Osaka Mathematical Journal
Vol. 8, No. 1, June, 1956.

On the Existence of Algebraically Closed Algebraic Extensions

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The existence of algebraically closed algebraic extensions of an A-algebraic system has been discussed by K. Shoda, in his paper [1], under the assumption that A-algebraic systems satisfy the fundamental conditions I, II, III and IV. He has obtained the following result: In order that there exists an algebraically closed algebraic extension of an A-algebraic system $\mathfrak{A}$, it is necessary and sufficient that $\mathfrak{A}$ satisfies the following three conditions:

a) If $\mathfrak{B}$ is an algebraic extension of $\mathfrak{A}$, and $\mathfrak{C}$ is an algebraic extension of $\mathfrak{B}$, then $\mathfrak{C}$ is an algebraic extension of $\mathfrak{A}$.

b) If $\mathfrak{B}$ is an algebraic extension of $\mathfrak{A}$, and α is algebraic over $\mathfrak{B}$, then $\mathfrak{B}(\alpha)$ is an algebraic extension of $\mathfrak{B}$.

c) If $\mathfrak{B}$ is an algebraic extension of $\mathfrak{A}$, then each polynomial $f(x)$ of $\mathfrak{B}(x)$ has a splitting system.

However, it seems to the writer that there is little difference between the existence of algebraically closed algebraic extensions and the condition c), and that the conditions I, II, III and IV are not essential regarding such a problem. The main purpose of the present paper is to prove the following theorem:

Theorem. *In order that there exists an algebraically closed algebraic extension of an A-algebraic system $\mathfrak{A}$, it is sufficient that $\mathfrak{A}$ satisfies the following three conditions*:

(A) *If $\mathfrak{B}$ is an algebraic extension of $\mathfrak{A}$, and $\mathfrak{C}$ is an algebraic extension of $\mathfrak{B}$, then $\mathfrak{C}$ is an algebraic extension of $\mathfrak{A}$.*

(B) *Let $\mathfrak{B}$ be an algebraic extension of $\mathfrak{A}$, and $\mathfrak{C}$ an extension of $\mathfrak{B}$ which is generated by $\mathfrak{B}$ and elements $\alpha, \beta, \cdots$. If $\mathfrak{C}$ has no $\mathfrak{B}$-polynomial-congruence, and the elements $\alpha, \beta, \cdots$ are algebraic over $\mathfrak{B}$, then $\mathfrak{C}$ is an algebraic extension of $\mathfrak{B}$.*

(C) *Let $\mathfrak{B}$ and $\mathfrak{C}$ be algebraic extensions of $\mathfrak{A}$, and let $\mathfrak{B}$ be contained in $\mathfrak{C}$. If $\mathfrak{B}$ is $\mathfrak{A}$-isomorphic to $\mathfrak{C}$, then $\mathfrak{B}=\mathfrak{C}$.*

In this theorem, only the sufficient condition is obtained. However, the foundation of the existence of the algebraically closed algebraic extension and the substance of the condition c) will come to be clear by this theorem and its proof.

The writer wishes to express his best thanks to Prof. K. Shoda for his kind encouragement and advice during the preparation of the present paper.

§ 1. Terminology, notations and some lemmas. First we shall explain terminology and notations which will be used in the following.

Let $\mathfrak{A}$ be a subsystem of an A-algebraic system $\mathfrak{B}$—hereafter the system V of finitary compositions and the family A of composition-identities will be fixed—then $\mathfrak{B}$ is called an *extension* of $\mathfrak{A}$. A congruence θ of $\mathfrak{B}$ is called an *$\mathfrak{A}$-polynomial-congruence,* if $\theta \neq 0$ and there are no distinct elements in $\mathfrak{A}$ which are congruent modulo θ. For $\alpha, \beta, \cdots \in \mathfrak{B}$, we denote by $\mathfrak{A}(\alpha, \beta, \cdots)$ the subsystem of $\mathfrak{B}$ generated by $\mathfrak{A}$ and $\alpha, \beta, \cdots$. An element α is said to be *algebraic* over $\mathfrak{A}$ if there is no $\mathfrak{A}$-polynomial-congruence of $\mathfrak{A}(\alpha)$, and $\mathfrak{B}$ is called an *algebraic extension* of $\mathfrak{A}$ if each element in $\mathfrak{B}$ is algebraic over $\mathfrak{A}$. $\mathfrak{A}$ is said to be *algebraically closed* if any element which is algebraic over $\mathfrak{A}$ belongs to $\mathfrak{A}$. Let $\mathfrak{B}$ and $\mathfrak{C}$ be two extensions of $\mathfrak{A}$. If there exists an isomorphism or a homomorphism of $\mathfrak{B}$ onto $\mathfrak{C}$ such that each element in $\mathfrak{A}$ is fixed, then we say that $\mathfrak{C}$ is *$\mathfrak{A}$-isomorphic* or *$\mathfrak{A}$-homomorphic* to $\mathfrak{B}$, and write $\mathfrak{B} \overset{\mathfrak{A}}{\cong} \mathfrak{C}$ or $\mathfrak{B} \overset{\mathfrak{A}}{\sim\!\!>} \mathfrak{C}$. Let $\mathfrak{B}$ be an extension of $\mathfrak{A}$, and θ a congruence of $\mathfrak{B}$. Then $\theta(\mathfrak{A})$ denotes the congruence of $\mathfrak{A}$ naturally defined by θ.

