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On Regular Neighbourhoods of 2-Manifolds in 4-Euclidean Space. I

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Introduction

In 1921 L. Antoine [1] dealt with the embedding of sets in a Euclidean space R and he pointed out that there are three categories of the embeddings. Let P, Q be (topologically) equivalent sets in R. The first category: There is an orientation preserving homeomorphism onto $\psi: R \to R$ such that $\psi(P) = Q$. We say that P, Q are congruent. The second category: There are neighbourhoods $U(P)$, $U(Q)$ of P, Q respectively such that there exists an orientation preserving homeomorphism onto $\psi: U(P) \to U(Q)$ such that $\psi(P) = Q$. We say that P, Q are semicongruent. The third category: P, Q are neither congruent nor semicongruent.

The present paper deals with the second category of the piecewise linear embedding of polyhedral manifolds in R^n $(n=3, 4)$. The questions studied are mostly local in character. We often use some of the results and methods due to J.H.C. Whitehead [9] and V.K.A.M. Gugenheim [6. I, 6. II]. I am greatly indebted to their papers.

The exposition is as follows: In section 1 the results which are well known and will be used in the rest of the paper are stated. The results in section 2 are analogous appropriate to congruence of theorems due to Whitehead concerning regular neighbourhoods of polyhedra in a Euclidean space. Section 3 contains the fundamental definitions and lemmas. In section 4 we deal with $(n-1)$–manifolds in R^n $(n=3, 4)$ and show that the equivalent manifolds are semicongruent (Theorem 2). In section 5 we deal with 2–manifolds in R^4 (Theorem 3) and characterize the semicongruence classes of equivalent oriented 2–manifolds in R^4 (Theorem 4). Section 6 contains some geometric applications of the above considerations.

1. Preliminaries

1.1. R^n will stand throughout this paper for n–dimensional metric

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Euclidean space (in the statements using this and other points sets the superscripts which denote the dimensionalities will often be omitted when they are clear from the context). By a *simplex* we shall mean a closed Euclidean simplex, and by a *complex*, a rectilinear closed locally finite simplicial subcomplex of some Euclidean space (we only deal with finite complexes except for R). Let K be a complex; we denote by $|K|$ the point set covered by the simplexes of K; such a point set will be called a *polyhedron* and K a *partition* of the polyhedron. For convenience sake we very often use K instead of $|K|$.

Polyhedra having isomorphic partitions will be said to be *equivalent*. Let K, L be isomorphic partitions of polyhedra P, Q. Let $\phi : P \to Q$ be the homeomorphism obtained by mapping each simplex of K linearly onto its correlate in L. We call ϕ a *piecewise linear homeomorphism onto*, or PLO. Let $Q \subset T$, where T is a polyhedron. Then the mapping $\psi : P \to T$ defined by $\psi(x) = \phi(x)$ for $x \in P$ is called a *piecewise linear homeomorphism into*, or PLI. The identity map be denoted by 1.

By I we shall denote the closed interval $0 \leqq t \leqq 1$. Let P, Q be polyhedra and let

$$\phi_I : P \times I \to Q \times I$$

be a PLI such that $\phi_I(x, t) = (y, t)$, where $x \in P$, $y \in Q$ and $t \in I$. If x, y and t are related as above, we write $y = \phi_t(x)$, and have thus defined a map $\phi_t : P \to Q$ which is a PLI and is such that

$$\phi_I(x, t) = (\phi_t(x), t) .$$

In the above situation, the PLI ϕ_I or the family of PLI ϕ_t are called an *into isotopy* between the PLI ϕ_0 and ϕ_1, which are said to be into isotopic; we write $\phi_0 \sim \phi_1$. If ϕ_I is a PLO or equivalently, if each ϕ_t is a PLO, then we refer to an *onto isotopy*, we say that ϕ_0, ϕ_1 are onto isotopic and write $\phi_0 \approx \phi_1$. Both $\sim$ and $\approx$ are equivalence relations [6. I].

1. 2. By a *q-element* E^q we shall mean a polyhedron equivalent to a q-simplex Δ^q, by a *q-sphere* S^q one equivalent to the boundary of a $(q+1)$-simplex.

Let K be a complex and H a set of simplexes in K. We denote by $N(H, K)$ the set of all simplexes of K which contain one or more simplexes in H and referred to as the *star* of H in K. The set of simplexes of K which are faces not intersecting H in some simplexes of $N(H, K)$ is denoted by $L(H, K)$ and referred to as the *link* of H in K.

A (combinatorial) q-manifold M^q is defined as a complex M^q such that $|N(\Delta, M)|$ is a q-element for every simplex $\Delta \subset M$. Alternatively, it is characterized as follows: Let x be any vertex of M. Then $|L(x, M)|$ is a $(q-1)$-element if $x \in \dot{M}$, a $(q-1)$-sphere if $x \in IM$, where $\dot{M}$ will denote the boundary of M and $IM = M - \dot{M}$.

From now on manifold will mean connected manifold. Let M^q be an orientable q-manifold. An orientation preserving PLI $\phi: N^q \to M^q$, where $N^q \subset M^q$ is an orientable submanifold, is said to be positive in M^q. If a PLO $\phi: M^q \to M^q$ is positive in M^q we call it a +PLO.

The following is well known:

(a) Let S be a sphere and $\phi: S \to S$ a +PLO. Then $\phi \approx 1$ [6. 1].

(b) Let E be an element and $\phi: E \to E$ a +PLO. Then $\phi \approx 1$ [6. I].

(c) Let M^2 be an orientable closed 2-manifold and $\phi: M^2 \to M^2$ a PLO such that ϕ induces an inner automorphism of the fundamental group of M^2. Then $\phi \approx 1$ [4].

(d) Let M^q be an orientable q-manifold and E_i^q $(i=1, 2)$ q-elements in IM^q. Let $P \subset M^q - E_1^q - E_2^q$ be a polyhedron which does not disconnect M^q and let $\phi: E_1^q \to E_2^q$ be a given PLO which is positive in M^q. There is a +PLO $\psi: M^q \to M^q$ such that $\psi | P = 1$, $\psi | E_1^q = \phi$ and $\psi \approx 1$ [6. I, Theorem 3 and also see its Remark].

(e) Let M^q be a q-manifold $(q > 1)$ and x_i, y_i $(i = 1, \cdots, k)$ points in IM^q. Then there is a +PLO $\phi: M^q \to M^q$ such that $\phi(x_i) = y_i$ for each i and $\phi \approx 1$ (it is trivial, see [8]).

1. 3. Let M be an orientable manifold and P, Q polyhedra of M. We say that P, Q are *congruent* in M, $P \equiv Q$ in M, if there is a +PLO $\phi: M \to M$ such that $\phi(P) = Q$. In statements using this and consequential definitions the words in M will often be omitted when they are clear. In the rest part of the paper by polyhedron we shall either mean polyhedron P in the same sense as up to now; or else oriented polyhedron $\boldsymbol{P}$, in which latter case all PLI, equivalence and congruence relations are taken to be consistent with the given orientations.

The following is due to Gugenheim [6. II]. A polyhedron $\boldsymbol{P}$ is said to be *locally embedded* in an orientable n-manifold M^n if there is an n-element $E^n \subset M^n$ such that $P \subset IE^n$ and P does not disconnect M^n. By $\{\boldsymbol{P}, \boldsymbol{M}^n\}$ we shall denote a polyhedron $\boldsymbol{P}$ locally embedded in M^n and the oriented manifold $\boldsymbol{M}^n$. We call $\{\boldsymbol{P}, \boldsymbol{M}^n\}$ a *pair*. We say that the pair $\{\boldsymbol{P}, \boldsymbol{M}^n\}$ and $\{\boldsymbol{Q}, \boldsymbol{N}^n\}$ are congruent, $\{\boldsymbol{P}, \boldsymbol{M}^n\} \equiv \{\boldsymbol{Q}, \boldsymbol{N}^n\}$, if, since $\boldsymbol{P}$ is locally embedded, there is a +PLO $\phi: \boldsymbol{E}^n \to \boldsymbol{N}^n$, where $\boldsymbol{E}^n$ has the orientation induced by $\boldsymbol{M}^n$ such that $\phi(\boldsymbol{P}) = \boldsymbol{Q}$. Congruence is an equivalence relation between pairs.

Let Π be an equivalence class of polyhedra. By (Π, n) we denote the set of congruence classes of pairs $\{\boldsymbol{P}, \boldsymbol{M}^n\}$, where $\boldsymbol{P} \in \Pi$ and $\boldsymbol{M}^n$ is an oriented n-manifold.

For $q<n$, let $<q, n>$ denote the set of congruence classes of oriented q-elements of R^n, $<q, n>$ is a commutative semigroup and the zero element $0 \in <q, n>$ denotes the congruence class of *flat q-elements* which are congruent in R^n to an arbitrary oriented q-simplex. We shall take $<n, n>=0$ as a formal way of saying that all n-elements of R^n which have the orientations which are induced by a fixed orientation of R^n, are flat (see 1.2 (d)).

