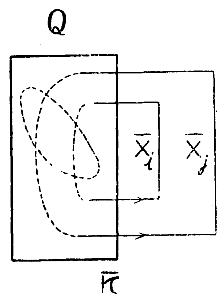

Fig. 1.

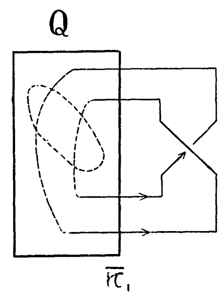

Fig. 2.

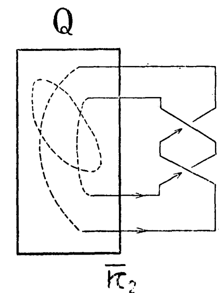

Fig. 3.

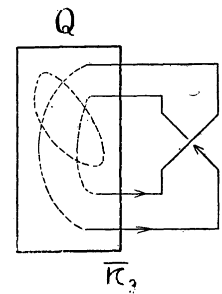

Fig. 4.

From this $\bar{\kappa}$ we introduce new links[4)] $\bar{\kappa}_1$, $\bar{\kappa}_2$ and $\bar{\kappa}_3$ as defined in Fig. 2, Fig. 3 and Fig. 4 respectively [4]. Let κ_1, κ_2 and κ_3 be links in E^3 such that they have $\bar{\kappa}_1$, $\bar{\kappa}_2$ and $\bar{\kappa}_3$ as their regular projections. Then it is clear that κ_1 and κ_3 are links of multiplicity $\mu-1$ and κ_2 is a link of multiplicity μ. Let $\nabla(t)$, $\nabla_1(t)$, $\nabla_2(t)$ and $\nabla_3(t)$ be the ∇-polynomials of κ, κ_1, κ_2 and κ_3 respectively.

Here we shall define $f(t)$, $g(t)$, $F(t)$ and $G(t)$ as follows:

$$\begin{aligned} f(t) &= \pm t^{p_1}\nabla_1(t)\,, \\ g(t) &= \pm t^{p}(1-t)\nabla(t)\,, \\ F(t) &= \pm t^{p_3}\nabla_3(t)\,, \qquad \text{and} \\ G(t) &= \pm t^{p_2}(1-t)\nabla_2(t) \end{aligned}$$

for suitably chosen integers p, p_1, p_2 and p_3.

From the assumption of induction $f(t)$ and $F(t)$ are symmetric polynomials of even reduced degree and $g(t)$ and $G(t)$ are skew symmetric polynomials and from the constructions of κ_1, κ_2 and κ_3 we have [4]

$$\begin{aligned} F(t) &= tf(t)+(t-1)g(t)\,, \\ G(t) &= (1-t)f(t)+g(t)\,. \end{aligned}$$

Hence, $f(t)$, $g(t)$, $F(t)$ and $G(t)$ satisfy the conditions of Lemma. Therefore, the difference of the reduced degrees of $f(t)$ and $g(t)$ is an odd integer. Since the reduced degree of $f(t)$ is even, the reduced degree of $g(t)$ is odd. By the definition of $g(t)$ the degree of $\nabla(t)$ is even. Thus, the proof of our theorem is complete.

4) A knot is considered as a link of multiplicity 1.

3. Given a link κ of multiplicity μ, κ can be spanned by an orientable surface F of genus h, represented by a disk to which is attached $2h+\mu-1$ bands $B_1, B_2, \cdots, B_{2h+\mu-1}$, whose corresponding projection si named by Torres the Seifert projection.

Let $B_1, B_2, \cdots, B_{2h}$ be canonical bands and let $B_{2h+1}, \cdots, B_{2h+\mu-1}$ be extra bands[5]. Let $a_1, a_2, \cdots, a_{2h+\mu-1}$ be simple closed curves which are drawn along each $B_1, B_2, \cdots, B_{2h+\mu-1}$. The curves $a_1, a_2, \cdots, a_{2h}$ are oriented such that a_{2i-1} crosses a_{2i} from left to right ($i=1, 2, \cdots, h$). The curves $a_{2h+1}, \cdots, a_{2h+\mu-1}$ have the same orientation as the orientation of X_1 as shown in Fig. 5.