We denote by $\mathfrak{A}(x_1, x_2, \cdots)$ the free A-product of an A-algebraic system $\mathfrak{A}$ and free elements (symbols) $x_1, x_2, \cdots$. Then an $\mathfrak{A}$-polynomial-congruence π of $\mathfrak{A}(x_1, x_2, \cdots)$ is called a *polynomial* of $\mathfrak{A}(x_1, x_2, \cdots)$. Let $P(x_1, x_2, \cdots)$ be a set of relations (identities) $p(x_1, x_2, \cdots)$ which defines a polynomial π of $\mathfrak{A}(x_1, x_2, \cdots)$, then $P(x_1, x_2, \cdots)$ is called a *system of polynomial relations* of π. If $P(x_1, x_2, \cdots)$, $P'(x_1, x_2, \cdots)$, $\cdots$ are all the systems of polynomial relations of a polynomial π of the free A-product $\mathfrak{A}(x_1, x_2, \cdots)$, then the set-union $P^*(x_1, x_2, \cdots) = P(x_1, x_2, \cdots) \vee P'(x_1, x_2, \cdots) \vee \cdots$ is, of course, a system of polynomial relations of the polynomial π. Such a system $P^*(x_1, x_2, \cdots)$ is called a *full system of polynomial relations* of the polynomial π. The residue class system of $\mathfrak{A}(x_1, x_2, \cdots)$ modulo π is denoted by $\mathfrak{A}(x_1, x_2, \cdots)/\pi$ or $\mathfrak{A}(x_1, x_2, \cdots)/P(x_1, x_2, \cdots)$. Moreover let $\mathfrak{B}$ be an extension of $\mathfrak{A}$, and elements $\alpha_1, \alpha_2, \cdots$ belong to $\mathfrak{B}$. If $\mathfrak{A}(x_1, x_2, \cdots)/P(x_1, x_2, \cdots) \overset{\mathfrak{A}}{\sim\!\!>} \mathfrak{A}(\alpha_1, \alpha_2, \cdots)$ by the mapping $x_1 \to \alpha_1$, $x_2 \to \alpha_2, \cdots$, we say that the elements $\alpha_1, \alpha_2, \cdots$ *satisfy* the system $P(x_1, x_2, \cdots)$, and write $P[\alpha_1, \alpha_2, \cdots]$. In particular, if π is a maximal polynomial of $\mathfrak{A}(x_1, x_2, \cdots)$, it is, of course, clear that $P[\alpha_1, \alpha_2, \cdots]$

means that $\mathfrak{A}(x_1, x_2, \cdots)/P(x_1, x_2, \cdots) \overset{\mathfrak{A}}{\cong} \mathfrak{A}(\alpha_1, \alpha_2, \cdots)$ by the mapping $x_1 \to \alpha_1$, $x_2 \to \alpha_2, \cdots$.

Finally we denote by ω_α the initial ordinal number corresponding to the cardinal number $\aleph_\alpha$. Moreover, when a set of ordinal numbers is used, we always use a well-ordered set by the order of ordinal numbers. For example, if $\{\mu_\nu \mid 1 \leqq \nu < \omega_\alpha\}$ is a set of ordinal numbers, then the order of μ_ν satisfies that $\nu_1 < \nu_2$ implies $\mu_{\nu_1} < \mu_{\nu_2}$.

Lemma 1. *For any polynomial π of the free A-product $\mathfrak{A}(x_1, x_2, \cdots)$, there exists a maximal polynomial φ containing π.*

Proof. Let P be the partially ordered set consisting of all polynomials containing π. And let $\pi_1 \leqq \pi_2 \leqq \cdots \leqq \pi_i \leqq \cdots$ be any subchain of P. Then it is clear that $\psi = \bigcup \pi_i$ is a congruence of $\mathfrak{A}(x_1, x_2, \cdots)$. Moreover ψ is a polynomial. Because, if ψ is not a polynomial, then there are distinct elements a, b in $\mathfrak{A}$ such that $a \overset{\psi}{\sim} b$. Hence there exists π_i such that $a \overset{\pi_i}{\sim} b$. This contradicts the assumption that π_i is a polynomial. Hence any subchain of P has an upper bound. Therefore P has a maximal element φ by Zorn's Lemma, i.e. there exists a maximal polynomial φ containing π.

Lemma 2. *Let $P(x_1, x_2, \cdots, x_n)$ be a system of polynomial relations of a polynomial π of the free A-product $\mathfrak{A}(x_1, x_2, \cdots, x_n)$, where n is a finite number. Let λ_0 be some transfinite ordinal number, and Π_0 a congruence of the free A-product $\mathfrak{A}(x_\nu \mid 1 \leqq \nu < \lambda_0)$ which is defined by all the relations of*

$$(*) \qquad P(x_{\nu_1}, x_{\nu_2}, \cdots x_{\nu_n}) \text{ for all } \nu_1, \nu_2, \cdots, \nu_n \text{ satisfying } 1 \leqq \nu_1 < \nu_2 < \cdots < \nu_n < \lambda_0.$$

If Π_0 is a polynomial of $\mathfrak{A}(x_\nu \mid 1 \leqq \nu < \lambda_0)$, then for any ordinal number λ_1 which is larger than n, the congruence Π_1 of the free A-product $\mathfrak{A}(x_\nu \mid 1 \leqq \nu < \lambda_1)$ which is defined by all the relations of

$$(**) \qquad P(x_{\nu_1}, x_{\nu_2}, \cdots, x_{\nu_n}) \text{ for all } \nu_1, \nu_2, \cdots, \nu_n \text{ satisfying } 1 \leqq \nu_1 < \nu_2 < \cdots < \nu_n < \lambda_1$$

is a polynomial of $\mathfrak{A}(x_\nu \mid 1 \leqq \nu < \lambda_1)$.

Proof. Suppose that Π_1 is not a polynomial. Then there exist distinct elements a, b in $\mathfrak{A}$ such that $a \overset{\Pi_1}{\sim} b$. And $a \overset{\Pi_1}{\sim} b$ means that we can reach from a to b, using some finite number of relations of (**) in $\mathfrak{A}(x_\nu \mid 1 \leqq \nu < \lambda_1)$, i.e. using in a free A-product $\mathfrak{A}(x_{\mu_1}, x_{\mu_2}, \cdots, x_{\mu_m})$ of $\mathfrak{A}$ and some finite elements $x_{\mu_1}, x_{\mu_2}, \cdots, x_{\mu_m}$ in $\{x_\nu \mid 1 \leqq \nu < \lambda_1\}$. Hence we have $a \overset{\Pi_0}{\sim} b$, using the relations of (*) in $\mathfrak{A}(x_1, x_2, \cdots, x_m)$. This contradicts the assumption.