For $q<n-1$, let (q, n) denote the set of congruence classes of oriented q-spheres of R^n, (q, n) is a commutative semigroup and the zero element $0 \in (q, n)$ denotes the congruence class of *flat q-spheres* which are congruent in R^n to an arbitrary oriented boundary of a $(q+1)$-simplex. Choose an orientation of R^n and give to every $(n-1)$-sphere of R^n the orientation induced by its interior; then $(n-1, n)$ denotes the congruence classes of $(n-1)$-spheres so oriented.

Then we may identify

$$(\mathrm{E}^q, n) \text{ and } <q, n> \text{ if } q<n$$
$$(\textstyle\sum^q, n) \text{ and } (q, n) \text{ if } q<n-1,$$

where E^q are the equivalence class of oriented q-elements and $\sum^q$ that of oriented q-spheres. The following is well known.

(a) Let $P, Q \subset R^n$ be polyhedra and let $\phi: R^n \to R^n$ be a $+$PLO such that $\phi(P)=Q$. Then there is an n-element E_0^n and $+$PLO $\phi_0: R^n \to R^n$ such that $\phi_0 | cl(R^n - E_0^n)=1$ and $\phi_0 | P = \phi | P$ [6. I, 5, 7].

(b) Let $P, Q \subset R^n$ be q-dimensional polyhedra, $2q+2 \leqq n$, and $\phi: P \to Q$ be a given PLO. Then there is a $+$PLO $\psi: R^n \to R^n$ such that $\psi | P = \phi$ [6. I].

(c)
$<1, n>$	$=0$	for each $n \geqq 1$	[6. I],
$<q, n>$	$=0$	for $2q+1 \leqq n$	[6. II],
$<2, n>$	$=0$	for $n \neq 4$	[6. II, 5],
$(1, 2)$	$=0$	well known,	
(q, n)	$=0$	for $2q+2 \leqq n$	[6. I],
$(2, 3)$	$=0$	Alexander's theorem	[5, 7].

2. Regular Neighbourhoods in R^n

2.1. If K is a complex, K' and K'' will stand for the first and second derived complexes of K. Let M^n be an n-dimensional manifold which contains K as a subcomplex. By a *regular neighbourhood* of K in M^n we shall mean a subcomplex $U(K, M)$ of M, such that $U(K, M)$

is an n-dimensional manifold and $U(K, M)$ contracts geometrically into K. The main results of Whitehead [9, p. 293] are;

(a) $N(K'', M'')$ is a regular neighbourhood.

(b) Any two regular neighbourhoods of K in M are equivalent.

(c) If K is geometrically collapsible, $U(K, M^n)$ is an n-element.

By $U(K)$ we often denote $U(K, M)$ when $M=R$.

2.2. Let M^q be a q-manifold and E^q a q-element such that

$$M^q \cap E^q = \dot{M}^q \cap \dot{E}^q = E^{q-1}$$

a $(q-1)$-element. We say that M^q and E^q have *regular contact* in E^{q-1}. A transformation $M^q \rightarrow M^q \cup E^q$, where M^q and E^q have regular contact in a $(q-1)$-element on the boundary of both, or the resultant of a finite sequence of such transformations, will be called a *regular expansion* of M^q [9, p. 291].

Let $M^q \subset R^n$ be a q-manifold and $F^q \subset M^q$ a q-element such that

$$\dot{M}^q \cap \dot{F}^q \supset F^{q-1},$$

a $(q-1)$-element. Let $E^q \subset R^n$ have regular contact with M^q in F^{q-1} and let the q-element $E^q \cup F^q$ be flat. Then we call E^q a *flat attachment* to M^q [6. I, p. 33].

Lemma. *If $M^n \subset R^n$ is an n-manifold, $E^n \subset R^n$ is an n-element and $M^n \rightarrow M^n \cup E^n$ is a regular expansion of M^n, then E^n is a flat attachment to M^n.*

Proof. Since the transformation $M^n \rightarrow M^n \cup E^n$ is a regular expansion, E^n meets M^n in an $(n-1)$-element F^{n-1} on the boundary of both. Let $F^n = U(F^{n-1}, M^n)$. By 2.1 (c) F^n is an n-element in M^n such that $\dot{M}^n \cap \dot{F}^n \supset F^{n-1}$. Since, by 1.2 (d), the n-element $E^n \cup F^n$ is flat, E^n is a flat attachment to M^n.

2.3. Theorem 1. *If P is a polyhedron in R^n, any two regular neighbourhoods $U_1(P)$ and $U_2(P)$ are congruent in R^n relative to P, that is to say, there is a $+PLO$ $\phi: R^n \rightarrow R^n$ such that $\phi(U_1(P)) = U_2(P)$ and $\phi | P = 1$.*

Proof. Let $|K| = P$ be a partition such that K is a subcomplex of R and each of the regular neighbourhood $U_i(P)$ contracts formally into K, $i=1, 2$ [9, p. 296]. By the second corollary to Lemma 10 of [9, p. 293], U_i'' expands regularly into $N(U_i'', R'')$. Since K expands formally into U_i, it follows from Lemma 11 of [9, p. 294] that $N(K'', R'')$ expands regularly into $N(U_i'', R'')$. Hence by Lemma 2.2 and Theorem 6 of [6. I]

$$U_i \equiv U_i'' \equiv N(U_i'', R'') \equiv N(K'', R'') .$$

Therefore there is a $+$PLO $\phi: R \to R$ such that $\phi(U_1) = U_2$. Since by 2.1 (a) we may assume that F^n in Lemma 2.2 which arises at each step of regular expansions is disjoint from P, $\phi | P = 1$.

2.4. Corollary 1. *If P, Q are congruent polyhedra in R^n, then $U(P)$ and $U(Q)$ are congruent in R^n.*

Proof. Let $\phi_1: R \to R$ be a $+$PLO such that $\phi_1(P) = Q$. Then $\phi_1(U(P))$ is an n-manifold which contracts geometrically into $\phi_1(P) = Q$, that is to say, $\phi_1(U(P))$ is a regular neighbourhood of Q in R. Hence, by Theorem 1, there is a $+$PLO $\phi_2: R \to R$ such that $\phi_2\phi_1(U(P)) = U(Q)$.

2.5. Corollary 2. *If P, Q are equivalent q-dimensional polyhedra in R^n, then $U(P)$ and $U(Q)$ are congruent and also equivalent provided $n \geqq 2q+2$.*

This follows from 1.3 (b) and 2.4 (see Theorem 24 of [9]).

2.6. REMARK. We can prove the following but since it is unnecessary in the rest of the paper, the proof is omitted:

If E^m is an m-element in R^n $(m \leqq n)$. Then the fundamental group of $R^n - E^m$ is the unity.

3. Local congruence and semicongruence.

3.1. Let $\boldsymbol{R}^n$ be R^n with a given orientation which we keep fixed throughout; all n-elements of R^n and their boundaries will be given the induced orientation.

Let $\boldsymbol{M}^q$ be an oriented q-manifold of R^n, and let $q < n$; we give to all q-elements of M^q and their boundaries the induced orientation.

Then we have for any vertex $x \in M'$ the pairs $\{\boldsymbol{N}(x, M''), \boldsymbol{R}\}$, $\{\boldsymbol{L}(x, M''), \boldsymbol{L}(x, R'')\}$. If $q < n-1$, $\{\boldsymbol{N}(x, M''), \boldsymbol{R}\}$ represents an element in $\langle q, n \rangle$, $\{\boldsymbol{L}(x, M''), \boldsymbol{L}(x, R'')\}$ represents an element in $(q-1, n-1)$ if $x \in IM$ and in $\langle q-1, n-1 \rangle$ if $x \in \dot{M}$.

3.2. DEFINITION 1. Let $\boldsymbol{M}^q$, $\boldsymbol{N}^q$ be equivalent oriented q-manifolds of R^n. We say that $\boldsymbol{M}^q$, $\boldsymbol{N}^q$ are *locally congruent* in R^n, if there is a $+$PLO $\phi: \boldsymbol{M}^q \to \boldsymbol{N}^q$ such that for any vertex $x \in M'$

$$\{\boldsymbol{L}(x, M''), \boldsymbol{L}(x, R'')\} \equiv \{\boldsymbol{L}(\phi(x), N''), \boldsymbol{L}(\phi(x), R'')\} .$$

If M^q, N^q are nonorientable, M^q, N^q are locally congruent when there is a PLO $\phi: M^p \to N^q$ as above, where the orientation of $\boldsymbol{L}(x, M'')$

is any one of the possible orientations of $L(x, M'')$ and that of $\boldsymbol{L}(\phi(x), N'')$ is one induced by ϕ and that of $\boldsymbol{L}(x, M'')$.