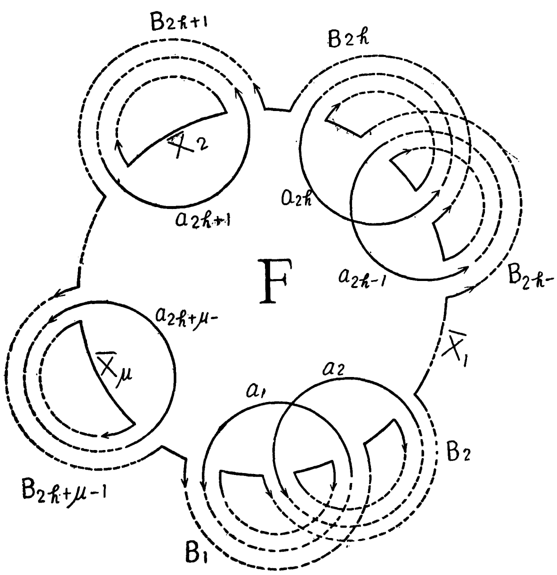

Fig. 5.

Then it is clear that a_i intersects a_j if and only if $i=2k-1$ and $j=2k$ $(1\leq k\leq h)$. If a_{2k-1} intersects a_{2k}, we can life a_{2k-1} in a neighborhood of the intersection and let the new curve be denoted by a'_{2k-1}. Let $v_{i,j}$ $(i, j=1, 2, \cdots, 2h+\mu-1)$ be equal to the number of times that a_i crosses over a_j from left to right minus the number of times that a_i crosses over a_j from right to left. Then we have easily the following relations:

$$v_{i,j}=v_{j,i}=\operatorname{link}(a_i, a_j) \qquad \text{if} \quad a_i \frown a_j=\phi$$
$$v_{2k-1,2k}=v_{2k,2k-1}+1=\operatorname{link}(a'_{2k-1}, a_{2k}) \qquad \text{if} \quad 1\leq k\leq h,$$

where link (a_i, a_j) denotes the linking number of a_i and a_j.

Then the ∇-polynomial $\nabla(t)$ of κ can be written as follows [1]:

5) See p. 63 of [1].

$$\pm t^{n}\nabla(t)=M_{h,\mu}(t)$$

$$(*)\qquad =\left|\begin{array}{ccccc|ccc}
v_{1,1}(1-t) & v_{1,2}(1-t)+t & \cdots & v_{1,2h-1}(1-t_1) & v_{1,2h}(1-t) & v_{1,2h+1}(1-t) & \cdots & v_{1,2h+\mu-1}(1-t) \\
v_{1,2}(1-t)-1 & v_{2,2}(1-t) & \cdots & v_{2,2h-1}(1-t) & v_{2,2h}(1-t) & v_{2,2h+1}(1-t) & \cdots & v_{2,2h+\mu-1}(1-t) \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
v_{2h-1,1}(1-t) & v_{2h-1,2}(1-t) & \cdots & v_{2h-1,2h-1}(1-t) & v_{2h-1,2h}(1-t)+t & v_{2h-1,2h+1}(1-t) & \cdots & v_{2h-1,2h+\mu-1}(1-t) \\
v_{2h,1}(1-t) & v_{2h,2}(1-t) & \cdots & v_{2h-1,2h}(1-t)-1 & v_{2h,2h}(1-t) & v_{2h,2h+1}(1-t) & \cdots & v_{2h,2h+\mu-1}(1-t) \\
\hline
v_{2h+1,1} & v_{2h+1,2} & \cdots & v_{2h+1,2h-1} & v_{2h+1,2h} & v_{2h+1,2h+1} & \cdots & v_{2h+1,2h+\mu-1} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
v_{2h+\mu-1,1} & v_{2h+\mu-1,2} & \cdots & v_{2h+\mu-1,2h-1} & v_{2h+\mu-1,2h} & v_{2h+\mu-1,2h+1} & \cdots & v_{2h+\mu-1,2h+\mu-1}
\end{array}\right|$$

Conversely if $v_{i,j}$ $(i, j=1, 2, \cdots, 2h+\mu-1)$ are given, then the ∇-polynomial, $\nabla(t)$, and a link of multiplicity μ, which has $\nabla(t)$ as the ∇-polynomial, can evidently be determined.

Now we have

Theorem 2. *Let κ be a link of multiplicity μ and let $X_1, X_2, \cdots, X_\mu$ be the components of κ. Let $l_{i,j}$ be the linking number of X_i and X_j, i.e.*

$$l_{i,j} = \mathit{link}\,(X_i, X_j) \qquad (i \neq j,\ \ i, j = 1, 2, \cdots, \mu).$$