Lemma 3. *Let ρ_0 be an ordinal number, and $P_\rho(x_1, x_2, \cdots, x_{n+1})$ a system of polynomial relations of a polynomial π_ρ of the free A-product $\mathfrak{A}(x_1, x_2, \cdots, x_{n+1})$ for all ρ satisfying $1 \leqq \rho < \rho_0$, where n is a finite number. Let λ_0 be some ordinal number which is infinitely larger than ρ_0, and Π_0 a congruence of the free A-product $\mathfrak{A}(x_\nu \mid 1 \leqq \nu < \lambda_0)$ which is defined by all the relations of*

(♯) $P_\rho(x_\rho, x_{\nu_1}, x_{\nu_2}, \cdots, x_{\nu_n})$ *for all* $\rho, \nu_1, \nu_2, \cdots, \nu_n$ *satisfying* $1 \leqq \rho < \rho_0$ *and* $\rho < \nu_1 < \nu_2 < \cdots < \nu_n < \lambda_0$.

If Π_0 is a polynomial of $\mathfrak{A}(x_\nu \mid 1 \leqq \nu < \lambda_0)$, then for any ordinal number λ_1 which is larger than $n+1$, the congruence Π_1 of the free A-product $\mathfrak{A}(x_\nu \mid 1 \leqq \nu < \lambda_1)$ which is defined by all the relations of

(♯♯) $P_\rho(x_\rho, x_{\nu_1}, x_{\nu_2}, \cdots, x_{\nu_n})$ *for all* $\rho, \nu_1, \nu_2, \cdots, \nu_n$ *satisfying* $1 \leqq \rho < \rho_0$ *and* $\rho < \nu_1 < \nu_2 < \cdots < \nu_n < \lambda_1$

is a polynomial of $\mathfrak{A}(x_\nu \mid 1 \leqq \nu < \lambda_1)$.

Proof. Suppose that Π_1 is not a polynomial. Then there exist distinct elements a, b in $\mathfrak{A}$ such that $a \overset{\Pi_1}{\sim} b$. And $a \overset{\Pi_1}{\sim} b$ means that we can reach from a to b, using some finite number of relations of (♯♯) in $\mathfrak{A}(x_\nu \mid 1 \leqq \nu < \lambda_1)$, i.e. using in a free A-product $\mathfrak{A}(x_{\mu_1}, x_{\mu_2}, \cdots, x_{\mu_m})$ of $\mathfrak{A}$ and some finite elements $x_{\mu_1}, x_{\mu_2}, \cdots, x_{\mu_m}$ in $\{x_\nu \mid 1 \leqq \nu < \lambda_1\}$. Hence we have $a \overset{\Pi_0}{\sim} b$, using the relations of (♯) in $\mathfrak{A}(x_1, x_2, \cdots, x_{\rho_0}, x_{\rho_0+1}, \cdots, x_{\rho_0+m-1})$. This contradicts the assumption.

Lemma 4. *Let $\mathfrak{A}$ be an A-algebraic system satisfying the conditions* (A) *and* (B). *If there exists no algebraically closed algebraic extension of $\mathfrak{A}$, then for any ordinal number ξ there exists a chain of algebraic extensions of $\mathfrak{A}$ as follows:*

$$\mathfrak{A} = \mathfrak{A}_0 \subset \mathfrak{A}_1 \subset \mathfrak{A}_2 \subset \cdots \subset \mathfrak{A}_\mu \subset \cdots \subset \mathfrak{A}_\xi,$$

where $\mathfrak{A}_\mu = \bigvee_{\nu<\mu} \mathfrak{A}_\nu$ if μ is a limit ordinal number, and $\mathfrak{A}_\mu = \mathfrak{A}_{\mu-1}(\alpha_{\mu-1})$ otherwise.

Proof. We can easily obtain this lemma, using the transfinite induction.

§ 2. **Proof of Theorem.** We can easily prove our theorem by the following two lemmas:

Lemma 5. *Assume that an A-algebraic system $\mathfrak{A}$ satisfies the condition* (B). *Then $\mathfrak{A}$ does not satisfy the condition* (C) *if there exists a congruence Φ of the free A-product $\mathfrak{A}(x_i \mid 1 \leqq i < \omega_0)$ such that $\Phi = \bigvee_{n<\infty} \Phi_n$, where Φ_n satisfy the following conditions:*

1) *Φ_n is a polynomial of $\mathfrak{A}(x_i \mid 1 \leqq i < \omega_0)$ which is defined by all the relations of $F_n(x_{i_1}, x_{i_2}, \cdots, x_{i_n})$ for all $i_1, i_2, \cdots, i_n$ satisfying $1 \leqq i_1 < i_2 < \cdots < i_n < \omega_0$, where $F_n(x_1, x_2, \cdots, x_n)$ is a system of polynomial relations of a maximal polynomial φ_n of the free A-product $\mathfrak{A}(x_1, x_2, \cdots, x_n)$.*

2) *$\Phi_n \leqq \Phi_{n+1}$ for all n.*

3) *Any x_i is not congruent to any element in $\mathfrak{A}$ modulo Φ_1.*

4) *If $i \neq j$, then x_i and x_j are not congruent modulo Φ_2.*

Proof. Suppose that there exists such a congruence Φ of $\mathfrak{A}(x_i \mid 1 \leqq i < \omega_0)$. First we shall show that Φ is a maximal polynomial of $\mathfrak{A}(x_i \mid 1 \leqq i < \omega_0)$. It is clear that Φ is a polynomial, since $\Phi = \bigvee_{n<\infty} \Phi_n$, and $\Phi_1 \leqq \Phi_2 \leqq \cdots \leqq \Phi_n \leqq \cdots$ is a chain of the polynomials. If Φ is not maximal, then there exists a polynomial ψ of $\mathfrak{A}(x_i \mid 1 \leqq i < \omega_0)$ which properly contains Φ, and hence there exists a relation $g(x_{i_1}, x_{i_2}, \cdots, x_{i_n})$ of ψ, which is not derived from the relations of Φ. The polynomial of $\mathfrak{A}(x_{i_1} x_{i_2}, \cdots, x_{i_n})$ defined by all the relations of $F_n(x_{i_1}, x_{i_2}, \cdots, x_{i_n})$ and the relation $g(x_{i_1}, x_{i_2}, \cdots, x_{i_n})$ properly contains the polynomial of $\mathfrak{A}(x_{i_1}, x_{i_2}, \cdots, x_{i_n})$ defined by all the relations of $F_n(x_{i_1}, x_{i_2}, \cdots, x_{i_n})$. This contradicts the assumption that $F_n(x_1, x_2, \cdots, x_n)$ is the system of polynomial relations of the maximal polynomial φ_n of $\mathfrak{A}(x_1, x_2, \cdots, x_n)$.