DEFINITION 2. Let $\boldsymbol{M}^q, \boldsymbol{N}^q$ be equivalent oriented q-manifolds of R^n. We say that $\boldsymbol{M}^q, \boldsymbol{N}^q$ are *semicongruent* in R^n if there are regular neighbourhoods $U(M^q)$, $U(N^q)$ and $+$PLO $\psi: \boldsymbol{U}(M^q) \to \boldsymbol{U}(N^q)$ such that $\psi(\boldsymbol{M}) = \boldsymbol{N}$. If M^q, N^q are nonorientable, M^q, N^q are semicongruent when there is a $+$PLO $\psi: \boldsymbol{U}(M^q) \to \boldsymbol{U}(N^q)$ such that $\psi(M) = N$.

REMARK. It is clear that local congruence is an equivalence relation. The reflexive and symmetric laws are clear for semicongruence and the transitive law is easily proved by Theorem 1. Thus semicongruence is an equivalence relation. Hence we may define local congruence classes and semicongruence classes of equivalent oriented (or nonorientable) manifolds in R^n. If $\boldsymbol{M}^q, \boldsymbol{N}^q$ of R^n are semicongruent, they are locally congruent. But whether in general the converse is true I do not know.

3.3. Lemma. *If $\boldsymbol{S}_i$ $(i=1, 2)$ are oriented 1-spheres of R^3 and ϕ is a $+PLO: \boldsymbol{S}_1 \to \boldsymbol{S}_2$. Then there is a $+PLO$ $\phi: \boldsymbol{N}(S_1'', R'') \to \boldsymbol{N}(S_2'', R'')$ such that $\psi | S_1 = \phi$.*

Proof. Let S_i $(i=1, 2)$ be subcomplexes of R^3. Let x_j be vertices of S_1' and $\phi(x_j) = y_j$ vertices of S_2' $(j=1, \cdots, p)$ which are ordered by the orientation of $\boldsymbol{S}_i$ $(i=1, 2)$. It is well known that $N(S_i'', R'')$ is the aggregate of 3-elements $N(z_{ij}, R'')$, where z_{ij} is a vertex of S_i', that is to say, $N(S_i'', R'') = N(\bigcup_{j=1}^{p} z_{ij}, R'')$ [9, p. 294]. Let $N(z_{ij}, R'') \cap N(z_{ij+1}, R'') = D_{ij}$,

$$(1) \qquad D_{ij} \begin{cases} = N(L(z_{ij}, S_i'') \cap L(z_{ij+1}, S_i''),\ L(z_{ij}, R'')) \\ = N(L(z_{ij}, S_i'') \cap L(z_{ij+1}, S_i''),\ L(z_{ij+1}, R'')) \end{cases}$$

is a 2-element $(i=1, 2.\ \ j=1, \cdots, p \bmod p)$.

Since $L(z_{ij}, S_i'')$ is two points and $L(z_{ij}, R'')$ is a 2-sphere such that $L(z_{ij}, S_i'') \subset L(z_{ij}, R'')$, by 1.2 (e) for each $j=1, \cdots, p$

$$(2) \qquad \{\boldsymbol{L}(x_j, S_1''),\ \boldsymbol{L}(x_j, R'')\} \equiv \{\boldsymbol{L}(y_j, S_2''),\ \boldsymbol{L}(y_j, R'')\}\,.$$

By (2) there is a $+$PLO $\psi_1'': \boldsymbol{L}(x_1, R'') \to \boldsymbol{L}(y_1, R'')$ such that $\psi_1''(\boldsymbol{L}(x_1, S_1'')) = \boldsymbol{L}(y_1, S_2'')$. Since $D_{ij-1} \cup D_{ij} \neq L(z_{ij}, R'')$ for $i=1,2$ and $j=1, \cdots, p \bmod p$, from (1) and Theorem 1 we have a $+$PLO $\rho_1: \boldsymbol{L}(y_1, R'') \to \boldsymbol{L}(y_1, R'')$ such that

$$\rho_1(\psi_1''(D_{1j})) = D_{2j} \quad j=1, p \qquad \text{and}$$
$$\rho_1 | L(y_1, S_2'') = 1\,.$$

Hence $\psi_1'': \boldsymbol{L}(x_1, R'') \to \boldsymbol{L}(y_1, R'')$ defined by $\psi_1' = \rho_1 \psi_1''$ is a $+$PLO such that $\psi_1'(D_{1j})=D_{2j}$ $(j=1, p)$ and $\psi_1'(\boldsymbol{L}(x_1, S_1''))=\boldsymbol{L}(y_1, S_2'')$. Then we extend ψ_1' to a $+$PLO $\psi_1: \boldsymbol{N}(x_1, R'')\to\boldsymbol{N}(y_1, R'')$ by the usual method (see 3.11 of [6. I]), we have ψ_1 such that $\psi_1|N(x_1, S_1'')=\phi$ and $\psi_1(D_{1j})=D_{2j}$ $(j=1, p)$.

As above by (2), we have a $+$PLO $\psi_2'': \boldsymbol{L}(x_2, R'')\to\boldsymbol{L}(y_2, R'')$ such that $\psi_2''(\boldsymbol{L}(x_2, S_1''))=\boldsymbol{L}(y_2, S_2'')$. From (1) and Theorem 1 we can assume that

$$\psi_2''(D_{1j})=D_{2j} \quad (j=1, 2) .$$

Since D_{21} is a 2-element and $\psi_1\psi_2''^{-1}|D_{21}$ is a $+$PLO$: D_{21}\to D_{21}$, by 1.2 (d) there is a $+$PLO $\rho_2: \boldsymbol{L}(y_2, R'')\to\boldsymbol{L}(y_2, R'')$ such that $\rho_2|cl(L(y_2, R'')-H_{12})=1$, $\rho_2|D_{21}=\psi_1\psi_2''^{-1}$, where H_{jk} is a regular neighbourhood of D_{2j} in $L(y_k, R'')$ which is disjoint from H_{j+1k}. Hence $\psi_2': \boldsymbol{N}(x_2, R'')\to\boldsymbol{N}(y_2, R'')$ defined by $\psi_2'=\rho_2\psi_2''$ on $L(x_2, R'')$ and the usual method (see 3.11 of [6. I]) on $N(x_2, R'')-L(x_2, R'')$, is a $+$PLO which is consistent with ψ_1 on D_{21}. Then $\psi_2: \boldsymbol{N}(x_1\cup x_2, R'')\to\boldsymbol{N}(y_1\cup y_2, R'')$ defined by

$$\psi_2=\psi_1 \quad \text{on} \quad \boldsymbol{N}(x_1, R''),$$
$$\psi_2=\psi_2' \quad \text{on} \quad \boldsymbol{N}(x_2, R'')$$

is a $+$PLO such that $\psi_2|N(x_1\cup x_2, S_1'')=\phi$ and $\psi(D_{1j})=D_{2j}$ $(j=p, 2)$.

Since the same construction can be applied succesively, we have a $+$PLO $\psi_{p-1}: \boldsymbol{N}(x_1\cup\cdots\cup x_{p-1}, R'')\to\boldsymbol{N}(y_1\cup\cdots\cup y_{p-1}, R'')$ such that $\psi_{p-1}|N(x_1\cup\cdots\cup x_{p-1}, S_1'')=\phi$ and $\psi_{p-1}(D_{1j})=D_{2j}$ $(j=p, p-1)$.

As above there is a $+$PLO $\psi_p'': \boldsymbol{L}(x_p, R'')\to\boldsymbol{L}(y_p, R'')$ such that $\psi_p''(\boldsymbol{L}(x_p, S_1''))=\boldsymbol{L}(y_p, S_2'')$ and $\psi_p''(D_{1j})=D_{2j}$ $(j=p, p-1)$. By twice applications of 1.2 (d) there is a $+$PLO $\rho_p: \boldsymbol{L}(y_p, R'')\to\boldsymbol{L}(y_p, R'')$ such that $\rho_p|D_{2p-1}\cup D_{2p}=\psi_{p-1}\psi_p''^{-1}$ and $\rho_p|cl(L(y_p, R'')-H_{p-1p}-H_{pp})=1$. Hence $\psi_p': \boldsymbol{N}(x_p, R'')\to\boldsymbol{N}(y_p, R'')$ defined by $\psi_p'=\rho_p\psi_p''$ on $L(x_p, R'')$ and the usual way on $N(x_p, R'')-L(x_p, R'')$ is a $+$PLO which is consistent with ψ_{p-1}. Then

$$\psi=\psi_p: \boldsymbol{N}(x_1\cup\cdots\cup x_p, R'')\to\boldsymbol{N}(y_1\cup\cdots\cup y_p, R'')$$

defined by taking

$$\psi=\psi_{p-1} \quad \text{on} \quad \boldsymbol{N}(x_1\cup\cdots\cup x_{p-1}, R'') \qquad \text{and}$$
$$\psi=\psi_p' \quad \text{on} \quad \boldsymbol{N}(x_p, R'')$$

is a $+$PLO such that $\psi|S_1=\phi$.

3.4. From 3.3, 1.2 (a) and Theorem 1 we have the well known result [3].

REMARK. *Any two oriented 1-spheres in* R^3 *are semicongruent in* R^3.

Therefore the topological type of any regular neighbourhood of 1-sphere in R^3 is the full torus, that is to say, the Cartesian product of a 2-element with a 1-sphere.