Let $\nabla(t)$ be the ∇-polynomial of κ. Then $\pm\nabla(1)$ is equal to the $(\mu-1)$-minor determinant of the following matrix A:

$$A=\begin{pmatrix} -\sum_{k=2}^{\mu} l_{1,k} & l_{1,2} & \cdots & l_{1,j} & \cdots & l_{1,\mu} \\ l_{2,1} & -\sum_{k=1,k\neq 2}^{\mu} l_{2,k} & \cdots & l_{2,j} & \cdots & l_{2,\mu} \\ \vdots & \vdots & & \vdots & & \vdots \\ l_{j,1} & l_{j,2} & \cdots & -\sum_{k=1,k\neq j}^{\mu} l_{j,k} & \cdots & l_{j,\mu} \\ \vdots & \vdots & & \vdots & & \vdots \\ l_{\mu,1} & l_{\mu,2} & \cdots & l_{\mu,j} & \cdots & -\sum_{k=2}^{\mu-1} l_{\mu,k} \end{pmatrix}.$$

Proof. From the definition of $v_{i,j}$ it is easy to see that

$$l_{i,j} = v_{2h+i-1,2h+j-1} = v_{2h+j-1,2h+i-1} = l_{j,i} \qquad \text{if} \quad i, j > 1 \quad i \neq j$$

$$l_{1,i} = -(v_{2h+i-1,2h+1} + \cdots + v_{2h+i-1,2h+i-1} + \cdots + v_{2h+i-1,2h+\mu-1}) \quad \text{if} \quad i > 1.$$

Hence, we have

$$v_{2h+i-1,2h+i-1} = -(l_{1,i} + l_{2,i} + \cdots + l_{i-1,i} + l_{i+1,i} + \cdots + l_{\mu,i}).$$

If we set $t=1$ in $(*)$, we have

$$\pm\nabla(1) = \left|\begin{array}{cc|cc|cc|ccc} 0 & 1 & & & & & & & \\ -1 & 0 & & & & & & & \\ \hline & & 0 & 1 & & & & 0 & \\ & & -1 & 0 & & & & & \\ \hline & 0 & & & 0 & 1 & & & \\ & & & & -1 & 0 & & & \\ \hline & & & & & & v_{2h+1,2h+1} & \cdots & v_{2h+1,2h+\mu-1} \\ & & * & & & & \vdots & & \\ & & & & & & v_{2h+\mu-1,2h+1} & \cdots & v_{2h+\mu-1,2h-\mu-1} \end{array}\right|$$

$$
= \begin{vmatrix} v_{2h+1,2h+1} & \cdots & v_{2h+1,2h+\mu-1} \\ \vdots & & \\ v_{2h+\mu-1,2h+1} & \cdots & v_{2h+\mu-1,2h+\mu-1} \end{vmatrix}
$$

$$
= \begin{vmatrix} -\sum_{k=1,k\neq 2}^{\mu} l_{2,i} & l_{2,3} & \cdots & l_{2,\mu} \\ l_{3,2} & -\sum_{k=1,k\neq 3}^{\mu} l_{3,k} & \cdots & l_{3,\mu} \\ \vdots & \vdots & & \vdots \\ l_{\mu,2} & l_{\mu,3} & \cdots & -\sum_{k=1}^{\mu-1} l_{\mu,k} \end{vmatrix}.
$$

Since it is clear that the rank of matrix A is $\mu-1$, the $(\mu-1)$-minor determinants of A coincide with $\pm\nabla(1)$. Thus the proof is complete.

Finally, we shall prove the following theorem.

Theorem 3. *Let $f(x)$ be a symmetric polynomial of even degree whose constant term is different from zero. Then there exists a link of any given multiplicity μ whose ∇-polynomial is $f(x)$.*

Proof. In order to construct a link κ of multiplicity μ whose ∇-polunomial is $f(x)$, we shall determine $v_{i,j}$ $(i, j = 1, 2, \cdots, 2h+\mu-1)$ in $(*)$.

To begin with, let

$$
v_{1,4},\, v_{3,6},\, \cdots,\, v_{2h-3,2h} \quad \text{and}
$$

$$
v_{2h+2,2h+2},\, \cdots,\, v_{2h+\mu-1,2h+\mu-1} \quad \text{and} \quad v_{2,2h+1}
$$

be equal to 1, let

$$
v_{1,3},\, v_{3,5},\, \cdots,\, v_{2h-3,2h-1} \quad \text{and} \quad v_{1,1}
$$

be undetermined, let

$$
v_{2h+1,2h+1}
$$

be equal to $-f(1)$, and let the others be equal to zero.