Now we shall prove that $\mathfrak{A}(\alpha_i \mid 1 \leqq i < \omega_0)$ is an algebraic extension of $\mathfrak{A}$, where $\mathfrak{A}(\alpha_i \mid 1 \leqq i < \omega_0)$ is A-isomorphic to $\mathfrak{A}(x_i \mid 1 \leqq i < \omega_0)/\Phi$ by the mapping $x_i \to \alpha_i$. Since Φ is the polynomial of $\mathfrak{A}(x_i \mid 1 \leqq i < \omega_0)$ and $F_1(x_i)$ is a system of polynomial relations of a maximal polynomial of $\mathfrak{A}(x_i)$, it is easily obtained that

$$\mathfrak{A}(\alpha_i) \overset{\mathfrak{A}}{\cong} \mathfrak{A}(x_i)/\Phi(\mathfrak{A}(x_i)) = \mathfrak{A}(x_i)/F_1(x_i) .$$

Hence α_i is algebraic over $\mathfrak{A}$. Since Φ is a maximal polynomial of $\mathfrak{A}(x_i \mid 1 \leqq i < \omega_0)$, and α_i is algebraic over $\mathfrak{A}$, it is easily varified that $\mathfrak{A}(\alpha_i \mid 1 \leqq i < \omega_0)$ is an algebraic extension of $\mathfrak{A}$ by the condition (B).

Next we shall show that $\mathfrak{A}(\alpha_i \mid 2 \leqq i < \omega_0)$ is $\mathfrak{A}$-isomorphic to $\mathfrak{A}(\alpha_i \mid 1 \leqq i < \omega_0)$. It is clear that $\mathfrak{A}(\alpha_i \mid 1 \leqq i < \omega_0) \overset{\mathfrak{A}}{\to} \mathfrak{A}(\alpha_i \mid 2 \leqq i < \omega_0)$ by the mapping $\alpha_i \to \alpha_{i+1}$. Moreover this $\mathfrak{A}$-homomorphism is an $\mathfrak{A}$-isomorphism, since Φ is a maximal polynomial of $\mathfrak{A}(x_i \mid 1 \leqq i < \omega_0)$

Finally we shall show that $\mathfrak{A}(\alpha_i \mid 1 \leqq i < \omega_0)$ properly contains $\mathfrak{A}(\alpha_i \mid 2 \leqq i < \omega_0)$. Suppose that $\alpha_1 = f(\alpha_2, \cdots, \alpha_n)$. Then we get $\alpha_2 =$

$f(\alpha_3, \cdots, \alpha_{n+1})$, since $\mathfrak{A}(\alpha_i | 2 \leqq i < \omega_0)$ is $\mathfrak{A}$-isomorphic to $\mathfrak{A}(\alpha_i | 1 \leqq i < \omega_0)$. Similarly we get $\alpha_1 = f(\alpha_3, \cdots, \alpha_{n+1})$, since $\mathfrak{A}(\alpha_i | i=1 \text{ or } 3 \leqq i < \omega_0)$ is $\mathfrak{A}$-isomorphic to $\mathfrak{A}(\alpha_i | 1 \leqq i < \omega_0)$. Hence $\alpha_1 = \alpha_2$. On the other hand,

$$\mathfrak{A}(\alpha_1, \alpha_2) \overset{\mathfrak{A}}{\cong} \mathfrak{A}(x_1, x_2)/\Phi(\mathfrak{A}(x_1, x_2)) = \mathfrak{A}(x_1, x_2)/F_2(x_1, x_2),$$

since Φ is a polynomial of $\mathfrak{A}(x_i | 1 \leqq i < \omega_0)$, and $F_2(x_1, x_2)$ is the system of polynomial relations of the maximal polynomial φ_2 of $\mathfrak{A}(x_1, x_2)$. Hence x_1 and x_2 are congruent modulo Φ_2. This contradicts the condition 4).

Now, it is obvious that $\mathfrak{A}$ does not satisfy the condition (C), by the existence of the above-mentioned algebraic extensions $\mathfrak{A}(\alpha_i | 1 \leqq i < \omega_0)$ and $\mathfrak{A}(\alpha_i | 2 \leqq i < \omega_0)$.

Lemma 6. *Let* $\mathfrak{A}$ *be an A-algebraic system satisfying the conditions* (A) *and* (B). *If there exists no algebracally closed algebraic extension of* $\mathfrak{A}$, *then there exists such a congruence* Φ *i.e. such polynomials* Φ_n *of* $\mathfrak{A}(x_i | 1 \leqq i < \omega_0)$ *as in Lemma* 5.

Proof. Let M_n be the set consisting of all the full systems of polynomial relations of all the maximal polynomials of the free A-product $\mathfrak{A}(x_1, x_2, \cdots, x_n)$, and $M = \bigvee_{n < \infty} M_n$. Moreover let $\aleph_\alpha$ be an infinite cardinal number which is larger than $\overline{\overline{M}}$, and $\aleph_\beta$ a cardinal number which is larger than $\aleph_\alpha^{\aleph_\alpha}$.