3.5. Lemma. *If $\boldsymbol{S}_i$ are 1-spheres of 3-spheres $\boldsymbol{S}_i^3$ $(i=1, 2)$ and ϕ is a $+PLO: \boldsymbol{S}_1 \cup \boldsymbol{N}(x_h, S_1^{3\prime\prime}) \cup \cdots \cup \boldsymbol{N}(x_k, S_1^{3\prime\prime}) \rightarrow \boldsymbol{S}_2 \cup \boldsymbol{N}(y_h, S_2^{3\prime\prime}) \cup \cdots \cup \boldsymbol{N}(y_k, S_2^{3\prime\prime})$ such that $\phi(\boldsymbol{S}_1)=\boldsymbol{S}_2$, $\phi(\boldsymbol{N}(x_h, S_1^{3\prime\prime}))=\boldsymbol{N}(y_h, S_2^{3\prime\prime})$, $\cdots$, $\phi(\boldsymbol{N}(x_k, S_1^{3\prime\prime}))=\boldsymbol{N}(y_k, S_2^{3\prime\prime})$ and $\phi(N(x_l, S_1^{3\prime\prime}) \cap N(x_m, S_1^{3\prime\prime}))=N(y_l, S_2^{3\prime\prime}) \cap N(y_m, S_2^{3\prime\prime})$ for $l \in (h, \cdots, k)$, $m \overline{\in} (h, \cdots, k)$, where x_j, $\phi(x_j)=y_j$ are the vertices of S_1', S_2' respectively $(j=1, \cdots, p)$ and $(h, \cdots, k) \neq (1, \cdots, p)$. Then there is a $+PLO$ $\psi: \boldsymbol{N}(S_1'', S_1^{3\prime\prime}) \rightarrow \boldsymbol{N}(S_2'', S_2^{3\prime\prime})$ such that $\psi | S_1 \cup N(x_h, S_1^{3\prime\prime}) \cup \cdots \cup N(x_k, S_1^{3\prime\prime})=\phi$, $\psi(\boldsymbol{N}(x_j, S_1^{3\prime\prime}))=\boldsymbol{N}(y_j, S_2^{3\prime\prime})$ for each $j=1, \cdots, p$ and $\psi(a_1)$, $\psi(b_1)$ are homotopic to a_2, b_2 respectively, where a_i, b_i is an appropriately oriented canonical curve system of the torus $\dot{N}(S_i'', S_i^{3\prime\prime})$ such that a_i is homotopic to zero in $N(S_i'', S_i^{3\prime\prime})$ $(i=1, 2)$.*

Proof. The first part of Lemma is clear from the proof of Lemma 3.3. Since a_1 is homotopic to zero in $N(S_1'', S_1^{3\prime\prime})$, $\psi(a_1)$ is also homotopic to zero in $N(S_2'', S_2^{3\prime\prime})$, that is to say, $\psi(a_1)$ is homotopic to a_2.

If $\psi(b_1)$ is not homotopic to b_2, it is homotopic to $a_2^l b_2$, where $l \neq 0$ is an integer. Since $(h, \cdots, k) \neq (1, \cdots, p)$, there is a number, say p, such that ϕ does not define on $N(x_p, S_1^{3\prime\prime})$. In the construction of ψ, we take a $+$PLO $\rho_p: \boldsymbol{L}(y_p, S_2^{3\prime\prime}) \rightarrow \boldsymbol{L}(y_p, S_2^{3\prime\prime})$ such that

$$\rho_p | D_{2p-1} \cup D_{2p} = \psi_{p-1} \psi_p''^{-1} \quad \text{and}$$
$$\rho_p | cl(L(y_p, S_2^{3\prime\prime}) - H_{p-1p} - H_{pp})$$

is an appropriate l-times rotation of the cylinder $cl(L(y_p, S_2^{3\prime\prime}) - H_{p-1p} - H_{pp})$ about S_2 which is the identity on the boundary of the cylinder and whose direction is the inverse of the orientation of a_2. Thus the resulting ψ is a $+$PLO: $\boldsymbol{N}(S_1'', S_1^{3\prime\prime}) \rightarrow \boldsymbol{N}(S_2'', S_2^{3\prime\prime})$ such that $\psi(b_1)$ is homotopic to b_2. Hence ψ is the required $+$PLO.

4. $(n-1)$-Manifolds in R^n $(n=3, 4)$

4.1. Let M^{n-1} be a manifold in R^n $(n=3, 4)$ and x_0, x_j vertices of M' which are the boundary of the 1-simplex (x_0, x_j) of M', then

$$N(x_0, M'') \cap N(x_j, M'') = C_j$$

is an $(n-2)$-element which is the dual cell of (x_0, x_j) in M'', and $N(x_0, R'') \cap N(x_j, R'') = E_j$ is an $(n-1)$-element which is the dual cell of (x_0, x_j) in R''. Let $x_1, \cdots, x_p$ be vertices of M' which belong to $L(x_0, M')$,

$\bigcup_{j=1}^{p} C_j = L(x_0, M'')$, an $(n-2)$-sphere if $x_0 \in IM$, an $(n-2)$-element if $x_0 \in \dot{M}$ and

$$\bigcup_{j=1}^{p} E_j = N(L(x_0, M''), L(x_0, R'')) .$$

4. 2. Lemma n. *Let S_i^{n-1} be $(n-1)$-spheres $(n=2, 3, 4)$, let $S_i^{n-2} \subset S_i^{n-1}$ be $(n-2)$-spheres $(i=1, 2)$ and $\phi: (S_1^{n-2} \cup E_{1h} \cup \cdots \cup E_{1k}) \to (S_2^{n-2} \cup E_{2h} \cup \cdots \cup E_{2k})$ be a $+PLO$ such that $\phi(S_1^{n-2}) = S_2^{n-2}$, $\phi(E_{1h}) = E_{2h}, \cdots, \phi(E_{1k}) = E_{2k}$ and $\phi(E_{1l} \cap E_{1m}) = E_{2l} \cap E_{2m}$ for $l \in (h, \cdots, k)$, $m \overline{\in} (h, \cdots, k)$, where E_{ij} is the star of vertex x_{ij} $(j=1, \cdots, p)$ of $(S_i^{n-2})''$ in $(S_i^{n-1})''$ such that $\bigcup_{j=1}^{p} E_{ij} = N((S_i^{n-2})'', (S_i^{n-1})'')$ and IE_{ij}, IE_{ik} are disjoint for $j \neq k$ and $\phi(x_{1j}) = x_{2j}$. Then there is a $+PLO$ $\psi: S_1^{n-1} \to S_2^{n-1}$ such that $\psi | S_1^{n-2} \cup E_{1h} \cup \cdots \cup E_{1k} = \phi$ and $\psi(E_{1j}) = E_{2j}$ for each j.*

Proof. We shall prove the Lemma by induction. Since the Lemma is clear for the case $n=2$, we wish to make the inductive step.

Let C_1^{n-1}, C_2^{n-1} be $(n-1)$-elements and $\rho: (C_1^{n-1})^{\cdot} \to (C_2^{n-1})^{\cdot}$ a +PLO, there is a +PLO $\eta: C_2^{n-1} \to C_2^{n-1}$ such that $\eta | (C_1^{n-1})^{\cdot} = \rho$ (see 3.12 of [6. I]). Since by $(n-2, n-1) = 0$ for $n=3, 4$, $cl(S_i^{n-1} - \bigcup_{j=1}^{p} E_{ij})$ is two $(n-1)$-elements, it is sufficient to prove the Lemma that there is a +PLO $\psi: (\bigcup_{j=1}^{p} E_{1j}) \to (\bigcup_{j=1}^{p} E_{2j})$ such that $\psi | S_1^{n-2} \cup E_{1h} \cup \cdots \cup E_{1k} = \phi$ and $\psi(E_{1j}) = E_{2j}$ for each j.

We construct a +PLO ψ by the stepwise extension of ϕ. Let E_{1q} be a star such that $q \neq h, \cdots, k$. Since $(n-3, n-2) = 0$ for $n=3, 4$, there is a +PLO $\psi_q'': (E_{1q})^{\cdot} \to (E_{1q})^{\cdot}$ such that $\psi_q''(\dot{D}_{1q}) = \dot{D}_{2q}$ where D_{iq} is the star of x_{iq} of $(S_i^{n-2})''$ in $(S_i^{n-2})''$. If there is an E_{1j} such that $j \in (h, \cdots, k)$ and $E_{1j} \cap E_{1q}$ is not empty. Since $\phi\psi_q''^{-1} | \dot{D}_{2q}$ is a +PLO, by 1.2 (d), $\phi\psi_q''^{-1} | \dot{D}_{2q} \approx 1$. By Theorem 5 in 6.2 there is a +PLO $\eta: (E_{2q})^{\cdot} \to (E_{2q})^{\cdot}$ such that $\eta | \dot{D}_{2q} = \phi\psi_q''^{-1} | \dot{D}_{2q}$. Then $\psi_q': (E_{1q})^{\cdot} \to (E_{2q})^{\cdot}$ defined by $\psi_q' = \eta\psi_q''$ is a +PLO such that $\psi_q' | \dot{D}_{q1} = \phi$. If $E_{1j} \cap E_{1q}$ is empty for each $j=h, \cdots, k$, we take $\psi_q' = \psi_q''$.