Then we have

$$M_{h,\mu}(t)=$$

$$\left|\begin{array}{cccccccc|ccc}
v_{1,1}(1-t) & t & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
-1 & 0 & \cdots & 0 & 0 & 0 & 0 & (1-t) & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & 0 & t & v_{2h-3,2h-1}(1-t) & (1-t) & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & v_{2h-3,2h-1}(1-t) & 0 & 0 & t & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & (1-t) & 0 & -1 & 0 & 0 & 0 & \cdots & 0 \\
\hline
0 & 1 & \cdots & 0 & 0 & 0 & 0 & -f(1) & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1
\end{array}\right| .$$

Interchange the second column with the $(2h+1)$-th column and add the $(2h+1)$-th row multiplied by $-t$ to the first row. Then, we have

$$-M_{h,\mu}(t)=\bar{M}_{h,\mu}(t)=$$

$$\left|\begin{array}{ccccccc}
v_{1,1}(1-t) & f(1)t & \cdots & 0 & 0 & 0 & 0 \\
-1 & 1-t & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & t & v_{2h-3,2h-1}(1-t) & (1-t) \\
0 & 0 & \cdots & -1 & 0 & 0 & 0 \\
0 & 0 & \cdots & v_{2h-3,2h-1}(1-t) & 0 & 0 & t \\
0 & 0 & \cdots & (1-t) & 0 & -1 & 0
\end{array}\right| .$$

Now we shall prove our theorem by a mathematical induction on the degree of $f(t)$.

If the degree of $f(x)$ is 2, i.e. if

$$f(t)=c_0t^2+c_1t+c_0\,,$$

then we set $h=1$.

Since

$$\bar{M}_{1,\mu}(t)=\begin{vmatrix} v_{1,1}(1-t) & f(1)t \\ -1 & (1-t)\end{vmatrix}=v_{1,1}t^2-(2v_{1,1}-f(1))t+v_{1,1}\,,$$

if we set $v_{1,1}=c_0$, then $|\nabla(t)|=|\bar{M}_{1,\mu}(t)|=|f(t)|$.

Suppose that our theorem is proved for the case where the degree of $f(t)$ is $2(n-1)$.

Then let

$$f(t) = c_0 t^{2n} + c_1 t^{2n-1} + \cdots + c_{n-1} t^{n+1} + c_n t^n + c_{n-1} t^{n-1} + \cdots + c_1 t + c_0 .$$

We set $h = n$.

Let $\bar{M}'_{n,\mu}(t)$ be the $(2n-1)$-minor determinant which is deduced from $\bar{M}_{n,\mu}(t)$ by striking out $(2n-1)$-th row and $(2n-1)$-th column. By a simple calculation we have

$$\bar{M}'_{n,\mu}(t) = -(1-t)^2 \bar{M}'_{n-1,\mu}(t)$$

and since $\bar{M}'_{1,\mu}(t) = (1-t)$, we have further

$$\bar{M}'_{n,\mu}(t) = (-1)^{n-1}(1-t)^{2n-1} .$$

Developing $\bar{M}_{n,k}(t)$ at the $(2n-1)$-th column, we have

$$\begin{aligned}\bar{M}_{n,\mu}(t) &= v_{2n-3,2n-1} t(1-t)^2 \bar{M}'_{n-1,\mu}(t) + \{t\bar{M}_{n-1,\mu}(t) - (1-t)^2 v_{2n-3,2n-1} \bar{M}'_{n-1,\mu}(t)\} \\ &= t\bar{M}_{n-1,\mu}(t) - (1-t)^3 v_{2n-3,2n-1} \bar{M}'_{n-1,\mu}(t) \\ &= t\bar{M}_{n-1,\mu}(t) + (-1)^{n-1} v_{2n-3,2n-1}(1-t)^{2n} .\end{aligned}$$

In order that

$$\bar{M}_{n,\mu}(t) = f(t) = c_0 t^{2n} + c_1 t^{2n-1} + \cdots + c_1 t + c_0 ,$$

it is sufficient that it holds

$$(**) \quad \bar{M}_{n-1,\mu}(t) = \frac{c_0 t^{2n} + c_1 t^{2n-1} + \cdots + c_1 t + c_0 + (-1)^n v_{2n-3,2n-1}(1-t)^{2n}}{t} .$$

If we set $v_{2n-3,2n-1} = (-1)^{n+1} c_0$, then the degree of the right hand side of (**) is $2(n-1)$. Hence, by the assumption of induction we can determine $v_{1,1}, v_{1,3}, \cdots, v_{2n-5,2n-3}$ satisfying (**). Therefore, $v_{1,1}, v_{1,3}, \cdots, v_{2n-3,2n-1}$ can be determined such that $|\nabla(t)| = |\bar{M}_{n,\mu}(t)| = |f(t)|$.

Hence, we have a link of multiplicity μ whose ∇-polynomial is $f(t)$, and the proof is complete.

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