(I) *Existence of* Φ_1. By Lemma 4, there exists an algebraic extension of $\mathfrak{A}$, which is adjoined with $\aleph_\beta$ elements not contained in $\mathfrak{A}$. Let $A_0 = \{\alpha_\mu | 1 \leqq \mu < \omega_\beta\}$ be the well-ordered set consisting of the above $\aleph_\beta$ elements. We define an equivalence relation θ of A_0 as follows: α_i and α_j are eqivalent under θ, if and only if $\mathfrak{A}(\alpha_i)$ and $\mathfrak{A}(\alpha_j)$ are $\mathfrak{A}$-isomorphic by the mapping $\alpha_i \to \alpha_j$, i.e. there exists a full system $P_1(x_1) \in M_1$ such that $P_1[\alpha_i]$ and $P_1[\alpha_j]$. Since $\overline{\overline{M}}_1 < \aleph_\alpha < \aleph_\beta$, there exists at least one class $A_1 = \{\alpha_{\mu_\nu} | 1 \leqq \nu < \omega_\beta\}$, whose cardinal number is $\aleph_\beta$, of the classification of A_0 defined by the above equivalence relation θ. Hereafter, we simply denote by $A_1 = \{\alpha_\nu | 1 \leqq \nu < \omega_\beta\}$ the class $A_1 = \{\alpha_{\mu_\nu} | 1 \leqq \nu < \omega_\beta\}$. And let $F_1(x_1)$ be the full system of polynomial relations of a maximal polynomial φ_1 which determines the class A_1. Then it is clear that

[*] $\quad F_1[\alpha_i]$ for all i satisfying $1 \leqq i < \omega_\beta$.

Now we shall define a congruence Φ_1 of $\mathfrak{A}(x_i | 1 \leqq i < \omega_0)$, using the above full system $F_1(x_1) \in M_1$. Since $\mathfrak{A}(A_1)$ is an algebraic extension of $\mathfrak{A}$ satisfying [*], it is verified that Φ_1 is a polynomial by Lemma 2. Since $\mathfrak{A}(x_i)/\Phi_1(\mathfrak{A}(x_i)) = \mathfrak{A}(x_i)/F_1(x_i) \overset{\mathfrak{A}}{\cong} \mathfrak{A}(\alpha_i)$ and $\alpha_i \notin \mathfrak{A}$, it is clear that any

x_i is not congruent to any element in $\mathfrak{A}$ module Φ_1.

(II) *Existence of* Φ_2. First we shall define an equivalence relation $\theta_{\omega_\alpha}=\bigcap\theta_\mu(1\leqq\mu<\omega_\alpha)$ of A_1 such that θ_μ are defined in the following fashions: The equivalence relation θ_1 is defined as follows: α_i and α_j are equivalent under θ_1, if and only if $\mathfrak{A}(\alpha_1, \alpha_i)$ and $\mathfrak{A}(\alpha_1, \alpha_j)$ are $\mathfrak{A}$-isomorphic by the mapping $\alpha_1\to\alpha_1$ and $\alpha_i\to\alpha_j$, i.e. there exists a fall system $P_2(x_1, x_2)\in M_2$ such that $P_2[\alpha_1, \alpha_i]$ and $P_2[\alpha_1, \alpha_j]$. Let $A_{1,\kappa}=\{\alpha_{\kappa_1}, \alpha_{\kappa_2}, \cdots\}$ be classes of the classification defined by θ_1. Then θ_2 is defined as follows: α_{κ_i} and $\alpha_{\kappa' j}$ are not equivalent under θ_2 if $\kappa\neq\kappa'$, and α_{κ_i} and α_{κ_j} are equivalent under θ_2 if and only if $\mathfrak{A}(\alpha_{\kappa_1}, \alpha_{\kappa_i})$ and $\mathfrak{A}(\alpha_{\kappa_1}, \alpha_{\kappa_j})$ are $\mathfrak{A}$-isomorphic by the mapping $\alpha_{\kappa_1}\to\alpha_{\kappa_1}$ and $\alpha_{\kappa_i}\to\alpha_{\kappa_j}$, i.e. there exists a full system $P_2(x_1, x_2)\in M_2$ such that $P_2[\alpha_{\kappa_1}, \alpha_{\kappa_i}]$ and $P_2[\alpha_{\kappa_1}, \alpha_{\kappa_j}]$. Moreover, if μ is not a limit ordinal number, then θ_μ is determined from $\theta_{\mu-1}$ in the same fashion that θ_2 is determined from θ_1 as above mentioned, and if μ is a limit ordinal number, then $\theta_\mu=\bigcap_{\nu<\mu}\theta_\nu$.

Now, since the number of all classes of the classification of A_1 defined by θ_{ω_α} is smaller than $\aleph_\beta$, there exists at least one class B consisting of $\aleph_\beta$ elements. Let $B_\mu=\{\alpha_{\kappa_1(\mu)}, \alpha_{\kappa_2(\mu)}, \cdots\}$ be the class containing B, of the classification defined by θ_μ, and let $P_{2,\mu}(x_1, x_2)\in M_2$ be the full system of polynomial relations of a maximal polynomial which determines the class $B_{\mu+1}$. Then the set $\{\alpha_{\kappa_1(\mu)}\,|\,1\leqq\mu<\omega_\alpha\}$ satisfies that

[#] $\quad P_{2,\mu}[\alpha_{\kappa_1(\mu)}, \alpha_{\kappa_1(\nu)}]$ for all μ, ν satisfying $1\leqq\mu<\nu<\omega_\alpha$.

Now we define a classification of $\{P_{2,\mu}(x_1, x_2)\}$ by the equality of systems of polynomial relations. Since $\overline{\overline{M}}_2<\aleph_\alpha$, there exists at least one class consisting of $\aleph_\alpha$ systems of polynomial relations:

$$F_2(x_1, x_2)=P_{2,\mu_1}(x_1, x_2)=P_{2,\mu_2}(x_1, x_2)=\cdots=P_{2,\mu_\nu}(x_1, x_2)=\cdots.$$

Corresponding to $\{P_{2,\mu_\nu}(x_1, x_2)\}$, we define a subset A_2 of the set $\{\alpha_{\kappa_1(\mu)}\,|\,1\leqq\mu<\omega_\alpha\}$ as follows:

$$A_2=\{\alpha_{\kappa_1(\mu_1)}, \alpha_{\kappa_1(\mu_2)}, \cdots, \alpha_{\kappa_1(\mu_\nu)}, \cdots\}=\{\alpha_{\kappa_1(\mu_\nu)}\,|\,1\leqq\nu<\omega_\alpha\}.$$

Hereafter, we simply denote by $A_2=\{\alpha_\nu\,|\,1\leqq\nu<\omega_\alpha\}$ the subset $A_2=\{\alpha_{\kappa_1(\mu_\nu)}\,|\,1\leqq\nu<\omega_\alpha\}$. Then by [#] it is verified that

[**] $\quad F_2[\alpha_i, \alpha_j]$ for all i, j satisfying $1\leqq i<j<\omega_\alpha$.