Since $\dot{E}_{iq}$, $\dot{D}_{iq}$ may correspond with S_i^{n-2}, S_i^{n-3} of Lemma for the case $n-1$, by the inductive hypothesis and the usual extension (see 3.11 of [6. I]) we have a +PLO $\psi_q^*: E_{1q} \to E_{2q}$ such that $\psi_q^* | ((S_1^{n-1} \cup E_{1h} \cup \cdots \cup E_{1k}) \cap E_{1q}) = \phi$ and $\psi_q^*(F_{1r}) = F_{2r}$, where F_{ir} is the star of vertex x_{ir} of $(\dot{D}_{iq})''$ in $(\dot{E}_{2q})''$.

Then $\psi_1: (S_1^{n-2} \cup E_{1h} \cup \cdots \cup E_{1k} \cup E_{1q}) \to (S_2^{n-2} \cup E_{2h} \cup \cdots \cup E_{2k} \cup E_{2q})$ defined by taking

$$\psi_1 = \phi \quad \text{on} \quad S_1^{n-2} \cup E_{1h} \cup \cdots \cup E_{1k}$$
$$\psi_1 = \psi_q^* \quad \text{on} \quad E_{1q}$$

is a +PLO which is the extension of ϕ over $S_1^{n-2}\cup E_{1h}\cup\cdots\cup E_{1k}\cup E_{1q}$ and $\psi_1(\boldsymbol{E}_{1q})=\boldsymbol{E}_{2q}$. By the same constructions we can extend ϕ to $\psi:(\bigcup_{j=1}^{p}\boldsymbol{E}_{1j})\to(\bigcup_{j=1}^{p}\boldsymbol{E}_{2j})$ such that $\psi(\boldsymbol{E}_{1j})=\boldsymbol{E}_{2j}$ for each $j=1,\cdots,p$. This completes the proof.

4.3. The following is evident.

Lemma n. *Let $\boldsymbol{S}_i^{n-1}$ be $(n-1)$-spheres $(n=2, 3, 4)$, let $\boldsymbol{C}_i^{n-2}\subset\boldsymbol{S}_i^{n-1}$ be $(n-2)$-elements $(i=1, 2)$ and let $\phi:(\boldsymbol{C}_1^{n-2}\cup\boldsymbol{E}_{1h}\cup\cdots\cup\boldsymbol{E}_{1k})\to(\boldsymbol{C}_2^{n-2}\cup\boldsymbol{E}_{2h}\cup\cdots\cup\boldsymbol{E}_{2k})$ be a +PLO such that $\phi(\boldsymbol{C}_1^{n-2})=\boldsymbol{C}_2^{n-2}$, $\phi(\boldsymbol{E}_{1h})=\boldsymbol{E}_{2h},\cdots,\phi(\boldsymbol{E}_{1k})=\boldsymbol{E}_{2k}$ and $\phi(E_{1l}\cap E_{1m})=E_{2l}\cap E_{2m}$ for $l\in(h,\cdots,k)$, $m\bar{\in}(h,\cdots,k)$, where E_{ij} is the star of vertex x_{ij} $(j=1,\cdots,p)$ of $(C_i^{n-2})''$ in $(S_i^{n-1})''$ such that $\bigcup_{j=1}^{p}E_{ij}=N((C_i^{n-2})'', (S_i^{n-1})'')$ and IE_{ij}, IE_{ik} are disjoint for $j\neq k$ and $\phi(x_{1j})=x_{2j}$. Then there is a +PLO $\psi:\boldsymbol{S}_1^{n-1}\to\boldsymbol{S}_2^{n-1}$ such that $\psi|C_1^{n-2}\cup E_{1h}\cup\cdots\cup E_{1k}=\phi$ and $\psi(\boldsymbol{E}_{1j})=\boldsymbol{E}_{2j}$ for each $j=1,\cdots,p$.*

4.4. Lemma. *If M_i $(i=1, 2)$ are $(n-1)$-manifolds in R^n $(n=3, 4)$ and $\phi:\boldsymbol{M}_1\to\boldsymbol{M}_2$ is a +PLO, then there is a +PLO $\psi:\boldsymbol{N}(M_1'', R'')\to\boldsymbol{N}(M_2'', R'')$ such that $\psi|M_1=\phi$.*

Proof. Let x_j be vertices of M_1' and $\phi(x_j)=y_j$ vertices of M_2' $(j=1,\cdots,m)$. Since the topological type of $L(x_1, M_1'')$ coincides with that of $L(y_1, M_2'')$ (see 4.1), there is a +PLO $\psi|L(x_1, M_1''):\ \boldsymbol{L}(x_1, M_1'')\to\boldsymbol{L}(y_1, M_2'')$ and $L(z_{ij}, R'')$ is an $(n-1)$-sphere, where z_{ij} is a vertex of M_i', by Lemma 4.2 or Lemma 4.3 there is a +PLO $\psi_1':\boldsymbol{L}(x_1, R'')\to\boldsymbol{L}(y_1, R'')$ such that $\psi_1'|L(x_1, M_1'')=\phi$ and $\psi(\boldsymbol{E}_{1a})=\boldsymbol{E}_{2a}$, where E_{ia} is the star of vertex z_{ia} of $L(z_{i1}, M_i'')$ in $L(z_{i1}, R'')$ (see 4.2 or 4.3). Then, extending ψ_1' to a +PLO $\psi_1:\boldsymbol{N}(x_1, R'')\to\boldsymbol{N}(y_1, R'')$ by the usual method, we have

$$\psi_1|N(x_1, M_1'')=\phi \quad\text{and}\quad \psi_1(E_{1a})=E_{2a} \quad\text{for each}\quad a\,.$$

As above, by Lemma 4.2 or 4.3 we have a +PLO $\psi_2'':\boldsymbol{L}(x_2, R'')\to\boldsymbol{L}(y_2, R'')$ such that $\psi_2''|N(x_1, R'')\cap L(x_2, R'')=\psi_1$, $\psi_2''|L(x_1, M_1'')=\phi$ and $\psi_2''(\boldsymbol{F}_{1b})=\boldsymbol{F}_{2b}$, where F_{ib} is the star of vertex z_{ik} of $L(z_{i2}, M_i'')$ in $L(z_{i2}, R'')$. Then we extend ψ_2'' to a +PLO $\psi_2':\boldsymbol{N}(x_2, R'')\to\boldsymbol{N}(y_2, R'')$ by the usual method which is consistent with ψ_1. Thus $\psi_2:\boldsymbol{N}(x_1\cup x_2, R'')\to\boldsymbol{N}(y_1\cup y_2, R'')$ defined by taking

$$\psi_2=\psi_1 \quad \text{on} \quad N(x_1, R'')\,,$$
$$\psi_2=\psi_2'\,, \quad \text{on} \quad N(x_2, R'')$$

is a +PLO such that

$$\psi_2 | N(x_1 \cup x_2, M_1'') = \phi, \quad \psi_2(\boldsymbol{E}_{1a}) = \boldsymbol{E}_{2a} \quad \text{and}$$
$$\psi_2(\boldsymbol{F}_{1b}) = \boldsymbol{F}_{2b} \quad \text{for each} \quad a, b.$$

Since the same construction can be applied to $x_3, \cdots, x_m$ succesively, we have a $+$PLO $\psi_m : \boldsymbol{N}(x_1 \cup \cdots \cup x_m, R'') \to \boldsymbol{N}(y_1 \cup \cdots \cup y_m, R'')$ such that $\psi_m | N(x_1 \cup \cdots \cup x_m, M_1'') = \phi$. Thus $\psi = \psi_m$ is the required $+$PLO.

4.5. From 4.4 we have

Theorem 2. *Any two equivalent $(n-1)$-manifolds in R^n $(n=3, 4)$ are semicongruent.*

REMARK. From Theorem 2, 2.1 (c) and 1.2 (d), we have $\langle 3.4 \rangle = 0$ [6. II].

5. Orientable 2-manifolds in $\boldsymbol{R}^4$

5.1. Let M be a 2-manifold in R^4 and x_0, x_j vertices of M' which are the boundary of the 1-simplex (x_0, x_j) of M'. Then

$$N(x_0, M'') \cap N(x_j, M'') = C_j$$

is a 1-element which is the dual cell of (x_0, x_j) in M'', and

$$N(x_0, R'') \cap N(x_j, R'') = E_j$$

is a 3-element which is the dual cell of (x_0, x_j) in R''.