Now we shall define a congruence Φ_2 of $\mathfrak{A}(x_i\,|\,1\leqq i<\omega_0)$, using the full system $F_2(x_1, x_2)\in M_2$ of polynomial relations of a maximal polynomial φ_2 of $\mathfrak{A}(x_1, x_2)$. Since $\mathfrak{A}(A_2)$ is an algebraic extension of $\mathfrak{A}$ satisfying [**], Φ_2 is a polynomial by Lemma 2. And we get $\Phi_1\leqq\Phi_2$, since

$F_2(x_i, x_j)$ contains $F_1(x_i)$ and $F_1(x_j)$. Moreover, since $\mathfrak{A}(x_i, x_j)/\Phi_2(\mathfrak{A}(x_i, x_j)) = \mathfrak{A}(x_i, x_j)/F_2(x_i, x_j) \overset{\mathfrak{A}}{\simeq} \mathfrak{A}(\alpha_i, \alpha_j)$ for all i, j satisfying $1 \leqq i < j < \omega_0$, x_i and x_j are not congruent modulo Φ_2 if $i \neq j$.

(III) *Existence of* Φ_3. A congruence ψ_1 of the free A-product $\mathfrak{A}(x_\mu | 1 \leqq \mu < \omega_\beta)$ is defined by all the relations of

$$F_2(x_i, x_j) \text{ for all } i, j \text{ satisfying } 1 \leqq i < j < \omega_\beta.$$

Then it is evident that ψ_1 is a polynomial of $\mathfrak{A}(x_\mu | 1 \leqq \mu < \omega_\beta)$ by Lemma 2, since Φ_2 is a polynomial of $\mathfrak{A}(x_i | 1 \leqq i < \omega_0)$. Hence there exists a maximal polynomial ψ_1^* containing ψ_1 by Lemma 1. By the condition (B), $\mathfrak{B}_1 = \mathfrak{A}(\alpha_\mu | 1 \leqq \mu < \omega_\beta)$ which is $\mathfrak{A}$-isomorphic to $\mathfrak{A}(x_\mu | 1 \leqq \mu < \omega_\beta)/\psi_1^*$ by the mapping $x_\mu \rightarrow \alpha_\mu$ is an algebraic extension of $\mathfrak{A}$, since ψ_1^* is a maximal polynomial and α_μ is algebraic over $\mathfrak{A}$. And we get $\alpha_\mu \neq \alpha_\nu$ if $\mu \neq \nu$, since $F_2[\alpha_\mu, \alpha_\nu]$ for all μ, ν satisfying $1 \leqq \mu < \nu < \omega_\beta$. Putting $\mathfrak{A}_1 = \mathfrak{A}(\alpha_1)$, we have $\mathfrak{B}_1 = \mathfrak{A}_1(\alpha_\mu | 1 < \mu < \omega_\beta)$. Thus, in the same fashion as in the proof of the existence of $F_2(x_1, x_2)$ in (II), we obtain that there exist a full system $P_{3,1}(x_1, x_2, x_3) \in M_3$ and a subset $\{\alpha_{\mu_\nu} | 1 < \nu < \omega_\alpha\}$ of $\{\alpha_\mu | 1 < \mu < \omega_\beta\}$ such that $P_{3,1}[\alpha_1, \alpha_{\mu_i}, \alpha_{\mu_j}]$ for all i, j satisfying $1 < i < j < \omega_\alpha$. And it is clear that $P_{3,1}(x_1, x_i, x_j)$ contains $F_2(x_1, x_i)$, $F_2(x_1, x_j)$ and $F_2(x_i, x_j)$.

Next, a congruence ψ_2 of the free A-product $\mathfrak{A}(x_\mu | 1 \leqq \mu < \omega_\beta)$ is defined by all the relations of

$$P_{3,1}(x_1, x_i, x_j) \text{ for all } i, j \text{ satisfying } 1 < i < j < \omega_\beta.$$

Then, it is verified that ψ_2 is a polynomial of $\mathfrak{A}(x_\mu | 1 \leqq \mu < \omega_\beta)$ by Lemma 3, since the congruence of $\mathfrak{A}(x_\nu | 1 \leqq \nu < \omega_\alpha)$ defined by all the relations of $P_{3,1}(x_1, x_i, x_j)$ for all i, j satisfying $1 < i < j < \omega_\alpha$ is a polynominal of $\mathfrak{A}(x_\nu | 1 \leqq \nu < \omega_\alpha)$. Hence there exists a maximal polynominal ψ_2^* containing ψ_2 by Lemma 1. By the condition (B), $\mathfrak{B}_2 = \mathfrak{A}(\alpha_\mu | 1 \leqq \mu < \omega_\beta)$ which is $\mathfrak{A}$-isomorphic to $\mathfrak{A}(x_\mu | 1 \leqq \mu < \omega_\beta)/\psi_2^*$ by the mapping $x_\mu \rightarrow \alpha_\mu$ is an algebraic extension of $\mathfrak{A}$, since ψ_2^* is a maximal polynomial and α_μ is algebraic over $\mathfrak{A}$. Moreover $\alpha_\mu \neq \alpha_\nu$ for all μ, ν satisfying $1 \leqq \mu < \nu < \omega_\beta$. If we put $\mathfrak{A}_2 = \mathfrak{A}(\alpha_2)$ and $\mathfrak{B}_2^* = \mathfrak{A}_2(\alpha_\mu | 2 < \mu < \omega_\beta)$, then, in the same fashion as in the proof of the existence of $F_2(x_1, x_2)$ in (II), we obtain that there exist a full system $P_{3,2}(x_1, x_2, x_3) \in M_3$ and a subset $\{\alpha_{\mu_\nu} | 2 < \nu < \omega_\alpha\}$ of $\{\alpha_\mu | 2 < \mu < \omega_\beta\}$ such that $P_{3,2}[\alpha_2, \alpha_{\mu_i}, \alpha_{\mu_j}]$ for all i, j satisfying $2 < i < j < \omega_\alpha$.

Next, a congruence ψ_3 of the free A-product $\mathfrak{A}(x_\mu | 1 \leqq \mu < \omega_\beta)$ is defined by all the relations of

$$P_{3,1}(x_1, x_i, x_j) \text{ for all } i, j \text{ satisfying } 1<i<j<\omega_\beta,$$
$$P_{3,2}(x_2, x_i, x_j) \text{ for all } i, j \text{ satisfying } 2<i<j<\omega_\beta.$$

Then ψ_3 is a polynomial of $\mathfrak{A}(x_\mu | 1 \leqq \mu < \omega_\beta)$ by Lemma 3, since the congruence of $\mathfrak{A}(x_\nu | 1 \leqq \nu < \omega_\alpha)$ defined by all the relations of $P_{3,1}(x_1, x_i, x_j)$ for all i, j satisfying $1<i<j<\omega_\alpha$ and $P_{3,2}(x_2, x_i, x_j)$ for all i, j satisfying $2<i<j<\omega_\alpha$ is a polynomial of $\mathfrak{A}(x_\nu | 1 \leqq \nu < \omega_\alpha)$. Hence there exists a maximal polynomial $\psi_3{}^*$ containing ψ_3. By the condition (B), $\mathfrak{B}_3 = \mathfrak{A}(\alpha_\mu | 1 \leqq \mu < \omega_\beta)$ which is $\mathfrak{A}$-isomorphic to $\mathfrak{A}(x_\mu | 1 \leqq \mu < \omega_\beta)/\psi_3{}^*$ by the mapping $x_\mu \to \alpha_\mu$ is an algebraic extension of $\mathfrak{A}$, And $\alpha_\mu \neq \alpha_\nu$ for all μ, ν satisfying $1 \leqq \mu < \nu < \omega_\beta$. Let $\mathfrak{A}_3 = \mathfrak{A}(\alpha_3)$ and $\mathfrak{B}_3{}^* = \mathfrak{A}_3(\alpha_\mu | 3 < \mu < \omega_\beta)$. Then there exist a full system $P_{3,3}(x_1, x_2, x_3) \in M_3$ and a subset $\{\alpha_{\mu_\nu} | 3 < \nu < \omega_\alpha\}$ of $\{\alpha_\mu | 3 < \mu < \omega_\beta\}$ such that $P_{3,3}[\alpha_3, \alpha_{\mu_i}, \alpha_{\mu_j}]$ far all i, j satisfying $3<i<j<\omega_\alpha$.

Now, let μ_0 be any ordinal number which is smaller than ω_α. If all the full systems $P_{3,\mu}(x_1, x_2, x_3) \in M_3$ $(1 \leqq \mu < \mu_0)$ are determined, then a full system $P_{3,\mu_0}(x_1, x_2, x_3) \in M_3$ is also determined in the following fashion: A congruence ψ_{μ_0} of the free A-product $\mathfrak{A}(x_\mu | 1 \leqq \mu < \omega_\beta)$ is determined by all the relations of

$$P_{3,\mu}(x_\mu, x_i, x_j) \text{ for all } \mu, i, j \text{ satisfying } 1 \leqq \mu < \mu_0 \text{ and } \mu<i<j<\omega_\beta.$$

Then the congruence ψ_{μ_0} is a polynomial of $\mathfrak{A}(x_\mu | 1 \leqq \mu < \omega_\beta)$. Because, if μ_0 is not a limit ordinal number, then it is clearly obtained that ψ_{μ_0} is a polynomial in the same fashion as in the case of ψ_2 or ψ_3, and if μ_0 is a limit ordinal number, then ψ_{μ_0} is a polynomial also, since $\psi_{\mu_0} = \bigvee_{\mu<\mu_0} \psi_\mu$, and $\psi_1 \leqq \psi_2 \leqq \cdots \leqq \psi_\mu \leqq \cdots$ $(\mu < \mu_0)$ is a chain of the polynomials of $\mathfrak{A}(x_\mu | 1 \leqq \mu < \omega_\beta)$. Therefore a full system $P_{3,\mu_0}(x_1, x_2, x_3) \in M_3$ is determined in the same fashion that $P_{3,1}(x_1, x_2, x_3)$, $P_{3,2}(x_1, x_2, x_3)$, $\cdots$ are determined. Thus, it is verified that there exist full systems $P_{3,\mu}(x_1, x_2, x_3) \in M_3$ $(1 \leqq \mu < \omega_\alpha)$ and an algebraic extension $\mathfrak{B}_{\omega_\alpha} = \mathfrak{A}(\alpha_\mu | 1 \leqq \mu < \omega_\beta)$ of $\mathfrak{A}$ such that

[##] $$P_{3,\mu}[\alpha_\mu, \alpha_i, \alpha_j] \text{ for all } \mu, i, j \text{ satisfying } 1 \leqq \mu < \omega_\alpha \text{ and } \mu<i<j<\omega_\beta.$$

We now define a classification of $\{P_{3,\mu}(x_1, x_2, x_3)\}$ by the equality of systems of polynomial relations. Since $\overline{\overline{M}}_3 < \aleph_\alpha$, there exists at least one class consisting of $\aleph_\alpha$ systems of polynomial relations:

$$F_3(x_1, x_2, x_3) = P_{3,\mu_1}(x_1, x_2, x_3) = P_{3,\mu_2}(x_1, x_2, x_3) = \cdots = P_{3,\mu_\nu}(x_1, x_2, x_3) = \cdots .$$