Let $x_1, \cdots, x_p$ be vertices of M' which belong to $L(x_0, M')$,

$$\bigcup_{j=1}^{p} C_j = L(x_0, M'') = \begin{cases} \text{a 1-sphere if} & x_0 \in IM \\ \text{a 1-element if} & x_0 \in \dot{M} \end{cases}$$

and
$$\bigcup_{j=1}^{p} E_j = N(L(x_0, M''), L(x_0, R'')).$$

5.2. Lemma. *Let $\boldsymbol{S}_i^3$ be 3-spheres, let $\boldsymbol{S}_i^1 \subset \boldsymbol{S}_i^3$ $(i=1, 2)$ be 1-spheres such that $\{\boldsymbol{S}_1^1, \boldsymbol{S}_1^3\} \equiv \{\boldsymbol{S}_2^1, \boldsymbol{S}_2^3\}$ and let $\phi : (\boldsymbol{S}_1^1 \cup \boldsymbol{E}_{1h} \cup \cdots \cup \boldsymbol{E}_{1k}) \to (\boldsymbol{S}_2^1 \cup \boldsymbol{E}_{2h} \cup \cdots \cup \boldsymbol{E}_{2k})$ be a $+$PLO such that $\phi(\boldsymbol{S}_1^1) = \boldsymbol{S}_2^1$, $\phi(\boldsymbol{E}_{1h}) = \boldsymbol{E}_{2h}, \cdots, \phi(\boldsymbol{E}_{1k}) = \boldsymbol{E}_{2k}$ and $\phi(E_{1l} \cap E_{1m}) = E_{2l} \cap E_{2m}$ for $l \in (h, \cdots, k)$, $m \bar{\in} (h, \cdots, k)$ and $(h, \cdots, k) \neq (1, \cdots, p)$, where E_{ij} is $N(z_{ij}, S_i^{3\prime\prime})$ in Lemma 3.5 $(j=1, \cdots, p)$. Then there is a $+$PLO $\psi : \boldsymbol{S}_1^3 \to \boldsymbol{S}_2^3$ such that $\psi | S_1^1 \cup E_{1h} \cup \cdots \cup E_{1k} = \phi$ and $\psi(\boldsymbol{E}_{1j}) = \boldsymbol{E}_{2j}$ for each $j=1, \cdots, p$.*

Proof. Since $\{\boldsymbol{S}_1^1, \boldsymbol{S}_1^3\} \equiv \{\boldsymbol{S}_2^1, \boldsymbol{S}_2^3\}$, there is a $+$PLO $\theta : \boldsymbol{S}_2^3 \to \boldsymbol{S}_1^3$ such that $\theta(\boldsymbol{S}_2^1) = \boldsymbol{S}_1^1$ and $\theta(N(S_2^1, S_2^{3\prime\prime}) = N(S_1^{1\prime\prime}, S_1^{3\prime\prime})$. Let a, b be the canonical curve system of $\dot{N}(S_2^{1\prime\prime}, S_2^{\prime\prime})$ such that a is homotopic to zero in $N(S_2^{1\prime\prime}, S_2^{3\prime\prime})$, $\theta(a)$, $\theta(b)$ is also the canonical curve system of $\dot{N}(S_1^{1\prime\prime}, S_1^{3\prime\prime})$. By Lemma 3.5 there is a $+$PLO $\psi' : \boldsymbol{N}(S_1^{1\prime\prime}, S_1^{3\prime\prime}) \to \boldsymbol{N}(S_2^{1\prime\prime}, S_2^{3\prime\prime})$ such that

$\psi' | S_1^1 \cup E_{1h} \cup \cdots \cup E_{1k} = \phi$ and $\psi'(\boldsymbol{E}_{1j}) = \boldsymbol{E}_{2j}$ for each j and $\psi'\theta(a)$, $\psi'\theta(b)$ are homotopic to a, b respectively. Thus $\psi'\theta | \dot{N}(S_2^{1\prime\prime}, S_2^{3\prime\prime}) : \dot{N}(S_2^{1\prime\prime}, S_2^{3\prime\prime}) \to \dot{N}(S_2^{1\prime\prime}, S_2^{3\prime\prime})$ is a PLO which induces the identical automorphism of the fundamental group of $\dot{N}(S_2^{1\prime\prime}, S_2^{3\prime\prime})$. By 1.2 (c), $\psi'\theta | \dot{N}(S_2^{1\prime\prime}, S_2^{3\prime\prime}) \approx 1$. By Theorem 5 in 6.2, there is an isotopy $\lambda_t : S_2^3 \to S_2^3$ such that $\lambda_0 = 1$, $\lambda_1 | \dot{N}(S_2^{1\prime\prime}, S_2^{3\prime\prime}) = \psi'\theta | \dot{N}(S_2^{1\prime\prime}, S_2^{3\prime\prime})$. Then $\psi : \boldsymbol{S}_1^3 \to \boldsymbol{S}_2^3$ defined by taking

$$\begin{aligned} \psi &= \psi' && \text{on } N(S_1^{1\prime\prime}, S_1^{3\prime\prime}) \\ \psi &= \lambda_1\theta^{-1} && \text{on } cl(S_1^3 - N(S_1^{1\prime\prime}, S_1^{3\prime\prime})) \end{aligned}$$

is the required $+$PLO.

5.3. Since $(2, 3) = 0$, the following is evident.

Lemma. *Let $\boldsymbol{S}_i^3$ be 3-spheres, let $\boldsymbol{C}_i^1 \subset \boldsymbol{S}_i^3$ $(i=1, 2)$ be 1-elements and let $\phi : (\boldsymbol{C}_1^1 \cup \boldsymbol{E}_{1h} \cup \cdots \cup \boldsymbol{E}_{1k}) \to (\boldsymbol{C}_2^1 \cup \boldsymbol{E}_{2h} \cup \cdots \cup \boldsymbol{E}_{2k})$ be a $+$PLO such that $\phi(\boldsymbol{C}_1^1) = \boldsymbol{C}_2^1$, $\phi(\boldsymbol{E}_{1h}) = \dot{\boldsymbol{E}}_{2h}, \cdots, \phi(\boldsymbol{E}_{1k}) = \boldsymbol{E}_{2k}$ and $\phi(E_{1l} \cap E_{1m}) = E_{2l} \cap E_{2m}$ for $l \in (h, \cdots, k)$, $m \in (h, \cdots, k)$, where E_{ij} is analogous to Lemma 5.2 $(j=1, \cdots, p)$. Then there is a $+$PLO $\psi : \boldsymbol{S}_1^3 \to \boldsymbol{S}_2^3$ such that $\psi | S_1^1 \cup E_{1h} \cup \cdots \cup E_{1k} = \phi$ and $\psi(\boldsymbol{E}_{1j}) = \boldsymbol{E}_{2j}$ for each $j = 1, \cdots, p$.*

5.4. Lemma. *If $\boldsymbol{M}_1, \boldsymbol{M}_2$ are oriented 2-manifolds in R^4 and $\phi : \boldsymbol{M}_1 \to \boldsymbol{M}_2$ is a $+$PLO such that for any vertex $x \in M_1'$*

$$(*) \qquad \{\boldsymbol{L}(x, M_1''),\ \boldsymbol{L}(x, R'')\} \equiv \{\boldsymbol{L}(\phi(x),\ M_2''),\ \boldsymbol{L}(\phi(x),\ R'')\}$$

then there is a $+$PLO $\psi : \boldsymbol{N}(M_1'', R'') \to \boldsymbol{N}(M_2'', R'')$ such that $\psi | M_1 = \phi$.

Proof. Let x_i be vertices of M_1' and $\phi(x_i) = y_i$ vertices of M_2' $(i=1, \cdots, n)$. Since the topological type of $L(x_1, M_1'')$ coincides with that of $L(y_1, M_2'')$ and there is a $+$PLO $\phi | L(x_1, M_1'') : \boldsymbol{L}(x_1, M_1'') \to \boldsymbol{L}(y_1, M_2'')$ by 5.2 or 5.3 and $(*)$ there is a $+$PLO $\psi_1' : \boldsymbol{L}(x_1, R'') \to \boldsymbol{L}(y_1, R'')$ such that $\psi_1 | L(x_1, M_1'') = \phi$ and $\psi_1'(\boldsymbol{E}_{1j}) = \boldsymbol{E}_{2j}$ for each j (see 5.2 and 5.3). Then, extending ψ_1' to a $+$PLO $\psi_1 : \boldsymbol{N}(x_1, R'') \to \boldsymbol{N}(y_1, R'')$ by the usual method, we have $\psi_1 | N(x_1, M_1'') = \phi$ and $\psi_1(E_{1j}) = E_{2j}$ for each j.

Since the analogous construction in 4.4 can be applied to $x_2, \cdots, x_{n-1}$ succesively whenever the conditions of Lemma 3.5 are fulfilled, we may assume that there is a $+$PLO $\psi_{n-1} : \boldsymbol{N}(x_1, \cdots, x_{n-1}, R'') \to \boldsymbol{N}(y_1 \cup \cdots \cup y_{n-1}, R'')$ such that $\psi_{n-1} | N(x_1 \cup \cdots \cup x_{n-1}, M_1'') = \phi$ and $\psi_{n-1}(\boldsymbol{G}_{1j}) = \boldsymbol{G}_{2j}$ for each j where G_{1j}, G_{2j} are E_{1j}, E_{2j} (of $N(x_n, S_1^{3\prime\prime})$, $N(y_n, S_2^{3\prime\prime})$) in Lemmas 5.2 and 5.3.

If $x_n \in \dot{M}$, by using 5.3 we can continue the construction and have the required $+$PLO $\psi = \psi_n$. If $x_n \in IM$, by $(*)$ there is $+$PLO $\theta : \boldsymbol{L}(y_n, R'') \to \boldsymbol{L}(x_n, R'')$ such that $\theta(\boldsymbol{L}(y_n, M_2'')) = \boldsymbol{L}(x_n, M_1'')$ and further

$\theta^{-1}|L(x_n, M_1'')=\phi$. Let $N(L(x_i, M_1''), L(x_i, R''))=N_{i1}$, $N(L(y_i, M_2''), L(y_i, R''))=N_{i2}$ and a, b a canonical curve system of $\dot{N}_{n2}$ such that a is homotopic to zero in N_{n2}, θa, θb is a canonical curve system of $\dot{N}_{n1}$ such that θa is homotopic to zero in N_{n1}.

If $\psi_{n-1}\theta|\dot{N}_{n2}:\dot{N}_{n2}\rightarrow\dot{N}_{n2}$ induces the identical automorphism of the fundamental group of $\dot{N}_{n2}$, by the method of 5.2 we have the required +PLO as in 4.4.

If $\psi_{n-1}\theta|\dot{N}_{n2}$ does not induce the identical automorphism of the fundamental group of $\dot{N}_{n2}$, $\psi_{n-1}\theta(b)$ is homotopic to $a^l b$, where $l\neq 0$ is an integer. Let (y_n, y_u, y_v) be a 2-simplex of M_2' whose vertices are y_n, y_u and y_v. Let $\dot{N}_{n2}\cap\dot{N}_{u2}=B_{nu}$, $\dot{N}_{n2}\cap\dot{N}_{v2}=B_{nv}$, $\dot{N}_{u2}\cap\dot{N}_{v2}=B_{uv}$, they are the cylinders. We deform ψ_{n-1} into a +PLO $\psi'_{n-1}:N(\bigcup_{i=1}^{n-1}x_i, M_1'')\rightarrow N(\bigcup_{i=1}^{n-1}y_i, M_2'')$ such that $\psi'_{n-1}|\boldsymbol{N}_{n1}:\boldsymbol{N}_{n1}\rightarrow\boldsymbol{N}_{n1}$ is a +PLO which arises from ψ_{n-1} and $\frac{l}{2}$-times rotations of B_{nu} and B_{nv} about $\dot{L}(y_n, M_2'')$ whose directions are the inverse of the orientation of a as in 3.5. Further $\psi'_{n-1}|(N_{u1}\cap N_{v1}):(N_{u1}\cap N_{v1})\rightarrow(N_{u2}\cap N_{v2})$ is a +PLO which arises from ψ_{n-1} and $\frac{l}{2}$-times rotations of B_{uv} about $L(y_u, M_2'')$ whose direction is the orientation of a as in 3.5. Since the rotations of B_{nu}, B_{uv} (B_{nv}, B_{uv}) do not induce to make the torsion of $\dot{N}_{u2}(\dot{N}_{v2})$, ψ'_{n-1} really exists such that $\psi'_{n-1}|N(\bigcup_{i\neq n,u,v}x_i, R'')=\psi_{n-1}$, $\psi'_{n-1}|N(\bigcup_{i=1}^{n-1}x_i, M'')=\phi$ and $\psi'_{n-1}\theta(b)$ is homotopic to b in $\dot{N}_{n2}$.

Then by 5.2 we can construct the required $\psi=\psi_n$.

5.5. REMARK. For nonorientable 2-manifolds the Lemma is also true with slight modifications of the expression.

5.6. From 5.4, 5.5 and Remark in 3.2, we have

Theorem 3. *Any two equivalent 2-manifolds in R^4 are semicongruent if and only if they are locally congruent.*

5.7. The following is well known [6. II, p. 135]. For a vertex x of a 2-manifold M' in R^4, $\{\boldsymbol{L}(x, M''), \boldsymbol{L}(x, R'')\}\neq 0$ is possible only if $x\in IM$. Hence there are k internal vertices $x_1, \cdots, x_k\in M'$ such that $\{\boldsymbol{L}(x_j, M''), \boldsymbol{L}(x_j, R'')\}\neq 0$. We call such points $x_1, \cdots, x_k$ *singular points* of M in R^4. If M has no singular point, we say that M is *locally flat* in R^4.

Lemma. *If $\boldsymbol{M}_i$ $(i=1, 2)$ are equivalent oriented 2-manifolds in R^4. Then $\boldsymbol{M}_1$, $\boldsymbol{M}_2$ are semicongruent if and only if there is a one-one corre-*

spondence between the singular points $x_1, \cdots, x_k$ *of* M_1 *and the singular points* $y_1, \cdots, y_k$ *of* M_2 *such that for each* $j=1, \cdots, k$

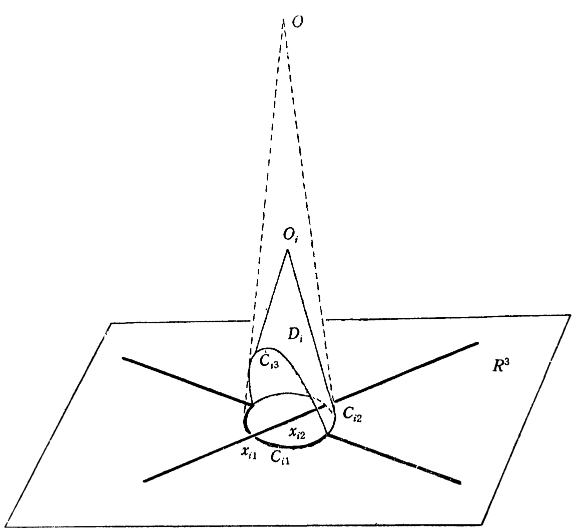
$$\{\boldsymbol{L}(x_j, M_1''), \boldsymbol{L}(x_j, R'')\} \equiv \{\boldsymbol{L}(y_j, M_2''), \boldsymbol{L}(y_j, R'')\} .$$

Proof. Let $\phi: \boldsymbol{M}_1 \to \boldsymbol{M}_2$ be an arbitrary PLO. Since there is an orientation reversing PLO: $\boldsymbol{M}_i \to \boldsymbol{M}_i$, we may assume that $\phi: \boldsymbol{M}_1 \to \boldsymbol{M}_2$ be a +PLO. Let $y_j'=\phi(x_j)$, by 1.2 (e) there is a +PLO $\rho: \boldsymbol{M}_2 \to \boldsymbol{M}_2$ such that $\rho(y_j')=y_j$ for each j. Since the local congruence of M_1, M_2 arises from $\rho\phi: \boldsymbol{M}_1 \to \boldsymbol{M}_2$, the sufficiency of Lemma is established by 5.4. The necessity of Lemma is evident.

5.8. Lemma. *Let* $\{\boldsymbol{S}^1, \boldsymbol{S}^3\}$ *be an arbitrary knot. Then there is a 2-sphere* $\boldsymbol{S}^2$ *in* R^4 *which has the only one singular point* O *such that*

$$\{\boldsymbol{L}(O, S^{2\prime\prime}), \boldsymbol{L}(O, R'')\} \equiv \{\boldsymbol{S}^1, \boldsymbol{S}^3\} .$$

Proof. We take an R^3 of R^4 such that there is a S^1 in R^3 and $\{\boldsymbol{S}^1, \boldsymbol{R}^3\} \equiv \{\boldsymbol{S}^1, \boldsymbol{S}^3\}$. Then we take an R^2 of R^3 and consider the regular projection of the knot $\boldsymbol{S}^1$ onto R^2. Let $y_1, \cdots, y_n$ be the double points of the projection and x_{i1} and x_{i2} the under and the upper points which



correspond to y_i $(i=1, \cdots, n)$. Let C_{i1} be a 1-element such that $IC_{i1} \ni x_{i1}$ and E_i a sufficient small 2-element such that $S^1 \cap E_i = C_{i1} \cup x_{i2}$ and $C_{i2}=cl(\dot{E}_i - C_{i1})$. Since we can assume that $(S^1 - \bigcup_{i=1}^{k} C_{i1}) \cup (\bigcup_{i=1}^{k} C_{i2}) = S_0$, for some $k \leqq n$, is unknotted, there is a 2-element D_0 of R^3 such that $\dot{D}_0 = S_0$. Let O be a point in $R^4 - R^3$ and D_i' a cone of the vertex O and the base $\dot{E}_i$ and $C_{i3} \subset D_i'$ a 1-element such that $\dot{C}_{i3} = \dot{C}_{i1}$ and C_{i1} corresponds topologically to C_{i3} by the generating lines of D_i', $i = 1, \cdots, k$ (see Fig). Let O_i be a point of $R^4 - (R^3 \cup C_{i3})$ which belongs to the interior of the cone of the vertex O and the base E_i, $i=1, \cdots, k$. Let D_i be a cone of the vertex O_i and the base $C_{i3} \cup C_{i2}$. Let D be a cone of the vertex O and the base $(S^1 - \bigcup_{i=1}^{k} C_{i1}) \cup (\bigcup_{i=1}^{k} C_{i3})$. It follows from the construction that $D \cup (\bigcup_{i=1}^{k} D_i) \cup D_0$ is a 2-sphere S^2. If we orient S^2 by the orientation of $\boldsymbol{S}^1$, we can easily see by Theorem 6 [6. I] that $\boldsymbol{S}^2$ has only one singular point O such that $\{\boldsymbol{L}(O, S^{2\prime\prime}), \boldsymbol{L}(O, R'')\} \equiv \{\boldsymbol{S}^1, \boldsymbol{S}^3\}$.