Corresponding to $\{P_{3,\mu_\nu}(x_1, x_2, x_3)\}$, we define a subset A_3 of the set $\{\alpha_\mu | 1 \leqq \mu < \omega_\beta\}$ as follows:

$$A_3 = \{\alpha_{\mu_1}, \alpha_{\mu_2}, \cdots, \alpha_{\mu_\nu}, \cdots\} = \{\alpha_{\mu_\nu} | 1 \leqq \nu < \omega_\alpha\} .$$

Hereafter, we simply denote by $A_3 = \{\alpha_\nu | 1 \leqq \nu < \omega_\alpha\}$ the set $A_3 = \{\alpha_{\mu_\nu} | 1 \leqq \nu < \omega_\alpha\}$. Then, by [##], it is easily verified that

[***] $F_3[\alpha_i, \alpha_j, \alpha_k]$ for all i, j, k satisfying $1 \leqq i < j < k < \omega_\alpha$.

Now we shall define a congruence Φ_3 of $\mathfrak{A}(x_i | 1 \leqq i < \omega_0)$, using the full system $F_3(x_1, x_2, x_3) \in M_3$ of polynomial relations of a maximal polynomial φ_3 of $\mathfrak{A}(x_1, x_2, x_3)$. Since $\mathfrak{A}(A_3)$ is an algebraic extension of $\mathfrak{A}$ satisfying [***], Φ_3 is a polynomial by Lemma 2. And we get $\Phi_2 \leqq \Phi_3$, since $F_3(x_i, x_j, x_k)$ contains $F_2(x_i, x_j)$, $F_2(x_i, x_k)$ and $F_2(x_j, x_k)$.

(IV) *Existence of* Φ_n. Suppose the existences of the full systems $F_1(x_1), F_2(x_1, x_2), \cdots, F_{n-1}(x_1, x_2, \cdots, x_{n-1})$. Then we can prove the existence of Φ_n in the almost same fashion as in the proof of the existence of Φ_3.

§ 3. **Some remarks.** It is, of course, clear that the above-mentioned theorem is a generalization of the existence theorem of an algebraically closed algebraic extension of a field. Moreover, we shall show that the following corollary which has been obtained in [3] is also obtained from our theorem:

Corollary. *If any extension of an A-algebraic system* $\mathfrak{A}$ *satisfies the fundamental conditions I, III and IV, and the condition that each subsystem is normal, then there exists an algebraically closed algebraic extension of* $\mathfrak{A}$.

This corollary is easily obtained by the following lemmas:

Lemma 7. *If any extension of* $\mathfrak{A}$ *satisfies the fundamental conditions I and IV, and the conditions that each subsystem is normal, then* $\mathfrak{A}$ *satisfies the following condition*:

(B*) *Let* $\mathfrak{B}$ *be an algebraic extension of* $\mathfrak{A}$. *If an extension* $\mathfrak{C}$ *of* $\mathfrak{B}$ *has no* $\mathfrak{B}$*-polynomial-congruence, then* $\mathfrak{C}$ *is an algebraic extension of* $\mathfrak{B}$.

Proof. Suppose that $\mathfrak{C}$ is not an algebraic extension of $\mathfrak{B}$. Then there exists a splitting extension $\mathfrak{S}$ of $\mathfrak{B}$ which is contained in $\mathfrak{C}$. And $\mathfrak{S}$ has a $\mathfrak{B}$-polynomial-congruence θ. Now let $\mathfrak{N}$ be a normal subsystem of $\mathfrak{S}$ corresponding to θ. Then it is clear that a congruence of $\mathfrak{C}$ whose normal subsystem is $\mathfrak{N}$ is a $\mathfrak{B}$-polynomial-congruence of $\mathfrak{C}$. This contradicts the assumption.

Lemma 8. *The condition* (B*) *implies the conditions* (A) *and* (B).

Proof. It is clear that (B*) implies (B). Hereafter we shall prove that (B*) implies (A). Let θ be any congruence of $\mathfrak{C}$. Then θ is not a $\mathfrak{B}$–polynomial-congruence, since $\mathfrak{C}$ is an glgebraic extension of $\mathfrak{B}$. Hence $\theta(\mathfrak{B})$ is not an $\mathfrak{A}$–polynomial-congruence, since $\mathfrak{B}$ is an algebraic extension of $\mathfrak{A}$. Hence θ is not an $\mathfrak{A}$–polynomial-congruence of $\mathfrak{C}$. Therefore, $\mathfrak{C}$ is an algebraic extension of $\mathfrak{A}$ by the condition (B*).

Lemma 9. *If any extension of* $\mathfrak{A}$ *satisfies the fundamental conditions I, III and IV, and the condition that each subsystem is normal, then* $\mathfrak{A}$ *satisfies the condition* (C).

Proof. Suppose that $\mathfrak{B}\overset{\mathfrak{A}}{\cong}\mathfrak{C}$ and $\mathfrak{B}\subseteq\mathfrak{C}$. Let α be any element in $\mathfrak{C}$, and let β be an element in $\mathfrak{B}$ corresponding to α by the isomorphism $\mathfrak{B}\overset{\mathfrak{A}}{\cong}\mathfrak{C}$. Then it is clear that $\mathfrak{A}(\alpha)\overset{\mathfrak{A}}{\cong}\mathfrak{A}(\beta)$, and there exists a maximal polynomial $f(x)$ of $\mathfrak{A}(x)$ such that $\mathfrak{A}(\alpha)\overset{\mathfrak{A}}{\cong}\mathfrak{A}(x)/f(x)$. Hence α and β are roots of $f(x)$, and hence we get $\mathfrak{A}(\alpha)=\mathfrak{A}(\beta)$ by Shoda's Lemma in [3]. Accordingly $\mathfrak{B}\ni\alpha$, i.e. $\mathfrak{B}\supseteq\mathfrak{C}$. Therefore $\mathfrak{B}=\mathfrak{C}$.

REMARK. The theorem in [2] is obtained by Lemma 9, without the assumption with respect to the operator ring.

(Received March 24, 1956)

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