5.9. Theorem 4. *There is a one-one correspondence between the semicongruence classes of equivalent oriented 2-manifolds in R^4 and the (unordered) finite sets of knot types, where the empty set stands for the unknotted type.*

Proof. By Lemma in 5.7 it is sufficient to prove the Theorem that for any orientable 2-manifold M_0 and any set of knot types $(\kappa_1, \cdots, \kappa_k)$ there is a 2-manifold $\boldsymbol{M}$ in R^4 such that $\boldsymbol{M}$ is equivalent to M_0 and $\boldsymbol{M}$ has the singular points O_i whose knot types are κ_i $(i=1, \cdots, k)$.

If $(\kappa_1, \cdots, \kappa_k)$ is the empty set, $\boldsymbol{M}$ is a 2-manifold in $R^3 \subset R^4$ equivalent to M_0. Since a 1-sphere of a 2-sphere is flat in a 3-sphere which contains the 2-sphere and $\langle 1, 3 \rangle = 0$, $\boldsymbol{M}$ is locally flat in R^4.

If $(\kappa_1, \cdots, \kappa_k)$ is a nonempty set, there are by 5.9 mutually disjoint 2-spheres $\boldsymbol{S}_i^2$ in R^4 which have only one singular point O_i whose knot types are κ_i respectively $(i=1, \cdots, k)$ and a 2-manifold $\boldsymbol{M}_*$ in R^3 of R^4 equivalent to $\boldsymbol{M}_0$. Let $c_i \subset R^3$ $(i=1, \cdots, k)$ be mutually disjoint 1-elements such that for each i, $C_i \cap M_*$ is one of the two end points and in IM_*, $C_i \cap S_i$ is the other end point and in IS_i and C_i are disjoint form S_j $(i \neq j)$. Let H_i be an appropriate small regular neighbourhood of C_i in R^3. M_* and S_i divide H_i into three 3-elements, let E_i be a 3-element of H_i which contains C_i. Since by appropriate modifications of H_i the orientations of $\boldsymbol{M}_*$, $\dot{\boldsymbol{E}}_i$, $\boldsymbol{S}_i$ will be coherent on their intersections, $\boldsymbol{M} = \boldsymbol{M}_* + \bigcup_{i=1}^{k} (\dot{\boldsymbol{E}}_i + \boldsymbol{S}_i)$ is an oriented 2-manifold, where the addition is the algebraic addition of complexs. It is easily seen that $\boldsymbol{M}$ is equivalent to M_0 and has the singular points O_i whose types are κ_i $(i=1, \cdots, k)$. Thus $\boldsymbol{M}$ is the required manifold.

5.10. REMARK. In the next paper we shall prove the following: *If $\boldsymbol{M}_i^2$ $(i=1, 2)$ are equivalent oriented 2-manifolds in R^4 such that they are not semicongruent and M_i^3 $(i=1, 2)$ are the boundaries of regular neighbourhoods $U(M_i^2)$ in R^4, then M_1^3 and M_2^3 are not necessarily equivalent.*

6. Applications

6. 1. Since by Theorem 2 in 4. 5, the regular neighbourhood of any two equivalent $(n-1)$-manifolds in R^n $(n=3, 4)$ are equivalent, the following will be seen easily (from Lemmas 4. 2, 4. 3 and elementary considerations).

Let M be an $(n-1)$-manifold in R^n $(n=3, 4)$ and C^1 a 1-element which will be thought of as a cone of the vertex $O \in IC^1$ and the base $B=\dot{C}^1$ and C^2 a 2-element which will be thought of as a cone of the vertex $O \in \dot{C}^2$ and the case $B=$1-element C^1 of $\dot{C}^2$ such that $O \overline{\in} C^1$. Then

$$U(M, R) = (\bigcup_{x \in IM} C^1 \times x) \cup (\bigcup_{x \in \dot{M}} C^2 \times x),$$

where we identify (O, x) with x.

Since C^1, C^2 are the cone of the vertex O and the base B, $C^i = \bigcup_{t \in I} B_t$ $(i=1, 2)$, where $B_t = \bigcup_{y \in B} y_t$ and y_t is a point of the segment $\overline{Oy}$ such that $\overline{yy_t} : \overline{y_t O} = t : 1-t$, thus $C^i = \bigcup_{y \in B,\, t \in I} y_t$, where $y_1 = O$ and $y_0 = y$.

6. 2. Theorem 5. *If M^{n-1} is an $(n-1)$-manifold in R^n $(n=3, 4)$ and $\phi : M^{n-1} \to M^{n-1}$ is a PLO such that $\phi \approx 1$, then there is a $+$PLO $\psi : \boldsymbol{R}^n \to \boldsymbol{R}^n$ such that $\psi | M = \phi$.*

Proof. Let $\phi_t : M \to M$, $t \in I$, be an isotopy between $\phi (=\phi_1)$ and $1 (=\phi_0)$. Then $\psi : \boldsymbol{R} \to \boldsymbol{R}$ defined by taking

$$\psi(z) = z \quad \text{if} \quad z \in R - U(M, R),$$
$$\psi(z) = (y_t, \phi_t(x)), \quad \text{if} \quad z \in U(M, R) \quad \text{and} \quad z \in C^i \times x$$

such that $z = (y_t, x)$, is a $+$PLO $: \boldsymbol{R} \to \boldsymbol{R}$ such that $\psi | M = \phi$.

Corollary. *If H^{n-1} is an $(n-1)$-sphere or an $(n-1)$-element in R^n $(n=3, 4)$ and $\phi : \boldsymbol{H}^{n-1} \to \boldsymbol{H}^{n-1}$ is a $+$PLO, then there is a $+$PLO $\psi : \boldsymbol{R}^n \to \boldsymbol{R}^n$ such that $\psi | H = \phi$.*

This follows from Theorem 5 and 1. 2 (d) or 1. 2 (b).

6. 3. Since the oriented 2-manifold in R^3 is locally flat in R^4, the topological type of any regular neighbourhood of an orientable 2-manifold M in R^4 which is locally flat is the Cartesian product of a 1-element with a regular neighbourhood of M in R^3. Hence we have the following: Let $C^2 (C^3)$ be a 2-element (3-element) which will be thought of as a cone of the vertex $O \in IC^2$ $(O \in \dot{C}^3)$ and the base $B = \dot{C}^2$ (B=2-element C^2 of $\dot{C}^3$ and $C^2 \bar{\ni} O$). Then $U(M, R) = (\bigcup_{x \in IM} C^2 \times x) \cup (\bigcup_{x \in \dot{M}} C^3 \times x)$, where we identify $(0, x)$ with x. By the similar argument to Theorem 5, we have

Theorem 6. *If M^{n-2} is an orientable $(n-2)$-manifold in R^n $(n=3, 4)$ which is locally flat if $n=4$ and $\phi: M^{n-2} \to M^{n-2}$ is a PLO such that $\phi \approx 1$, then there is a +PLO $\psi: \boldsymbol{R}^n \to \boldsymbol{R}^n$ such that $\psi | M = \phi$.*

Corollary. *If H^{n-2} is an $(n-2)$-sphere or an $(n-2)$-element in R^n $(n=3, 4)$ which is locally flat if $n=4$ and $\phi: \boldsymbol{H}^{n-2} \to \boldsymbol{H}^{n-2}$ is a +PLO, then there is a +PLO $\psi: \boldsymbol{R}^n \to \boldsymbol{R}^n$ such that $\psi | H = \phi$.*

REMARK. If $n=4$, the condition of local flatness in Theorem 6 and its corollary is essential. By 1.2 (e) we easily have the following: *Let M be an orientable 2-manifold R^4. For any PLO $\phi: M \to M$ such that $\phi \approx 1$, there is a +PLO $\psi: \boldsymbol{R}^4 \to \boldsymbol{R}^4$ such that $\psi | M = \phi$ if and only if M is locally flat.*

6.4. The concept to be *free* defined by K. Borsuk [2] is an invariant under semicongruence. From the considerations above mentioned, we easily have the following:

Theorem 7. *Any $(n-1)$-manifold and any orientable $(n-2)$-manifold which are locally flat in R^n are free in R^n $(n=3, 4)$.*

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